Matriz fundamental:

Sea (E) $\dot{X} = A(t)X$ con $X(t) \in \mathbb{R}^n$, $A(t) \in M_{n\times n}(\mathbb{R})$

La matriz fundamental asociada a (E) es por definición la solución a la ecuación

$$(E^*) \begin{cases} \dot{M} = A(t)M \\ M(0) = Id_{n\times n} \end{cases} \qquad M(t) \in M_{n\times n}(\mathbb{R})$$

La vamos a denotar $M(t)$.

Otra forma de pensar a la matriz fundamental es la siguiente: $M(t) = \left(X_1(t) \middle| X_2(t) \middle| X_3(t) \middle| \cdots \middle| X_n(t) \right)$ $X_i(t) \in \mathbb{R}^n$

Cuando multiplicamos $A(t)$ por $M(t)$:

$$A(t)M(t) = A(t)\left(X_1(t)\,\middle|\,X_2(t)\,\middle|\,\cdots\,\middle|\,X_n(t)\right) = \left(A(t)X_1(t)\,\middle|\,\cdots\,\middle|\,A_n(t)X_n(t)\right)$$

$$\dot{M}(t) = \left(\dot{x}_1(t)\,\middle|\,\cdots\,\middle|\,\dot{x}_n(t)\right) =$$

$$M(0) = Id = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & 0 & & \vdots \\ & \vdots & & 0 \\ 0 & 0 & & 1 \end{pmatrix}$$

De la igualdad entre columnas obtenemos:

$$(E^*) = \begin{cases} \dot{X}_1 = A(t)X_1, \ \dot{X}_2(t) = A(t)X_2, \ \ldots, \ \dot{X}_n = A(t)X_n \\ X_1(0) = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ X_2(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ \ldots, \ X_n(0) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \end{cases}$$

(E^*) = n veces (E)

agregando n condiciones iniciales que forman la base canónica.

Objetivo: Si queremos resolver (E) con una condición inicial cualquiera $X(0) = X_0 \in \mathbb{R}^n$, basta hacer $X(t) = M(t)X_0$

(Principio de superposición)

importante respetar el orden

$$\dot{M} = A(t)M$$

Prueba: $X(0) = \overbrace{M(0)}^{Id} X_0 = Id\, X_0 = X_0$

$$\dot{X} = (M(t)X_0)' = \dot{M}(t)X_0 = A(t)\overbrace{M(t)X_0}^{X(t)} = A(t)X(t) \Rightarrow$$

$X(t)$ es solución de (E).

Resumen: $(E_0)\begin{cases}\dot{X} = A(t)X \\ X(0) = X_0\end{cases}$ la solución a (E_0)

es $$X(t) = \begin{pmatrix}\text{Matriz} \\ \text{fundamental}\end{pmatrix} \times \begin{pmatrix}\text{condición} \\ \text{inicial}\end{pmatrix} = M(t)X_0 .$$

Esto muestra la dependencia lineal de la solución $X(t)$ con respecto a la condición inicial:

La solución a $\begin{cases}\dot{X} = AX \\ X(0) = cX_0 + c'X_0'\end{cases}$ es

$$X(t) = M(t)\left(cX_0 + c'X_0'\right) = c\,\overbrace{M(t)X_0} + c'\,\overbrace{M(t)X_0'}$$

$$= c\,Y(t) + c'\,Z(t)$$ con Y y Z soluciones a

$$\begin{cases}\dot{Y} = A(t)Y \\ Y(0) = X_0\end{cases} \qquad \begin{cases}\dot{Z} = A(t)Z \\ Z(0) = X_0'\end{cases}$$

Ejemplo fácil: $n=1$ $\quad \underline{\dot{x} = a x}$

en $\mathbb{R}$, $a(t) = a \in \mathbb{R}$ constante.

$$\begin{cases} \dot{x} = ax \\ x(0) = 1 \end{cases}$$ llamando $M(t)$ a la solución es $M(t) = e^{at}$

Y la solución de $\begin{cases} \dot{x} = ax \\ x(0) = x_0 \end{cases}$ $x(t) = e^{at} x_0$

$= M(t) x_0$

es independiente de la condición inicial

caso 1) $x_0 = \boxed{2}$ $\quad x_1(t) = M(t)2$

caso 2) $x_0 = \boxed{3}$ $\quad x_2(t) = M(t)3$

caso 3) $x_0 = 5$ $\quad x_3(t) = M(t)5$

$$x_3(t) = x_1(t) + x_2(t)$$

Ejemplo: $A = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ cuál es la matriz fundamental de (E) $\dot{X} = AX$

$n=2$, en $\mathbb{R}^2$

Base canónica: $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\begin{cases} \dot{x} = \alpha x \\ \dot{y} = \beta y \end{cases}$$

La solución general es: $x(t) = x_0 e^{\alpha t}$

$y(t) = y_0 e^{\beta t}$

$$\begin{cases} \dot{X} = AX \\ X(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{cases} \qquad \begin{cases} \dot{X} = AX \\ X(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{cases}$$

$$X_1(t) = \begin{pmatrix} e^{\alpha t} \\ 0 \end{pmatrix} \qquad X_2(t) = \begin{pmatrix} 0 \\ e^{\beta t} \end{pmatrix}$$

$$\boxed{M(t) = \begin{pmatrix} e^{\alpha t} & 0 \\ 0 & e^{\beta t} \end{pmatrix}}$$

$$X(0) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

$$X(t) = \underbrace{\begin{pmatrix} e^{\alpha t} & 0 \\ 0 & e^{\beta t} \end{pmatrix}}_{M(t)} \underbrace{\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}}_{X_0} = \begin{pmatrix} x_0 e^{\alpha t} \\ y_0 e^{\beta t} \end{pmatrix}$$

Mas adelante: $M(t) = e^{At}$ exponencial de matrices.

Ejemplo: $A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ matriz de Jordan

$$\begin{cases} \dot{X} = AX \\ X(0) = X_0 \end{cases} \qquad \begin{cases} \dot{x} = \lambda x + y \\ \dot{y} = \lambda y \\ x(0) = x_0,\ y(0) = y_0 \end{cases} \rightsquigarrow \begin{cases} \dot{x} = \lambda x + y_0 e^{\lambda t} \\ x(0) = x_0 \\ y(t) = y_0 e^{\lambda t} \end{cases}$$

$$\rightsquigarrow \begin{cases} \overbrace{e^{-\lambda t}\dot{x} - \lambda e^{-\lambda t} x}^{(e^{-\lambda t} x)^{\cdot}} = y_0 \\ x(0) = x_0 \\ y(t) = y_0 e^{\lambda t} \end{cases} \rightsquigarrow \begin{cases} e^{-\lambda t} x = y_0 t + k \\ x(0) = x_0 \\ y(t) = y_0 e^{\lambda t} \end{cases}$$

$$x(t) = e^{\lambda t}(y_0 t + k) \qquad x(0) = k = x_0$$

$$\begin{cases} x(t) = e^{\lambda t}(y_0 t + x_0) \\ y(t) = e^{\lambda t} y_0 \end{cases}$$

$$X(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} & ? & \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

$$e^{\lambda t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

$$\Rightarrow \boxed{M(t) = e^{\lambda t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}} = e^{\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} t}$$

Ejemplo: $A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$

$$x(t) = r_0 e^{at} \cos(\theta_0 - bt)$$

$$y(t) = r_0 e^{at} \operatorname{sen}(\theta_0 - bt)$$

$$x(0) = r_0 \cos(\theta_0) = x_0$$

$$y(0) = r_0 \operatorname{sen}(\theta_0) = y_0$$

$$\cos(\theta_0 - bt) = \, ?$$

$$\operatorname{sen}(\theta_0 - bt) = \, ?$$

$$e^{i(\theta_0 - bt)} = \overbrace{\cos(\theta_0 - bt)}^{R} + i \overbrace{\operatorname{sen}(\theta_0 - bt)}^{I}$$

$$\parallel$$

$$(e^{i\theta_0} e^{-ibt}$$

$$\parallel$$

$$(\cos\theta_0 + i\operatorname{sen}\theta_0)(\cos(-bt) + i\operatorname{sen}(-bt))$$

$$\parallel$$

$$\overbrace{\cos(\theta_0)\cos(-bt) - \operatorname{sen}\theta_0 \operatorname{sen}(-bt)}^{R} + i\overbrace{(\operatorname{sen}\theta_0 \cos(-bt) + \cos\theta_0 \operatorname{sen}(-bt))}^{I}$$

$$r_0 \cos(\theta_0 - bt) = r_0 \left(\cos(\theta_0)\cos(-bt) - \operatorname{sen}\theta_0 \operatorname{sen}(-bt) \right)$$

$$= x_0 \cos(-bt) - y_0 \operatorname{sen}(-bt)$$

$$r_0 \operatorname{sen}(\theta_0 - bt) = r_0 \left(\operatorname{sen}\theta_0 \cos(-bt) + \cos\theta_0 \operatorname{sen}(-bt) \right)$$

$$= y_0 \cos(-bt) + x_0 \operatorname{sen}(-bt)$$

$$X(t) = e^{at} \begin{pmatrix} x_0 \cos(-bt) - y_0 \operatorname{sen}(-bt) \\ y_0 \cos(-bt) + x_0 \operatorname{sen}(-bt) \end{pmatrix} = e^{at} \begin{pmatrix} \cos(-bt) & -\operatorname{sen}(-bt) \\ \operatorname{sen}(-bt) & \cos(-bt) \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

$$= e^{at} \begin{pmatrix} \cos(bt) & \operatorname{sen}(bt) \\ -\operatorname{sen}(bt) & \cos(bt) \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = M(t) \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

Resumen:

$$\boxed{M(t) = e^{at}\begin{pmatrix} \cos(bt) & \operatorname{sen}(bt) \\ -\operatorname{sen}(bt) & \cos(bt) \end{pmatrix}}$$

$$A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

$$e^{\begin{pmatrix} a & b \\ -b & a \end{pmatrix} t}$$

Resumen en dimensión 2:

$$A = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \qquad A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \qquad A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

$$\underbrace{M(t) = \begin{pmatrix} e^{\alpha t} & 0 \\ 0 & e^{\beta t} \end{pmatrix} \qquad M(t) = e^{\lambda t}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \qquad M(t) = e^{at}\begin{pmatrix} \cos(bt) & \operatorname{sen}(bt) \\ -\operatorname{sen}(bt) & \cos(bt) \end{pmatrix}}_{X(t) = M(t)X_0}$$

Exponencial de una matriz:

Recordar la serie de Taylor de la exponencial real:

$$e^{x} = \frac{1}{0!} + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

Definimos la exponencial de una matriz A como la serie:

$$e^{A} = \sum_{k=0}^{\infty} \frac{A^k}{k!} \rightsquigarrow \text{es convergente (lo admitimos)}$$

Matiz fundamental como exponencial de una matriz:

Afirmación: sea (E) $\dot{X} = AX$ con A constante

$$\Rightarrow \quad M(t) = e^{At}$$

Prueba: $N(t) = e^{At}$

$$N(0) = e^{\overbrace{A\cdot 0}^{0}} = e^{0} = I + \overbrace{\frac{0}{1!} + \frac{0^2}{2!} + \cdots}^{0} = I \quad ✓$$

$$\dot{N}(t) = \frac{d}{dt}\left[e^{At}\right] = \frac{d}{dt}\left[I + At + \frac{\overbrace{(At)^2}^{A^2t^2}}{2!} + \frac{(At)^3}{3!} + \cdots\right]$$

No lo justifique

$$= 0 + \overbrace{A + A^2 t + \frac{A^3 t^2}{2!} + \cdots} = A\left[I + At + \frac{(At)^2}{2!} + \cdots\right] = Ae^{At} = AN(t)$$

$$\Rightarrow \quad \dot{N} = AN \quad \Rightarrow \quad N(t) = M(t). \quad XX$$

Conclusión: La solución de $\begin{cases} \dot{X} = AX \\ X(0) = X_0 \end{cases}$ es $X(t) = e^{At} X_0$

Calculo de e^{A}

Afirmación 1: Si $B = P^{-1}AP \Rightarrow e^{B} = P^{-1}e^{A}P$

$$e^{B} = I + B + \frac{B^2}{2!} + \frac{B^3}{3!} + \cdots = I + P^{-1}AP + \frac{(P^{-1}A\cancel{P})(\cancel{P^{-1}}AP)}{2!}$$

$$+ \frac{(P^{-1}A\cancel{P})(\cancel{P^{-1}}A\cancel{P})(\cancel{P^{-1}}AP)}{3!} + \cdots$$

$$= P^{-1}IP + P^{-1}AP + \frac{P^{-1}A^2P}{2!} + \frac{P^{-1}A^3P}{3!} + \cdots = P^{-1}\left[I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots\right]P = P^{-1}e^{A}P \;\#$$

Afirmación 2: $A = \left(\begin{array}{c|c} B_{n\times n} & 0 \\ \hline 0 & C_{m\times m} \end{array}\right) \Rightarrow e^A = \left(\begin{array}{c|c} e^B & 0 \\ \hline 0 & e^C \end{array}\right)$

$$A^k = \left(\begin{array}{c|c} B^k & \\ \hline & C^k \end{array}\right)$$

Ejemplo: $A = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ $\qquad e^A = \begin{pmatrix} e^\alpha & 0 \\ 0 & e^\beta \end{pmatrix}$

Ejemplo: $A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$ A es diagonalizable

Valores propios: $(1-\lambda)(3-\lambda) \begin{array}{l} \nearrow 1 \\ \searrow 3 \end{array}$

Vectores propios:

$$A - I = \begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix} \Longrightarrow \begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2y \\ 2y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow y = 0$$

$$v_1 = (1, 0)$$

$$A - 3I = \begin{pmatrix} -2 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2x + 2y \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x = y$$

$$v_2 = (1, 1)$$

$$P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad P^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \quad B = P^{-1} A P = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$$

$$e^{Bt} = \begin{pmatrix} e^t & 0 \\ 0 & e^{3t} \end{pmatrix} = P^{-1} e^{At} P \Rightarrow e^{At} = P \begin{pmatrix} e^t & 0 \\ 0 & e^{3t} \end{pmatrix} P^{-1}$$

$$e^{At} = P\begin{pmatrix} e^t & 0 \\ 0 & e^{3t} \end{pmatrix} P^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} e^t & 0 \\ 0 & e^{3t} \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} e^t & -e^t \\ 0 & e^{3t} \end{pmatrix}$$

$$e^{At} = \begin{pmatrix} e^t & -e^t + e^{3t} \\ 0 & e^{3t} \end{pmatrix} \leftarrow \text{matriz fundamental}$$

$$X(t) = \begin{pmatrix} e^t & -e^t + e^{3t} \\ 0 & e^{3t} \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x_0 e^t + y_0(-e^t + e^{3t}) \\ y_0 e^{3t} \end{pmatrix}$$