$\dot{X} = AX$

$X = \begin{pmatrix} x \\ y \end{pmatrix} \quad A = \begin{pmatrix} 3 & -4 \\ 2 & -1 \end{pmatrix}$

$\lambda = \frac{2 \pm \sqrt{4 - 4 \cdot 5}}{2} = 1 \pm \sqrt{-4}$

$= 1 \pm 2i$

$$\boxed{\lambda = 1 + 2i \quad \bar{\lambda} = 1 - 2i}$$

Espacio propio asociado a

$\lambda = 1 - 2i$ es

$$\left\{ \left(z, \frac{1-i}{2} z \right) : z \in \mathbb{C} \right\} = \left\langle \overbrace{(2, 1-i)}^{v_\lambda} \right\rangle$$

Valores propios de A:

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & -4 \\ 2 & -1-\lambda \end{vmatrix} = (3-\lambda)(-1-\lambda) + 8$$

$$= -3 - 3\lambda + \lambda + \lambda^2 + 8 = \lambda^2 - 2\lambda + 5$$

Vectores propios:

$$A - (1+2i)I = \begin{array}{c} 1) \\ 2) \end{array} \begin{pmatrix} 2-2i & -4 \\ 2 & -2-2i \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$(1-i) \times 2) = 1) \qquad (2-2i)z - 4w = 0$

$$w = \frac{2(1-i)}{4} z = \frac{1-i}{2} z$$

Verificamos:

$$A v_\lambda = \begin{pmatrix} 3 & -4 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ 1-i \end{pmatrix} = \begin{pmatrix} 6-4(1-i) \\ 4-(1-i) \end{pmatrix} = \begin{pmatrix} 2+4i \\ 3+i \end{pmatrix}$$

$$\lambda v_\lambda = (1+2i) \boxed{\begin{pmatrix} 2 \\ 1-i \end{pmatrix}} = \begin{pmatrix} 2+4i \\ 1-i+2i(1-i) \end{pmatrix} = \begin{pmatrix} 2+4i \\ 3+i \end{pmatrix}$$

$$A v_\lambda = \lambda v_\lambda$$

$$v_{\bar\lambda} = \overline{v_\lambda} = \overline{\begin{pmatrix} 2 \\ 1-i \end{pmatrix}} = \boxed{\begin{pmatrix} 2 \\ 1+i \end{pmatrix}}$$

En $\mathbb{C}^2$: $\left\{ \begin{pmatrix} 2 \\ 1-i \end{pmatrix}, \begin{pmatrix} 2 \\ 1+i \end{pmatrix} \right\}$ base vect. p. de A

$$A \overline{v_\lambda} = \begin{pmatrix} 3 & -4 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ 1+i \end{pmatrix} = \begin{pmatrix} 6-4(1+i) \\ 4-(1+i) \end{pmatrix} = \begin{pmatrix} 2-4i \\ 3-i \end{pmatrix}$$

$$\bar\lambda\, \overline{v_\lambda} = (1-2i) \begin{pmatrix} 2 \\ 1+i \end{pmatrix} = \begin{pmatrix} 2-4i \\ 1+i-2i(1+i) \end{pmatrix} = \begin{pmatrix} 2-4i \\ 3-i \end{pmatrix}''$$

Tomamos parte real e imaginaria de v_λ: $v_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $v_2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

$$\boxed{P = \begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix}} \qquad P^{-1} = -\frac{1}{2} \begin{pmatrix} -1 & 0 \\ -1 & 2 \end{pmatrix}$$

$$Y = P^{-1} X$$

$$\boxed{X = PY}$$

$$P^{-1} A P = -\frac{1}{2} \begin{pmatrix} -1 & 0 \\ -1 & 2 \end{pmatrix} \underbrace{\begin{pmatrix} 3 & -4 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix}}$$

$\lambda = 1 + 2i$

$$\begin{pmatrix} \mathrm{Re}\lambda & \mathrm{Im}\lambda \\ -\mathrm{Im}\lambda & \mathrm{Re}\lambda \end{pmatrix} = D$$

$$\underbrace{-\frac{1}{2} \begin{pmatrix} -1 & 0 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 3 & 1 \end{pmatrix}}$$

$$-\frac{1}{2} \begin{pmatrix} -2 & -4 \\ 4 & -2 \end{pmatrix} \Rightarrow \boxed{P^{-1} A P = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}}$$

es de la forma $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$

Haciendo el cambio de base

$\dot{X} = AX$ es equivalente a $\dot{Y} = DY$ con $D = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$

cuyas soluciones son $Y = \begin{pmatrix} r_0 e^t \cos(\theta_0 - 2t) \\ r_0 e^t \sin(\theta_0 - 2t) \end{pmatrix}$ r_0, θ_0 c.i.

en polares

en Y.

$$X = PY = \begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} r_0 e^t \cos(\theta_0 - 2t) \\ r_0 e^t \sin(\theta_0 - 2t) \end{pmatrix} = \begin{pmatrix} 2 r_0 e^t \cos(\theta_0 - 2t) \\ r_0 e^t \left(\cos(\theta_0 - 2t) - \operatorname{sen}(\theta_0 - 2t) \right) \end{pmatrix}$$

$$X(0) = \begin{pmatrix} 2 r_0 \cos\theta_0 \\ r_0 (\cos\theta_0 - \operatorname{sen}\theta_0) \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

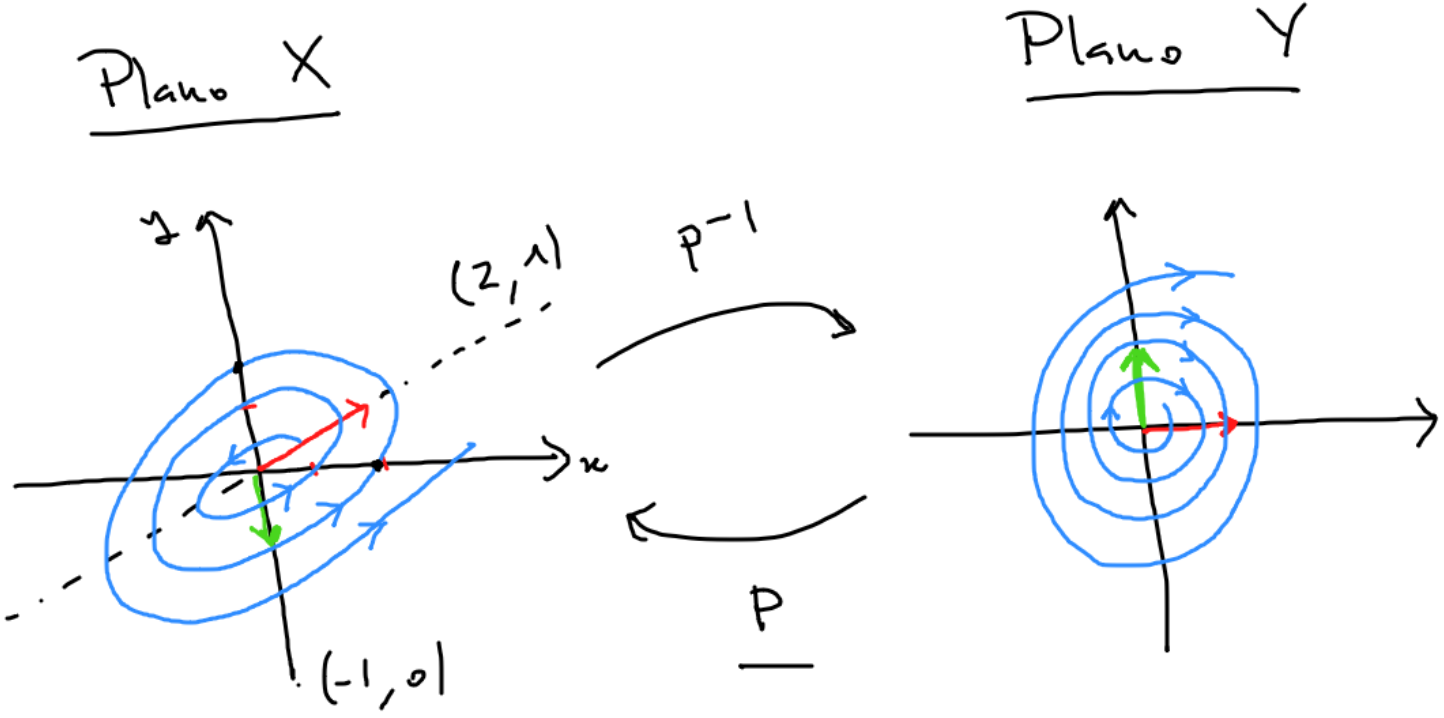
$\frac{1}{2}(1 - \tan\theta_0) = \frac{y_0}{x_0}$

$\theta_0 = \operatorname{atan}\left(1 - \frac{2y_0}{x_0}\right)$

Diagrama de fase:

$$A = \begin{pmatrix} 3 & -4 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 3x - 4y \\ 2x - y \end{pmatrix}$$

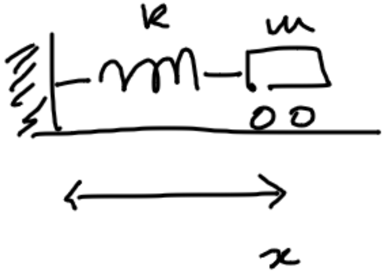


$$P = \begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix}$$

cambia orientación $\det P = -2 < 0$

$$\|Y\| = r_0 e^{t}$$

Resorte :



$$m\ddot{x} = -kx$$

$$m\ddot{x} + kx = 0$$

$$x(0) = x_0$$

$$\dot{x}(0) = v_0$$

$$\omega_0^2 = k/m$$

$$\ddot{x} + \omega_0^2 x = 0$$

$$X = \mathcal{L}(x)$$

$$x(t) = e^{rt}$$

$$\dot{x}(t) = re^{rt}$$

$$\ddot{x}(t) = r^2 e^{rt}$$

$$r^2 \cancel{e^{rt}} + \omega_0^2 \cancel{e^{rt}} = 0 \qquad \boxed{r^2 + \omega_0^2 = 0}$$

$$\boxed{r = \pm i\omega_0}$$

vino del cielo

$$x(t) = A\cos(\omega_0 t) + B\,\mathrm{sen}(\omega_0 t)$$

$$\mathcal{L}(\dot{x}) = s\,\mathcal{L}(x) - x_0$$

$$\mathcal{L}(\ddot{x}) = s\,\mathcal{L}(\dot{x}) - v_0$$

$$= s(sX - x_0) - v_0$$

$$= s^2 X - s x_0 - v_0$$

$$\mathcal{L}(\ddot{x}) + \omega_0^2 \underbrace{\mathcal{L}(x)}_{X} = \mathcal{L}(0) = 0$$

$$s^2 X - s x_0 - v_0 + \omega_0^2 X = 0$$

$$\left(s^2 + \omega_0^2\right) X = s x_0 + v_0$$

$$X = \frac{s x_0 + v_0}{s^2 + \omega_0^2}$$

$$x = \mathcal{L}^{-1}(X) = \mathcal{L}^{-1}\left(\frac{s x_0 + v_0}{s^2 + \omega_0^2}\right) = x_0 \underbrace{\mathcal{L}^{-1}\left(\frac{s}{s^2+\omega_0^2}\right)}_{\cos(\omega_0 t)} + \frac{v_0}{\omega_0} \underbrace{\mathcal{L}^{-1}\left(\frac{\omega_0}{s^2+\omega_0^2}\right)}_{\text{sen}(\omega_0 t)}$$

$$\boxed{x = x_0 \cos(\omega_0 t) + \frac{v_0}{\omega_0} \text{sen}(\omega_0 t)}$$

Reducir el orden: $\ddot{x} + w_0^2 x = 0$ en $\mathbb{R}$

$\boxed{y = \dot{x}}$, $\dot{y} = \ddot{x}$ $\rightarrow$ $\dot{y} + w_0^2 x = 0$

$\boxed{\dot{y} = -w_0^2 x}$

en $\mathbb{R}^2$

$$\begin{cases} \dot{x} = y \\ \dot{y} = -w_0^2 x \end{cases}$$

$$X = \begin{pmatrix} x \\ y \end{pmatrix} \qquad \dot{X} = \overbrace{\begin{pmatrix} 0 & 1 \\ -w_0^2 & 0 \end{pmatrix}}^{A} \begin{pmatrix} x \\ y \end{pmatrix} = AX$$

$$\dot{X} = AX$$

$$A = \begin{pmatrix} 0 & 1 \\ -w_0^2 & 0 \end{pmatrix}$$

$$\det(A - \lambda I) = \left| \begin{pmatrix} -\lambda & 1 \\ -w_0^2 & -\lambda \end{pmatrix} \right| = \lambda^2 + w_0^2$$

$$p(\lambda) = \lambda^2 + w_0^2$$

$$\lambda = \pm w_0 i$$

$$\boxed{\begin{array}{l} \lambda = w_0 i \\ \bar{\lambda} = -w_0 i \end{array}}$$

$$A - w_0 i I = \begin{pmatrix} -w_0 i & 1 \\ -w_0^2 & -w_0 i \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} -w_0 i z + w \\ -w_0^2 z - w_0 i w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-w_0 i z + w = 0 \qquad w = (w_0 i) z$$

Espacio propio: $\{ (z, (w_0 i) z) : z \in \mathbb{C} \}$

$$= \langle \underbrace{(1, w_0 i)}_{v_\lambda} \rangle$$

$$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$v_2 = \begin{pmatrix} 0 \\ w_0 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & 0 \\ 0 & w_0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1/w_0 \end{pmatrix} \overbrace{\begin{pmatrix} 0 & 1 \\ -w_0^2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & w_0 \end{pmatrix}}^{\begin{pmatrix} 0 & w_0 \\ -w_0^2 & 0 \end{pmatrix}} \quad \boxed{\begin{pmatrix} 0 & w_0 \\ -w_0 & 0 \end{pmatrix}}$$

$$\dot{Y} = \begin{pmatrix} 0 & w_0 \\ -w_0 & 0 \end{pmatrix} Y$$ y las soluciones:

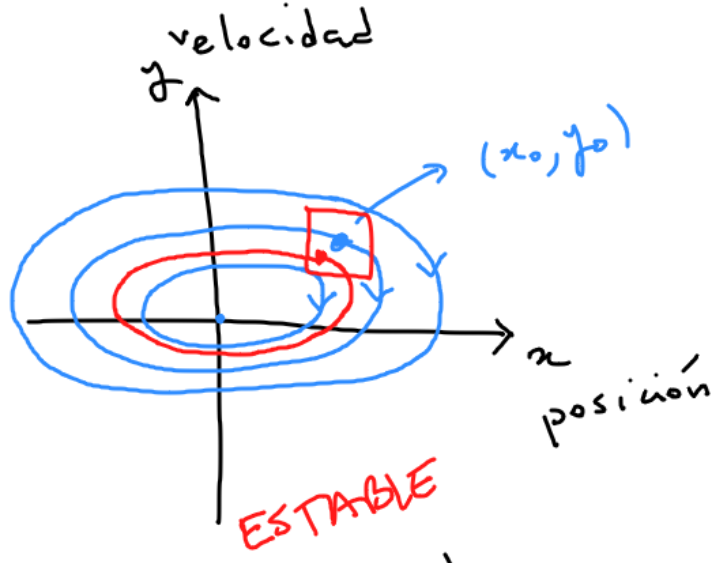
$$Y = \begin{pmatrix} r_0 \cos(\theta_0 - w_0 t) \\ r_0 \operatorname{sen}(\theta_0 - w_0 t) \end{pmatrix}$$

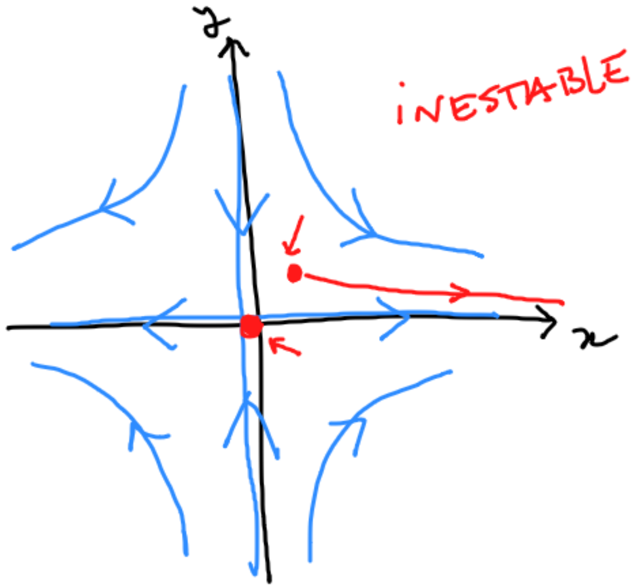
$$X = PY = \begin{pmatrix} 1 & 0 \\ 0 & w_0 \end{pmatrix} \begin{pmatrix} r_0 \cos(\theta_0 - w_0 t) \\ r_0 \operatorname{sen}(\theta_0 - w_0 t) \end{pmatrix}$$

$$= \begin{pmatrix} r_0 \cos(\theta_0 - w_0 t) \\ w_0 r_0 \operatorname{sen}(\theta_0 - w_0 t) \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\boxed{x(t) = r_0 \cos(\theta_0 - w_0 t)}$$

Estabilidad de soluciones para sistemas lineales (Cap. 3)





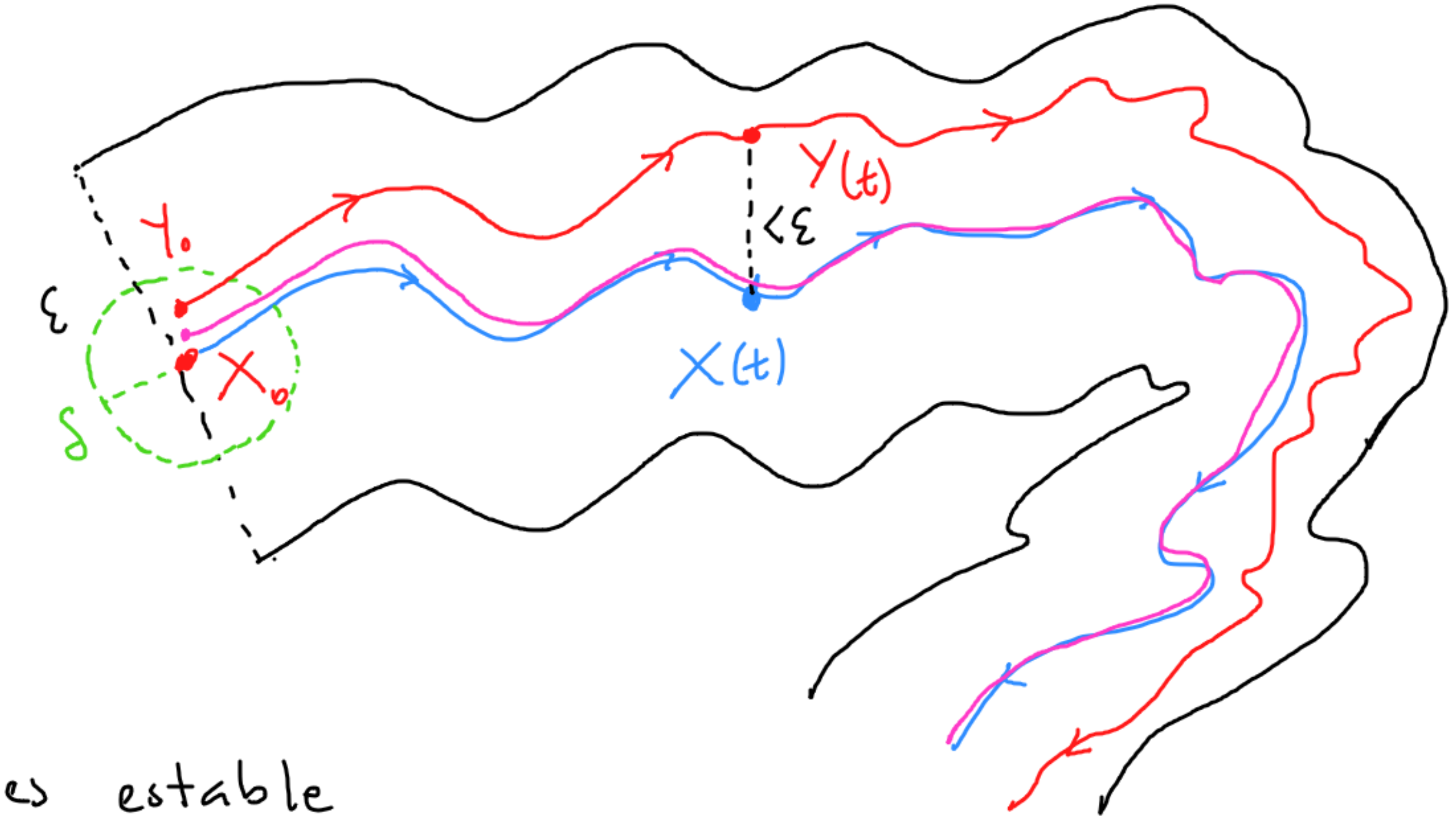
(Estable a futuro)

DEF: Consideremos X solución de (E) $\begin{cases} \dot{X} = AX \\ X(0) = X_0 \end{cases}$. Decimos que X es estable si: $\forall \varepsilon > 0, \exists \delta > 0$ t.q. si Y es otra solución $\dot{Y} = AY$ con $Y(0) = Y_0$ y $\| Y_0 - X_0 \| < \delta \Rightarrow \| Y(t) - X(t) \| < \varepsilon \quad \forall t \geq 0$.

Elijo un $\varepsilon > 0$

$(\forall \varepsilon > 0)$

Puedo encontrar δ?



<u>DEF</u>: Si $X(t)$ no es estable decimos que es <u>inestable</u>.

DEF: Decimos que $X(t)$ es asintóticamente estable si es estable y además si existe $\delta > 0$ t.q. $\forall$ sol. $Y(t)$ con $Y(0) = Y_0$, $\|Y_0 - X_0\| < \delta$ se cumple

$$\lim_{t \to +\infty} \|Y(t) - X(t)\| = 0$$

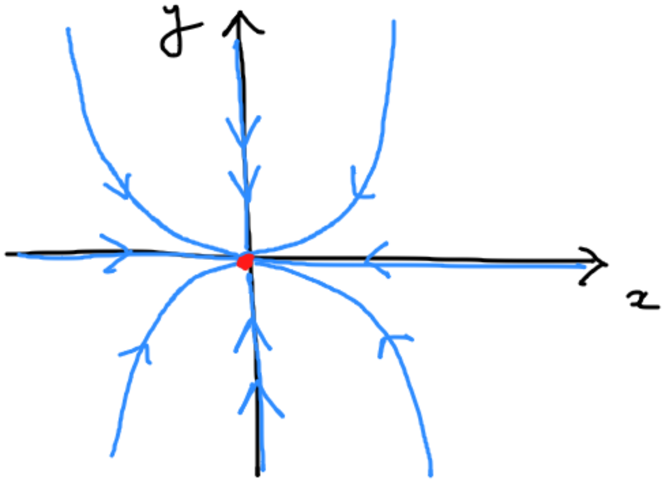
$$\begin{tikzcd}
 & & \text{Asintóticamente estable} \\
 & \text{Estable} \arrow[ur] \arrow[dr] & \\
X(t) \arrow[ur] \arrow[dr] & & \text{No asintóticamente estable} \\
 & \text{Inestable} &
\end{tikzcd}$$

Ejemplos:

1) $\dot{X} = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} X$ $\qquad X(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \ \forall t$

Es estable? Si

Asint. estable? Si

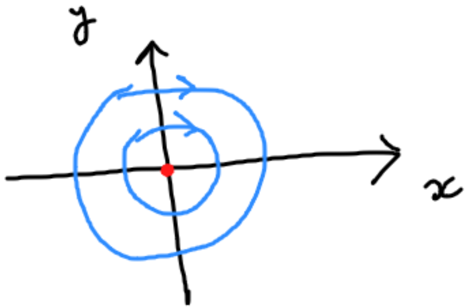


2) $\dot{X} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} X$

$X(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Estable? Si

Asint. estable? No



Ejercicio: si la solución $X(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\forall t$ es estable

$\Rightarrow$ todas las soluciones son estables.

$\boxed{\dot{X} = AX}$

Ejemplo: $A = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ $\quad \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x_0 e^{\alpha t} \\ y_0 e^{\beta t} \end{pmatrix} = \begin{pmatrix} e^{\alpha t} & 0 \\ 0 & e^{\beta t} \end{pmatrix} \overbrace{\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}}^{X_0}$

solución = matriz(t) × cond. inicial

$X(t)$ con $\overbrace{X(0)}^{X_0} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$

$Y(t)$ con $\underbrace{Y(0)}_{Y_0} = \begin{pmatrix} x_0' \\ y_0' \end{pmatrix}$

$$\|Y(t) - X(t)\| = \left\| \begin{pmatrix} x_0' e^{\alpha t} \\ y_0' e^{\beta t} \end{pmatrix} - \begin{pmatrix} x_0 e^{\alpha t} \\ y_0 e^{\beta t} \end{pmatrix} \right\|$$

$$= \left\| \begin{pmatrix} (x_0' - x_0) e^{\alpha t} \\ (y_0' - y_0) e^{\beta t} \end{pmatrix} \right\|$$

$$= \left\| \begin{pmatrix} e^{\alpha t} & 0 \\ 0 & e^{\beta t} \end{pmatrix} \underbrace{(Y_0 - X_0)} \right\|$$

$$\left\| \begin{pmatrix} e^{\alpha t} & 0 \\ 0 & e^{\beta t} \end{pmatrix} z \right\| < \varepsilon$$

si $\|z\| < \delta$.

Pasa si $\alpha \leq 0$ y $\beta \leq 0$.

$$\left\| \begin{pmatrix} e^{\alpha t} & 0 \\ 0 & e^{\beta t} \end{pmatrix} z \right\| \begin{cases} \text{acotado?} \\ \xrightarrow[t \to +\infty]{} 0 \text{ ?} \\ \xrightarrow[t \to +\infty]{} +\infty \text{ ?} \end{cases}$$

$X(t) = e^{At} X_0 \begin{pmatrix} x_0' - x_0 \\ y_0' - y_0 \end{pmatrix}$

$A = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$

$e^{At} = \begin{pmatrix} e^{\alpha t} & 0 \\ 0 & e^{\beta t} \end{pmatrix}$

"Comportamiento asintótico de $X_0 = 0$ ($X(t) = 0$ es estable:)

$$\left\| \begin{pmatrix} e^{\alpha t} & 0 \\ 0 & e^{\beta t} \end{pmatrix} Y_0 \right\| < \varepsilon \quad \text{si} \quad \|\overset{z}{\widetilde{Y_0}}\| < \delta$$

$$\left\| \begin{pmatrix} e^{\alpha t} & 0 \\ 0 & e^{\beta t} \end{pmatrix} (Y_0 - X_0) \right\| < \varepsilon \quad \text{si} \quad \|\underbrace{Y_0 - X_0}_{z}\| < \delta$$