Ecuaciones Lineales Autónomas en el plano

$$\dot{X} = AX$$

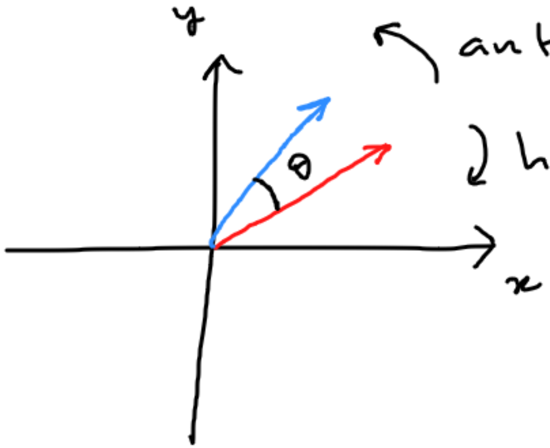
$$X = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \, , \quad A \in M_2(\mathbb{R})$$

$A \to$

- diagonalizable : $P^{-1}AP = \begin{pmatrix} \alpha & \\ & \beta \end{pmatrix}$ [α y β pueden ser iguales]
- Jordan : $P^{-1}AP = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$
- Rotohomotecia : $P^{-1}AP = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$

HOY

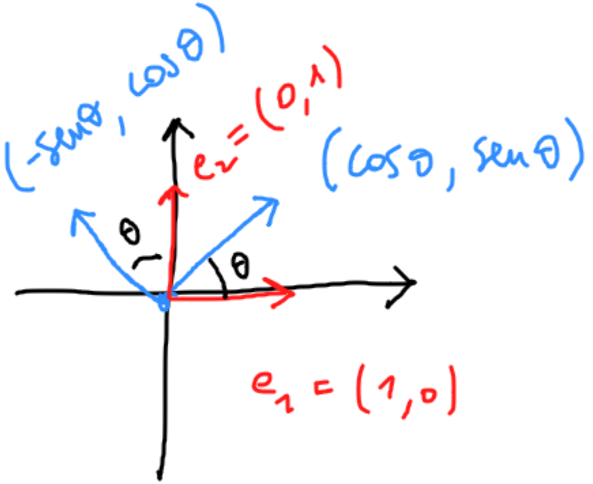
Recordatorio: matriz de rotación en el plano



↶ antihorario $\theta > 0$

↷ horario $\theta < 0$

$$\overbrace{a^2+b^2=1}^{\text{Esto se cumple solo para rotaciones}}$$



$$\text{Rotación} = \begin{pmatrix} \cos\theta & -\operatorname{sen}\theta \\ \operatorname{sen}\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

$$a = \cos\theta \qquad b = -\operatorname{sen}\theta$$

$A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ Valores propios:

$(b \neq 0)$

$$\det(A - \lambda I) = \begin{vmatrix} a-\lambda & b \\ -b & a-\lambda \end{vmatrix}$$

$$= (a-\lambda)^2 + b^2 \neq 0$$

para $\lambda \in \mathbb{R}$

Si $\lambda \in \mathbb{C}$:

$$(a-\lambda)^2 = -b^2$$

$$a - \lambda = \pm\sqrt{-b^2} = \pm i b \rightsquigarrow \boxed{\lambda = a \pm ib}$$

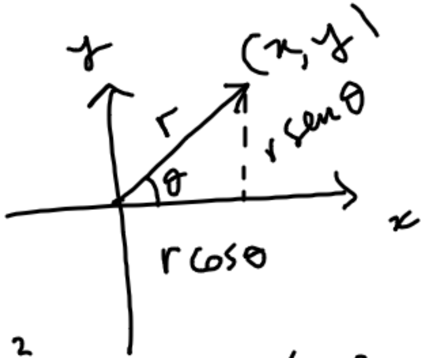
$$\lambda = a + ib, \quad \bar{\lambda} = a - ib$$

rotar un ángulo θ.

En el caso de la rotación: $a - ib = \cos\theta + i\,\mathrm{sen}\,\theta = e^{i\theta}$

Caso particular $\dot{X} = AX$ con $A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$

$$(E) \begin{cases} \dot{x} = ax + by \\ \dot{y} = -bx + ay \end{cases}$$



Coordenadas polares:

$$x = r\cos\theta$$
$$y = r\,\mathrm{sen}\,\theta$$

$$r^2 = x^2 + y^2 \xrightarrow[\text{derivo}]{} \cancel{2} r\dot{r} = \cancel{2} x\dot{x} + \cancel{2} y\dot{y}$$

$$r\dot{r} = x\dot{x} + y\dot{y}$$

$$\Big\Downarrow (E)$$

$$(E_r) \quad \boxed{\dot{r} = ar} \Longleftarrow \cancel{r}\dot{r} = x(ax+by) + y(-bx+ay)$$
$$= ax^2 + \cancel{bxy} - \cancel{bxy} + ay^2$$
$$= a(x^2+y^2) = ar^{\cancel{2}}$$

$$x = r\cos\theta \xrightarrow{\text{derivo}} \dot{x} = \dot{r}\cos\theta - r\,\text{sen}\,\theta\,\dot{\theta} \xrightarrow{(E_r)} \dot{x} = ar\cos\theta - r\,\text{sen}\,\theta\,\dot{\theta}$$

$(E_r)\ \dot{r} = ar$

$$\Big\Downarrow (E)$$

$$ax + by = a\underbrace{r\cos\theta}_{x} - \underbrace{r\,\text{sen}\,\theta}_{y}\,\dot{\theta}$$

$$\cancel{ax} + by = \cancel{ax} - y\dot{\theta}$$

$$\boxed{\dot{\theta} = -b}\ (E_\theta)$$

En resumen:

$(E)\ \begin{cases} \dot{x} = ax + by \\ \dot{y} = -bx + ay \end{cases}$ en polares es $\begin{cases} (E_r)\ \dot{r} = ar \\ (E_\theta)\ \dot{\theta} = -b \end{cases}$

La solución con condición inicial (r_0, θ_0) es

$$r(t) = r_0 e^{at} \quad \theta(t) = \theta_0 - bt$$

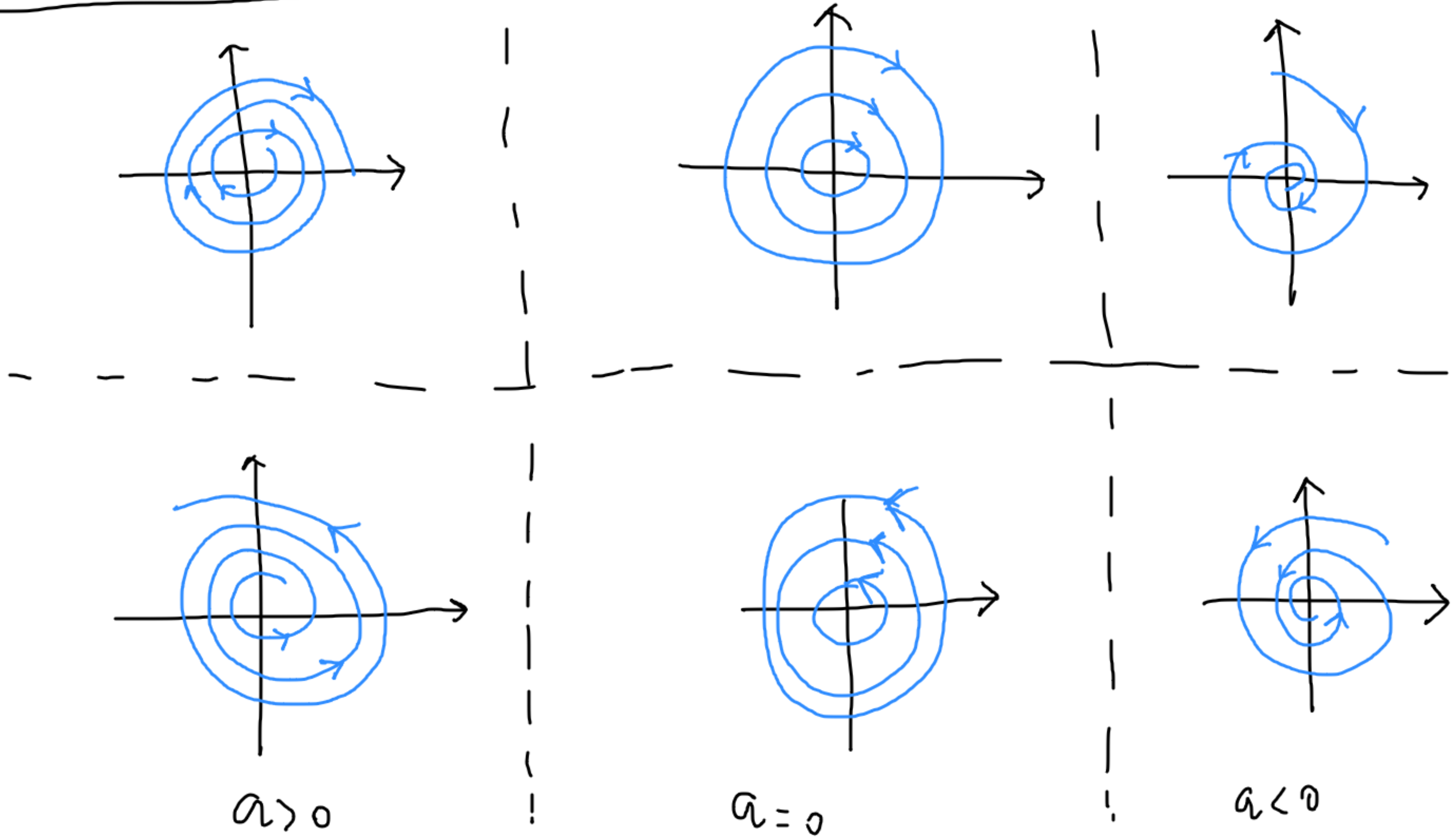
$$\boxed{\begin{aligned} x(t) &= r_0 e^{at}\cos(\theta_0 - bt) \\ y(t) &= r_0 e^{at}\,\text{sen}(\theta_0 - bt) \end{aligned}}$$

Diagrama de fase

$x(t) = r_0 e^{at} \cos(\theta_0 - bt)$ $\quad$ $y(t) = r_0 e^{at}\, \text{sen}(\theta_0 - bt)$

$b > 0$

$b < 0$



Cómo llevar una matriz con val.p. complejos a $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$

<u>Notación</u>: • $\mathbb{R}^2$ espacio vectorial ($\mathbb{R}$)

$$v \xrightarrow{\times\lambda} \lambda v \qquad \text{con } \lambda \in \mathbb{R}$$

• $\mathbb{C}^2 = \{(z,w) : z, w \in \mathbb{C}\}$ espacio vectorial sobre $\mathbb{C}$

$$v \xrightarrow{\times(a+ib)} (a+ib)v$$

$A \in M_2(\mathbb{R})$ la puedo pensar como $A \in M_2(\mathbb{C})$

porque $\mathbb{R} \subset \mathbb{C}$.

Supongamos que A tiene val. p. complejos $\lambda = a - ib$

$\overline{\lambda} = a + ib$

Calculamos los vect. p. complejos: $\underline{v_\lambda}\,,\ \underline{v_{\overline{\lambda}}} \in \mathbb{C}^2$

<u>Observación:</u> $A \in M_n(\mathbb{R}) \Rightarrow p(\lambda) = \det(A - \lambda I)$

$= a_0 + a_1\lambda + \cdots + a_n\lambda^n$

Es decir $p(\lambda)$ tiene coef. a_i reales con los $a_i \in \mathbb{R}$

$$\overline{a_0 + a_1\lambda + \cdots + a_n\lambda^n} = \overline{p(\lambda)} = \overline{0} = 0$$

$$\| $$

Los val. p. son las raíces de $p(\lambda) = 0$

Si λ es una raiz $p(\lambda) = 0 \Rightarrow p(\overline{\lambda}) = a_0 + a_1\overline{\lambda} + \cdots + a_n(\overline{\lambda})^n$

Si $p(\lambda) = 0 \Rightarrow p(\overline{\lambda}) = 0$

Vect. p. son $v_\lambda = (z, w) \in \mathbb{C}^2$ $\quad z, w, t, s \in \mathbb{C}$

$v_{\bar\lambda} = (t, s) \in \mathbb{C}^2$

$\overline{v_\lambda} = (\bar z, \bar w)$

$$A\overline{v_\lambda} = \overline{Av_\lambda} = \overline{\lambda v_\lambda} = \bar\lambda\, \overline{v_\lambda} \Rightarrow \boxed{\overline{v_\lambda} = v_{\bar\lambda}}$$

A tiene coef. reales

$\Rightarrow t = \bar z$ y $s = \bar w$

En resumen: Si (z, w) es el λ-vector propio de A

$\Rightarrow (\bar z, \bar w)$ es el $\bar\lambda$-vector propio de A

Tenemos que $\mathcal{C} = \{ (z, w), (\bar{z}, \bar{w}) \}$ es una base de vect. p de $\mathbb{C}^2$

La matriz A en la base $\mathcal{C}$ es $\begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix}$

$z = a + ib \qquad \bar{z} = a - ib$

$w = c + id \qquad \bar{w} = c - id$

combinación lineal complejos

$v_1 = \begin{pmatrix} a \\ c \end{pmatrix} \qquad v_\lambda = \begin{pmatrix} z \\ w \end{pmatrix} = v_1 + i v_2$

$v_2 = \begin{pmatrix} b \\ d \end{pmatrix} \qquad v_{\bar{\lambda}} = \begin{pmatrix} \bar{z} \\ \bar{w} \end{pmatrix} = v_1 - i v_2$

$\langle \{ v_1, v_2 \} \rangle \supseteq \langle \{ v_\lambda, v_{\bar{\lambda}} \} \rangle = \mathbb{C}^2 \Rightarrow \{ v_1, v_2 \}$ es una base de $\mathbb{C}^2$

Si $\{v_1, v_2\}$ es una base (sobre $\mathbb{C}$) $\Rightarrow$ es l.i. sobre $\mathbb{C}$.

$\Rightarrow$ es l.i. sobre $\mathbb{R}$

$\Rightarrow$ $\{v_1, v_2\}$ es una base de $\mathbb{R}^2$.

$v_1 = \begin{pmatrix} a \\ c \end{pmatrix}$ $v_2 = \begin{pmatrix} b \\ d \end{pmatrix}$ vectores reales.

$\mathcal{B} = \{v_1, v_2\}$

¿Cómo queda la matriz A en la base $\mathcal{B}$?

$v_\lambda = v_1 + i v_2$ $\qquad$ $v_1 = \frac{1}{2} v_\lambda + \frac{1}{2} v_{\bar{\lambda}}$ $\qquad$ $A v_1 = \frac{1}{2} A v_\lambda + \frac{1}{2} A v_{\bar{\lambda}}$

$v_{\bar{\lambda}} = v_1 - i v_2$ $\qquad$ $v_2 = \frac{1}{2i} v_\lambda - \frac{1}{2i} v_{\bar{\lambda}}$ $\qquad$ $= \frac{1}{2} \lambda v_\lambda + \frac{1}{2} \bar{\lambda} v_{\bar{\lambda}}$

$$A v_1 = \frac{1}{2} \lambda v_\lambda + \frac{1}{2} \bar{\lambda} v_{\bar{\lambda}} = \frac{1}{2} \lambda (v_1 + i v_2) + \frac{1}{2} \bar{\lambda} (v_1 - i v_2)$$

$$v_\lambda = v_1 + i v_2$$

$$v_{\bar{\lambda}} = v_1 - i v_2$$

$$= \left(\frac{1}{2} \lambda + \frac{1}{2} \bar{\lambda} \right) v_1 + \left(\frac{i\lambda}{2} - \frac{i\bar{\lambda}}{2} \right) v_2$$

$$= \alpha v_1 - \beta v_2$$

$$\lambda = \alpha + i\beta$$

$$\bar{\lambda} = \alpha - i\beta$$

$$A v_2 = \frac{1}{2i} A v_\lambda - \frac{1}{2i} A v_{\bar{\lambda}} = \frac{1}{2i} \lambda v_\lambda - \frac{1}{2i} \bar{\lambda} v_{\bar{\lambda}} = \frac{1}{2i} \lambda (v_1 + i v_2) - \frac{1}{2i} \bar{\lambda} (v_1 - i v_2)$$

$$= \underbrace{\left(\frac{\lambda}{2i} - \frac{\bar{\lambda}}{2i} \right)}_{\beta} v_1 + \underbrace{\left(\frac{\lambda}{2} + \frac{\bar{\lambda}}{2} \right)}_{\alpha} v_2 = \beta v_1 + \alpha v_2$$

Resumen total: $Av_1 = \alpha v_1 - \beta v_2$ "A" $= \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$

$Av_2 = \beta v_1 + \alpha v_2$ en $\mathcal{B}$

Partimos de A con val. p. complejos y llegamos a la forma $\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} = P^{-1}AP$

Ejercicio 3.b.i

$$A = \begin{pmatrix} -7 & 10 \\ -5 & 9 \end{pmatrix}$$

Val. prop.:

$$\begin{vmatrix} -7-\lambda & 10 \\ -5 & 9-\lambda \end{vmatrix} = -(7+\lambda)(9-\lambda) + 50$$

$$= \lambda^2 - 2\lambda - 13 = 0$$

$$\lambda = \frac{\cancel{2} \pm \sqrt{\cancel{4} + \cancel{4}\cdot 13}}{\cancel{2}} = 1 \pm \sqrt{+14}$$

? ? Mal el ejemplo

Ejemplo: $A = \begin{pmatrix} 3 & -4 \\ 2 & -1 \end{pmatrix}$

Val. prop.: $\begin{vmatrix} 3-\lambda & -4 \\ 2 & -1-\lambda \end{vmatrix} = (3-\lambda)(-1-\lambda)+8$

$= -3-3\lambda+\lambda+\lambda^2+8$

$= \lambda^2 - 2\lambda + 5$

$$\lambda = \frac{\cancel{2} \pm \sqrt{\cancel{4}-\cancel{4}\cdot 5}}{\cancel{2}} = 1 \pm 2i$$

$$\boxed{\begin{aligned} \lambda &= 1+2i \\ \bar{\lambda} &= 1-2i \end{aligned}}$$

$P = ?$

$$A-(1+2i)I = \begin{pmatrix} 3-(1+2i) & -4 \\ 2 & -1-(1+2i) \end{pmatrix}$$

$$= \begin{pmatrix} 2(1-i) & -4 \\ 2 & -2(1+i) \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\boxed{v_\lambda = (2, 1-i)}$

$v_1 = (2, 1)$

$v_2 = (0, -1)$

$(1+i)(1-i) = 2$

$2(1-i)z - 4w = 0$

$$\boxed{w = \frac{1-i}{2} z}$$

$P = \begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix}$

$P^{-1}AP$ "

$\left\{ \left(z, \frac{1-i}{2} z\right) : z \in \mathbb{C} \right\}$

$\xrightarrow{z=2}$ $= \langle (2, 1-i) \rangle$

$\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$