Ecuaciones autónomas lineales en el plano $\mathbb{R}^2$

$\dot{X} = AX$ $\quad X : \mathbb{R} \longrightarrow \mathbb{R}^2$ (curva en el plano)

$A \in M_2(\mathbb{R})$

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\quad X = \begin{pmatrix} x \\ y \end{pmatrix}$ $\quad \begin{cases} \dot{x} = ax + by \\ \dot{y} = cx + dy \end{cases}$

Si A →

- Diagonalizable — 2 vectores propios l.i
- Jordan — 1 vector propio
- Roto-homotecia — 0 vector propio

Cambio de base

$Y = P^{-1}X$

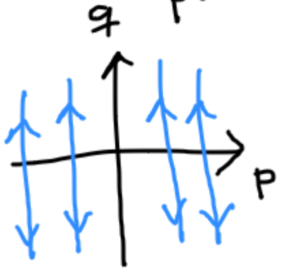
$\dot{Y} = \underbrace{(P^{-1}AP)}_{\text{mas sencilla que } A} Y$

Diagonalizable: $P^{-1}AP = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \rightsquigarrow$ Según los signos de α y β tenemos distintos diagramas de fase.

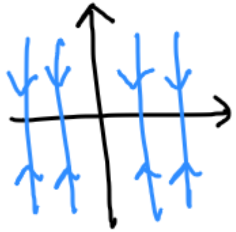
$Y = \begin{pmatrix} p \\ q \end{pmatrix}$

Si α ó β es cero:

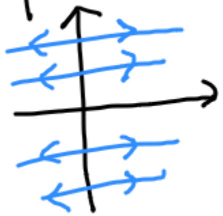
$\alpha = 0$, $\beta > 0$



$\alpha = 0$, $\beta < 0$



$\alpha > 0$, $\beta = 0$

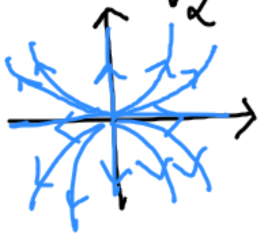


$\alpha < 0$, $\beta = 0$

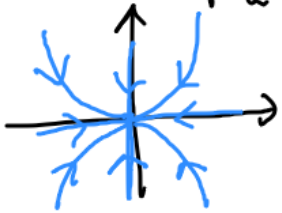
Si $\alpha \neq 0$ y $\beta \neq 0$:

mismo signo:

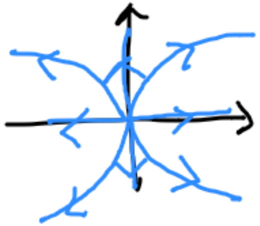
$\beta/\alpha > 1, +$



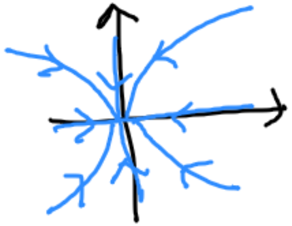
$\beta/\alpha > 1, -$



$\beta/\alpha < 1, +$

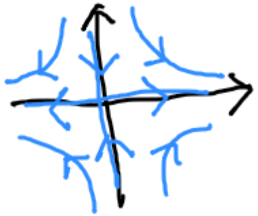


$\beta/\alpha < 1, -$

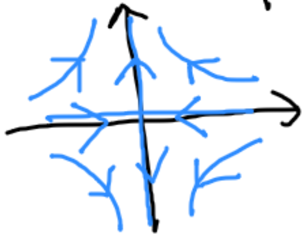


$\neq$ signo:

$\alpha > 0$ $\beta < 0$



$\alpha < 0$ $\beta > 0$



Matriz de Jordan: $\quad P^{-1}AP \to \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \leftarrow$

$\qquad\qquad\qquad\qquad\to \begin{pmatrix} \lambda & 0 \\ 1 & \lambda \end{pmatrix}$

$$P^{-1}AP = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

$$\begin{pmatrix} \dot{p} \\ \dot{q} \end{pmatrix} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} \lambda p + q \\ \lambda q \end{pmatrix} \qquad \begin{cases} \dot{p} = \lambda p + q \\ \dot{q} = \lambda q \end{cases}$$

$$q = q_0 e^{\lambda t}$$

$$\dot{p} = \lambda p + q_0 e^{\lambda t} \qquad \dot{p} - \lambda p = q_0 e^{\lambda t} \qquad \mu = e^{-\lambda t}$$

$$\dot{\mu} = -\lambda \mu$$

Factor integrante: $\quad \underbrace{\mu \dot{p} - \lambda \mu p}_{(\mu p)^{\cdot} = \dot{\mu} p + \cancel{\mu \dot{p}}} = q_0 e^{\lambda t} \mu \qquad \dot{\mu} \cancel{p} = -\lambda \mu \cancel{p}$

$$\mu = e^{-\lambda t}$$

$$(\mu p)^{\cdot} = q_0 e^{\lambda t} \mu = q_0 \cancel{e^{\lambda t}} \cancel{e^{-\lambda t}} = q_0$$

$$\int (\mu p)^{\cdot} dt = q_0 t + C$$

$$\boxed{\begin{cases} p = e^{\lambda t}(q_0 t + p_0) \\ q = q_0 e^{\lambda t} \end{cases}}$$

$$\mu p = q_0 t + C$$

$$p = \frac{1}{\mu}(q_0 t + C) = e^{\lambda t}(q_0 t + C)$$

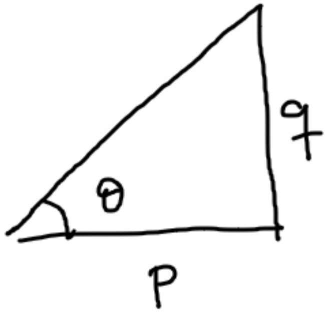
$$p_0 = p(0) = C \Rightarrow p = e^{\lambda t}(q_0 t + p_0)$$

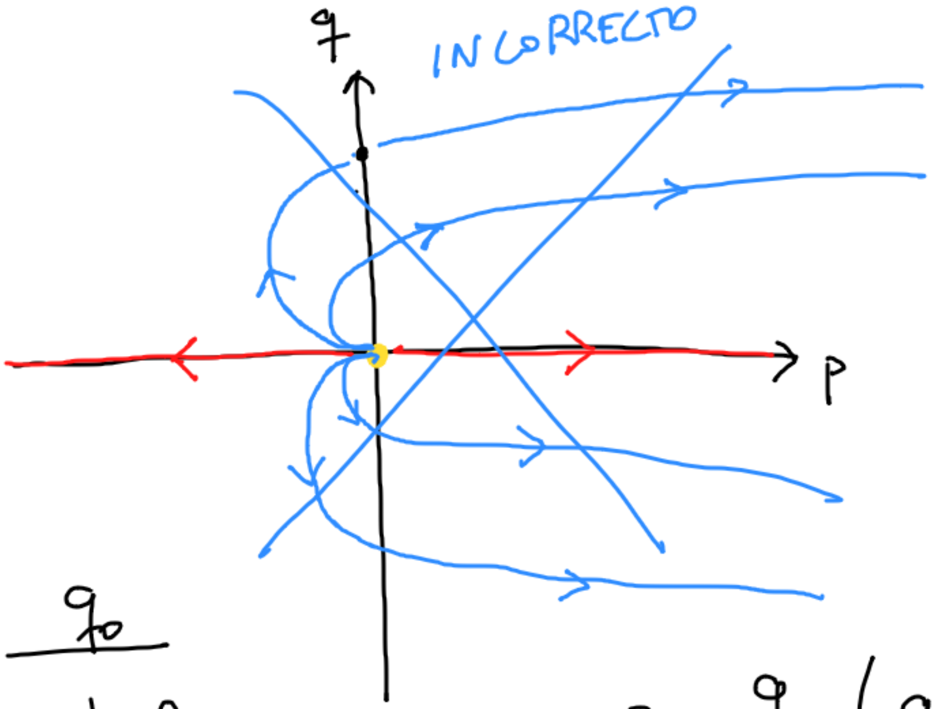
Solución:

$$\begin{pmatrix} \dot{p} \\ \dot{q} \end{pmatrix} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} \qquad \begin{cases} p = e^{\lambda t}(q_0 t + p_0) \\ q = e^{\lambda t} q_0 \end{cases}$$

Diagrama de fase:

$\underline{\underline{\lambda > 0}}$





$$\frac{q}{q_0} = e^{\lambda t}$$

$$t = \frac{1}{\lambda} \ln \frac{q}{q_0}$$

$$\tan\theta = \frac{q}{p} = \frac{\cancel{e^{\lambda t}} q_0}{\cancel{e^{\lambda t}}(q_0 t + p_0)} = \frac{q_0}{q_0 t + p_0}$$

$$p = \frac{q}{q_0}\left(\frac{q_0}{\lambda} \ln \frac{q}{q_0} + p_0\right)$$

$\theta \longrightarrow 0 \quad t \rightarrow \pm\infty \qquad p \rightarrow 0,\ q \rightarrow 0 \quad t \rightarrow -\infty$

$$p = e^{\lambda t}(q_0 t + p_0)$$

$$q = e^{\lambda t} q_0$$

$q_0 < 0$

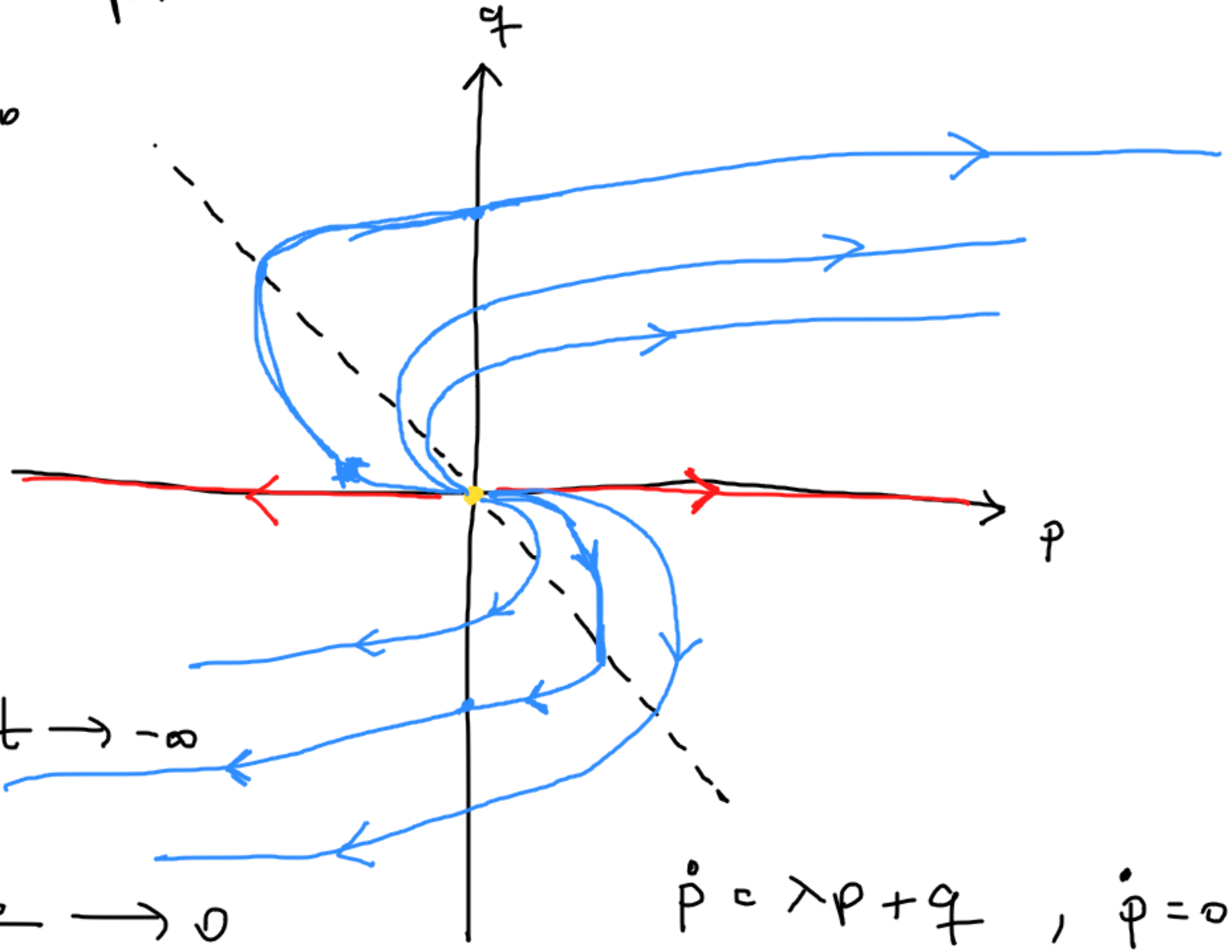
$p \to -\infty$

$q \to -\infty$

$q_0 > 0$

$p \to +\infty$

$q \to +\infty$ $\quad t \to +\infty$



$p \to 0,\ q \to 0 \quad t \to -\infty$

$$\tan\theta = \frac{q}{p} = \frac{q_0}{q_0 t + p_0} \longrightarrow 0 \qquad t \to \pm\infty$$

$\dot{p} = \lambda p + q$, $\dot{p} = 0$ $\quad \lambda p + q = 0$

$\dot{q} = \lambda q$ $\qquad\qquad q = -\lambda p$

Ejercicio 2. b). i) $\quad A = \begin{pmatrix} -1 & 1 \\ -4 & 3 \end{pmatrix}$

Valores propios: $\det(A - \lambda I) = \left| \begin{pmatrix} -1-\lambda & 1 \\ -4 & 3-\lambda \end{pmatrix} \right|$

$$= -(1+\lambda)(3-\lambda) + 4$$

$$= -\left[3 - \lambda + 3\lambda - \lambda^2\right] + 4$$

$$= -3 - 2\lambda + \lambda^2 + 4$$

$$= \lambda^2 - 2\lambda + 1$$

$$= (\lambda - 1)^2$$

$\lambda = 1$ es raiz doble

del pol. car.

$m_a(1) = 2$

$m_g(1) = \dim(\operatorname{Ker}(A - I))$

$= ?$

$$A - I = \begin{pmatrix} -2 & 1 \\ -4 & 2 \end{pmatrix}$$

$$\begin{pmatrix} -2 & 1 \\ -4 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2x + y \\ -4x + 2y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\operatorname{Ker}(A - I) = \{ (x, 2x) : x \in \mathbb{R} \} \qquad -2x + y = 0$$

$$y = 2x$$

$$= \langle (1, 2) \rangle$$

$$\dim \operatorname{Ker}(A - I) = 1 \qquad \overbrace{m_g(1)}^{1} < \overbrace{m_a(1)}^{2} \Rightarrow \text{Jordan.}$$

Vector propio es $v_1 = (1, 2)$

Busco una base $\{v_1, v_2\}$ t.q. $P^{-1}AP = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$$A v_1 = v_1 \quad A v_2 = v_1 + v_2 \rightsquigarrow A v_2 - v_2 = v_1$$

$$(A - I) v_2 = v_1$$

$$\boxed{\begin{array}{l} v_1 \text{ es el vector propio} \\ v_2 \text{ debe satisfacer } (A - I) v_2 = v_1 \end{array}}$$

$$v_2 = \begin{pmatrix} x \\ y \end{pmatrix} \qquad A - I = \begin{pmatrix} -2 & 1 \\ -4 & 2 \end{pmatrix}$$

$$v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad (A - I)\, v_2 = v_1$$

$$\begin{pmatrix} -2 & 1 \\ -4 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\begin{cases} -2x + y = 1 \\ -4x + 2y = 2 \end{cases} \qquad y = 1 + 2x$$

$x = 0$

$$v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Base de Jordan : $\boxed{\begin{array}{l} v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{array}}$

$$P = (v_1, v_2) = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$$

$$P^{-1} A P = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} -1 & 1 \\ -4 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}}_{\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}}$$

$$\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \longrightarrow \text{Jordan.}$$

$$p = e^{\lambda t}(q_0 t + p_0)$$
$$q = e^{\lambda t} q_0$$

$$p = e^{t}(q_0 t + p_0)$$
$$q = e^{t} q_0$$

$$p = e^t(q_0 t + p_0)$$
$$q = e^t q_0$$

$$P = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

$$Y = e^t\begin{pmatrix} q_0 t + p_0 \\ q_0 \end{pmatrix}$$

$$\begin{pmatrix} q_0 t + p_0 \\ q_0 \end{pmatrix}$$

$$X = PY = e^t \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = e^t \begin{pmatrix} q_0 t + p_0 \\ 2q_0 t + 2p_0 + q_0 \end{pmatrix}$$

$$X(0) = \begin{pmatrix} p_0 \\ 2p_0 + q_0 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

$$p_0 = x_0$$
$$q_0 = y_0 - 2x_0$$

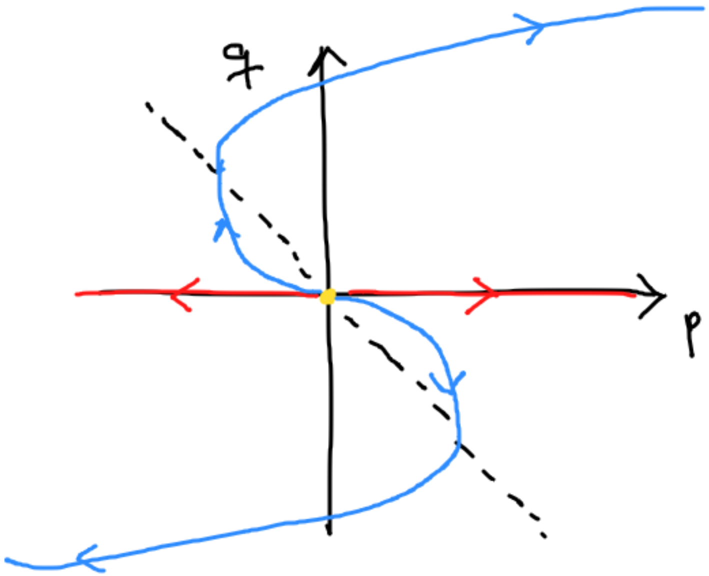
$$x = e^t\left((y_0 - 2x_0)t + x_0\right) \quad y = e^t\left(2(y_0 - 2x_0)t + y_0\right)$$

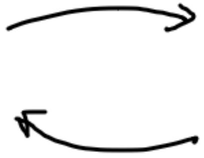
$$p = e^{t}(q_0 t + p_0)$$

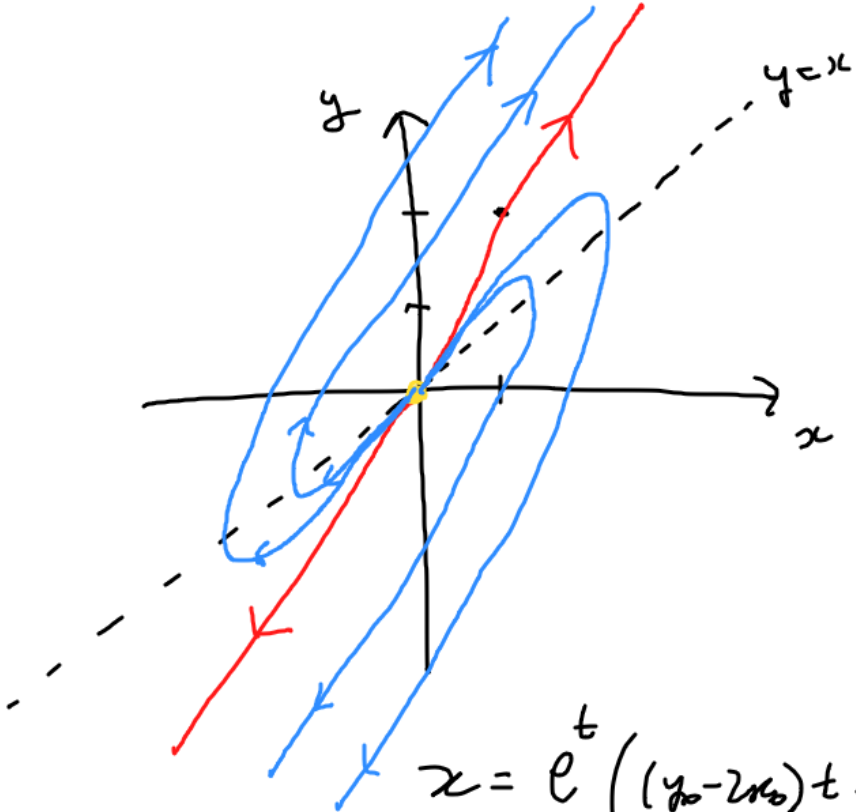
$$q = e^{t} q_0$$

$$P\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$$

$$\begin{pmatrix}1 & 0\\2 & 1\end{pmatrix} = P$$







$$x = e^{t}\left((y_0 - 2x_0)t + x_0\right)$$

$$y = e^{t}\left(2(y_0 - 2x_0)t + y_0\right)$$

$$\begin{pmatrix}1 & 0\\2 & 1\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$$

$$\begin{pmatrix}-1 & 1\\-4 & 3\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}-x+y\\-4x+3y\end{pmatrix}$$

$$y = x$$

Cómo encontrar una base de Jordan

Paso 1: Calcular el vp λ

Paso 2: Hallar un vector propio v_1

Paso 3: Resolver el sistema $(A-\lambda I)v_2 = v_1$

(incógnita es v_2)

$$P = (v_1, v_2) \qquad P^{-1}AP = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

$$X = PY.$$