Ecuaciones lineales – Métodos matriciales

(EL general) $\dot{X} = A(t)X + B(t)$

$X : (a,b) \longrightarrow \mathbb{R}^n \quad n \geq 1$

$A(t) \in M_n(\mathbb{R})$ matriz $n \times n$

$B(t) \in \mathbb{R}^n$ vector

Homogénea : $B(t) = 0 \quad \forall t$

Autónoma : $A(t) = A$ constante

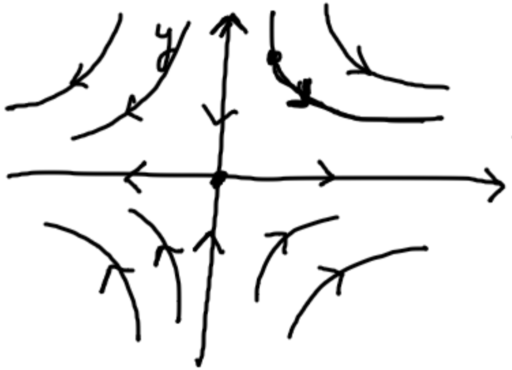
Caso particular : $X \in \mathbb{R}^2$, A constante $\in M_2(\mathbb{R})$ y $B(t) = 0 \ \forall t$

Ecuaciones lineales en el plano - Autónomas - Homogéneas

$X \in \mathbb{R}^2 \qquad A \in M_2(\mathbb{R}) \qquad B(t) = 0$

Útil dibujar el diagrama de fase: $X = \begin{pmatrix} x \\ y \end{pmatrix}$

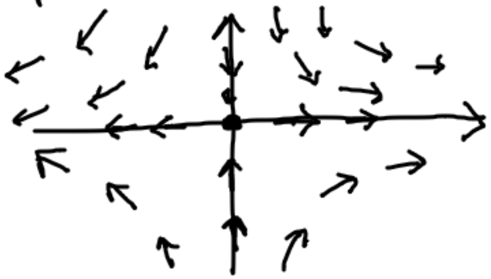
Perdemos la cinemática



dibujar las trayectorias de las soluciones indicando la dirección de movimiento.

Útil dibujar el campo de vectores

$$\dot{X} = \underbrace{AX}_{\text{campo de vectores}}$$

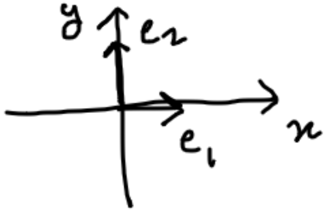


Cambio de base:

$\dot{X} = AX$ Derivada: Tranf. Lineal (función)

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

en la base canónica de $\mathbb{R}^2$



$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{cases} \dot{x} = ax + by \\ \dot{y} = cx + dy \end{cases}$$

$P \in M_2(\mathbb{R})$ constante.

Cambio lineal de variable: $Y = P^{-1}X$, $X = PY$

$$\dot{Y} = P^{-1}\dot{X} = P^{-1}AX = (P^{-1}AP)Y$$

$$\boxed{\dot{X} = AX \xrightarrow[Y = P^{-1}X]{} \dot{Y} = (P^{-1}AP)Y}$$

Objetivo: encontrar P tal que $P^{-1}AP$ es mas simple que A.

↳ conjugación

(*) A diagonalizable $\rightsquigarrow$ $P^{-1}AP$ es diagonal

A tiene forma de Jordan no diagonal $\rightsquigarrow$ $P^{-1}AP = \begin{pmatrix} 1 & \lambda \\ & 1 \end{pmatrix}$

si no $\rightsquigarrow$ $P^{-1}AP =$ rotación

Caso A diagonalizable:

Existe una base de vectores propios $\overbrace{v_\alpha \text{ y } v_\beta}^{\text{l.i.}}$

con valores propios α y β (que pueden ser iguales)

$$A v_\alpha = \alpha v_\alpha \qquad A v_\beta = \beta v_\beta$$

$$P = \left(\begin{array}{c} \text{vectores propios} \\ \text{como columnas} \end{array} \right) = \begin{pmatrix} v_\alpha & v_\beta \end{pmatrix} = \begin{pmatrix} x_\alpha & x_\beta \\ y_\alpha & y_\beta \end{pmatrix}$$

$$v_\alpha = x_\alpha e_1 + y_\alpha e_2 \qquad P \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x_\alpha \\ y_\alpha \end{pmatrix} = v_\alpha$$

$$v_\beta = x_\beta e_1 + y_\beta e_2 \qquad P \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} x_\beta \\ y_\beta \end{pmatrix} = v_\beta$$

$$P\begin{pmatrix} v_\alpha \text{ en la base} \\ \text{de v.p.} \end{pmatrix} = \begin{pmatrix} v_\alpha \text{ en la base} \\ \text{canónica } e_1, e_2 \end{pmatrix}$$

P como transf. lineal es la identidad

$$P\,(\text{coord. v.p.}) = (\text{Coord. base canónica})$$

P es una matriz de cambio de base.

$$AP = [\quad]_{c\,c}\,[\quad]_{c\,vp}$$

$$P^{-1}AP = [\quad]_{vp\,c}\,[\quad]_{c\,c}\,[\quad]_{c\,vp}$$

$P^{-1} A P$ = matriz diagonal $= D = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$

hay que pensarla en las coord. de los v.p.

En estas coordenadas la ecuación es: $\dot{Y} = DY$

$$Y = \begin{pmatrix} p \\ q \end{pmatrix} \qquad \begin{pmatrix} \dot{p} \\ \dot{q} \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} \alpha p \\ \beta q \end{pmatrix}$$

$$\boxed{\begin{cases} \dot{p} = \alpha p \\ \dot{q} = \beta q \end{cases}} \qquad \boxed{\begin{array}{l} p(t) = p_0 e^{\alpha t} \\ q(t) = q_0 e^{\beta t} \end{array}} \qquad Y = \begin{pmatrix} p_0 e^{\alpha t} \\ q_0 e^{\beta t} \end{pmatrix}$$

$$X = PY$$

Ejercicio 1 de la ficha 2

$$X = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\dot{X} = \underbrace{\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}}_{A} X$$

Polinomio caracteristico $\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ 0 & 3-\lambda \end{vmatrix}$

$= (1-\lambda)(3-\lambda)$

$\underline{\alpha = 1 \text{ y } \beta = 3}$

<u>Vectores propios</u>:

$\alpha = 1$ $\quad A - I = \begin{array}{cc} e_1 & e_2 \end{array} \begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix}$

$\operatorname{Ker}(A - I) = \{ (x, 0) : x \in \mathbb{R} \} = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle \quad v_\alpha = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\beta = 3$ $\quad A - 3I = \begin{pmatrix} -2 & 2 \\ 0 & 0 \end{pmatrix} = \{ (x, x) : x \in \mathbb{R} \} = \left\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\rangle \quad v_\beta = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad e_1 \quad e_2$

$$P = \underbrace{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}}_{} \qquad P^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

$$x(0) = p_0 + q_0 = x_0$$
$$y(0) = q_0 = y_0$$

$$P^{-1} A P = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}}$$

$$p_0 = x_0 - y_0$$
$$q_0 = y_0$$

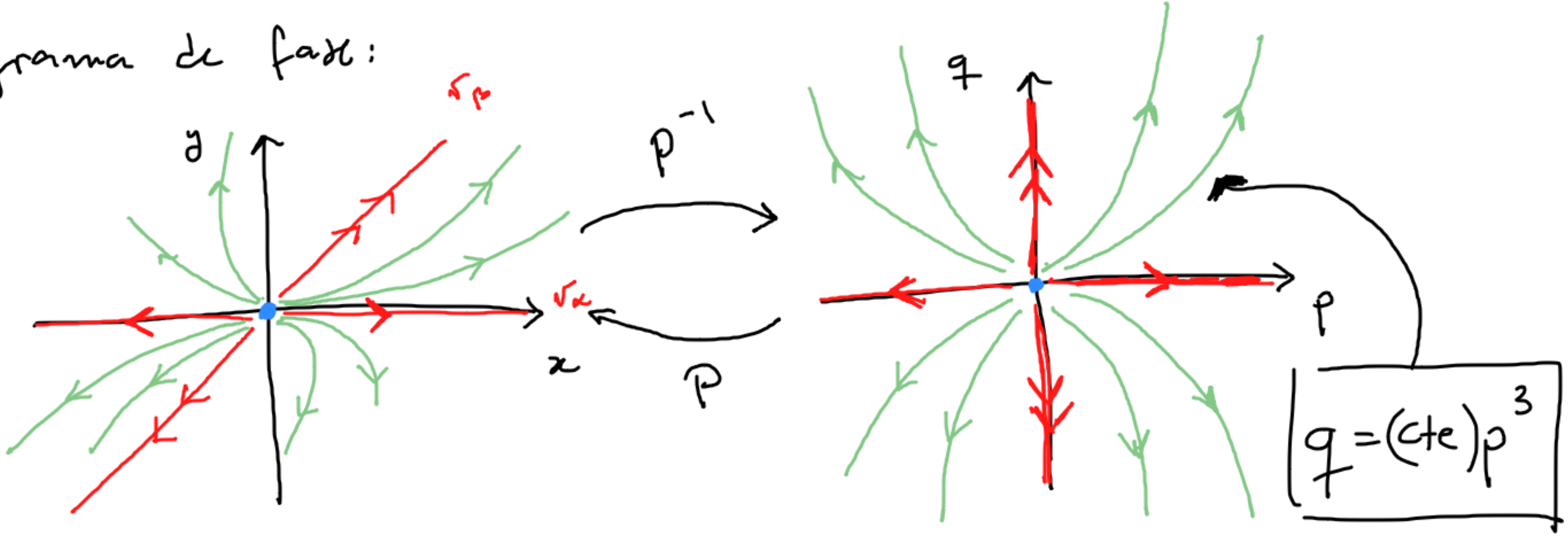
$$\begin{cases} x(t) = (x_0 - y_0) e^{t} + y_0 e^{3t} \\ y(t) = y_0 e^{3t} \end{cases}$$

$$= \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} = D$$

$$\begin{cases} \dot{x} = x + 2y \\ \dot{y} = 3y \end{cases} \rightsquigarrow \begin{cases} \dot{p} = p \\ \dot{q} = 3q \end{cases} \qquad \begin{array}{l} p(t) = p_0 e^{t} \\ q(t) = q_0 e^{3t} \end{array}$$

$$X = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_0 e^{t} \\ q_0 e^{3t} \end{pmatrix} = \begin{pmatrix} p_0 e^{t} + q_0 e^{3t} \\ q_0 e^{3t} \end{pmatrix} \qquad \begin{cases} x(t) = p_0 e^{t} + q_0 e^{3t} \\ y(t) = q_0 e^{3t} \end{cases}$$

Diagrama de fase:



"complicado"

canónica

"fácil"

v.p.

$$\begin{pmatrix} p_0 e^{t} \\ q_0 e^{3t} \end{pmatrix}$$

$$q = q_0 e^{3t} = q_0 \left(e^{t}\right)^3 = \frac{q_0}{p_0^3}\left(p_0 e^{t}\right)^3 = cte\, p^3$$

$$p(t) = p$$
$$q(t) = q$$

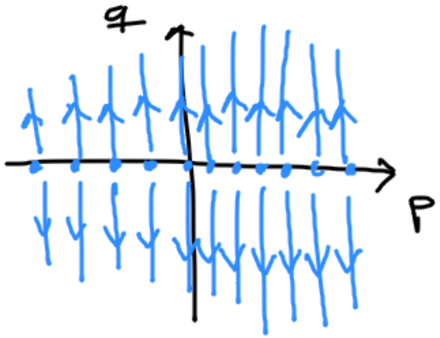
$$P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Diagramas de fase según α y β: (en el plano p, q)

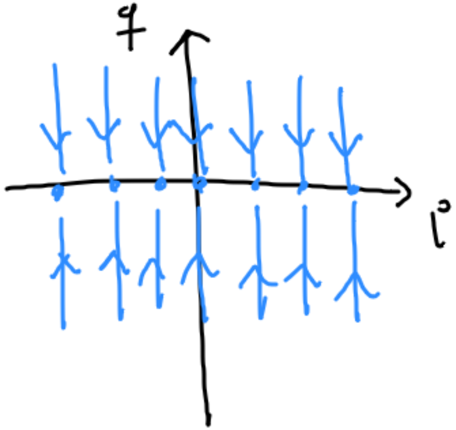
1) $\alpha = 0$

$p(t) = p_0$

$q(t) = q_0 e^{\beta t}$

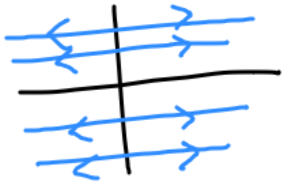


$\alpha = 0$, $\beta > 0$

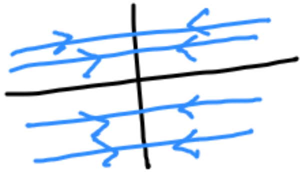


$\alpha = 0$, $\beta < 0$

2) $\beta = 0$



$\alpha > 0$, $\beta = 0$



$\alpha < 0$, $\beta = 0$

3) α y β son del mismo signo $\rightsquigarrow$ $\beta/\alpha > 0$

$$p = p_0 e^{\alpha t}$$

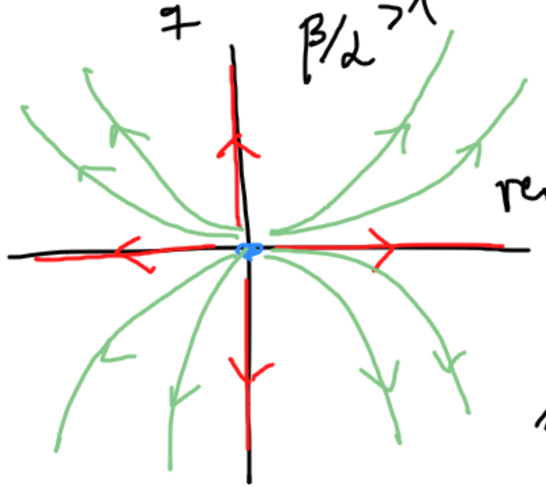
$$q = q_0 e^{\beta t} \qquad q = q_0 \left(e^{\alpha t}\right)^{\beta/\alpha}$$

$$= \frac{q_0}{p_0^{\beta/\alpha}} \left(p_0 e^{\alpha t}\right)^{\beta/\alpha}$$

$$= (cte)\, p^{\beta/\alpha}$$

Si $\alpha, \beta > 0$:

$\beta/\alpha > 1$



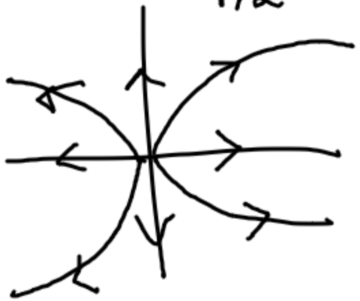
repulsor

"parábolas"

Trayectorias:

$$q = (cte)\, p^{\beta/\alpha}$$

$\beta/\alpha < 1$



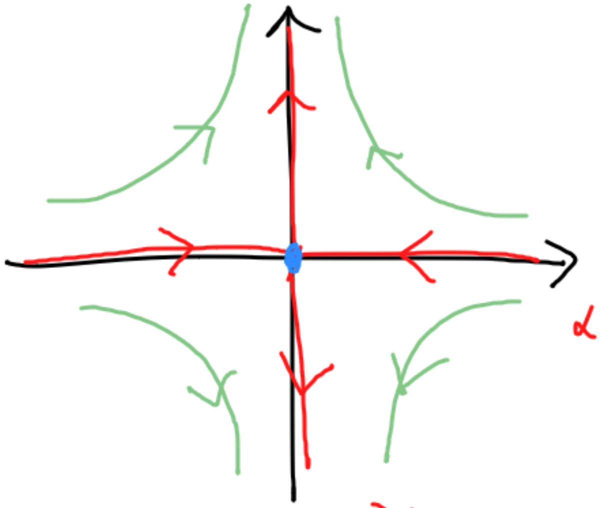
"raíz cuadrada"

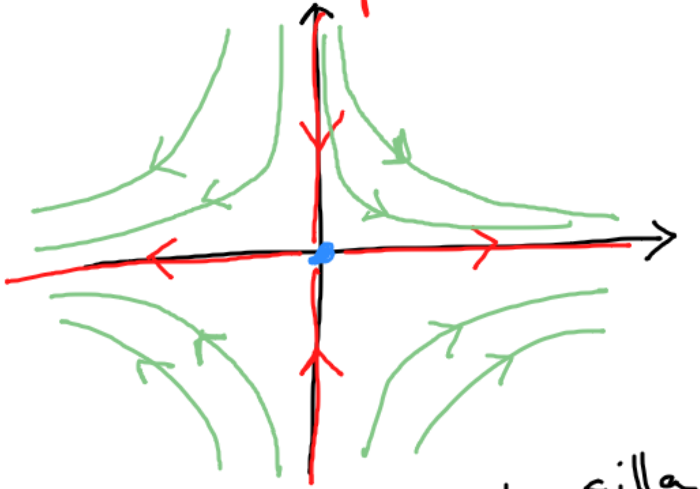
Si $\alpha, \beta < 0$: todas las flechas se invierten.

atractor

4) α y β tienen distinto signo: $\beta/\alpha < 0$

Trayectorias: $q = (cte)\, p^{\beta/\alpha}$





punto silla