Transformada de Laplace

$f : [0, +\infty) \longrightarrow \mathbb{R}$ $\qquad$ $\mathcal{L}(f)(s) = \int_0^{+\infty} e^{-st} f(t)\,dt$

orden exponencial

Ejemplos:

1) $\mathcal{L}(1)(s) = 1/s \;,\; s > 0$

2) $\mathcal{L}(e^{at} f(t)) = \mathcal{L}(f(t))(s-a)$

$\mathcal{L}(e^{at}) = \dfrac{1}{s-a} \;,\; s > a$

3) $\mathcal{L}(\cos(wt))(s) = \dfrac{s}{s^2+w^2}$ $\qquad$ $\mathcal{L}(\sin(wt))(s) = \dfrac{w}{s^2+w^2}$ $\qquad s > 0$

4) $\mathcal{L}(t^n)(s) = \dfrac{n!}{s^{n+1}} \;,\; s > 0$

5) $\mathcal{L}(f')(s) = s\,\mathcal{L}(f)(s) - f(0)$

Anti-transformada $\mathcal{L}^{-1}$

En las aplicaciones a ED vamos a tener que calcular la antitransformada de funciones racionales $R(s) = \dfrac{P(s)}{Q(s)}$

Dos pasos importantes: 1) Descomponer en fracciones simples

2) Usar la tabla de transformadas.

Ejemplo: $R(s) = \dfrac{A_1}{s-a_1} + \dfrac{A_2}{s-a_2} + \cdots + \dfrac{A_n}{s-a_n}$ (Paso 1)

(Paso 2)

$$\mathcal{L}^{-1}(R) = A_1 \mathcal{L}^{-1}\left(\frac{1}{s-a_1}\right) + A_2 \mathcal{L}^{-1}\left(\frac{1}{s-a_2}\right) + \cdots + A_n \mathcal{L}^{-1}\left(\frac{1}{s-a_n}\right)$$

$$= A_1 e^{a_1 t} + A_2 e^{a_2 t} + \cdots + A_n e^{a_n t}$$

Ejercicio 7.5

I) $F(s) = \dfrac{2}{3s+5}$ Ya está en fracción simple.

$$\mathcal{L}^{-1}(F)_{(t)} = \mathcal{L}^{-1}\left(\frac{2}{3s+5}\right)_{(t)} = \frac{2}{3}\mathcal{L}^{-1}\left(\frac{1}{s+5/3}\right) = \frac{2}{3}e^{-\frac{5}{3}t}$$

II) $F(s) = \dfrac{-2s+6}{s^2+64}$

$$= -2\,\frac{s}{s^2+8^2} + \frac{6}{8}\,\frac{8}{s^2+8^2}$$

$$\mathcal{L}^{-1}(F)(t) = -2\cos(8t) + 6/8\,\text{sen}(8t)$$

$$R(s) = \frac{1}{(s^2+1)(s-2)^2}$$

$$= \frac{As+B}{s^2+1} + \frac{C}{(s-2)^2} + \frac{D}{(s-2)}$$

$$R(s) = \frac{\overbrace{a_n s^n + \cdots + a_0}^{P(s)}}{\underbrace{b_m s^m + \cdots + b_0}_{Q(s)}}$$

$\xrightarrow{n \geq m?}$ Si Hacer división de polinomios

$\xrightarrow{n \geq m?}$ No. Fracciones simples

$$Q(s) = C\,(s-a_1)^{d_1} \cdots (s-a_\ell)^{d_\ell}\, q_1(s)^{r_1} \cdots q_k(s)^{r_k}$$

q_i = cuadráticos sin raíces.

$$R(s) = \frac{A_1}{(s-a_1)^{d_1}} + \frac{A_2}{(s-a_1)^{d_1-1}} + \cdots + \frac{A_{d_1-1}}{(s-a_1)} + \cdots$$

$$+ \frac{B_1}{q_1(s)^{r_1}} + \cdots + \frac{B_{r_1-1}}{q_1(s)} + \cdots$$

$\underline{III}$) $F(s) = \dfrac{e^{2s}}{s-3} = e^{2s} \cdot \dfrac{1}{s-3}$ $\qquad \boxed{\mathcal{L}^{-1}(F)(t) = e^{3t+6} A(t)}$

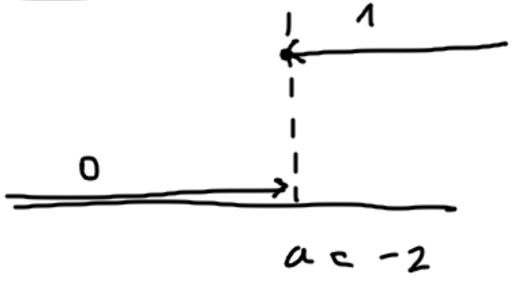


Tabla de las notas del curso:

$$A(t) = \begin{cases} 0 & \text{si } 0 \le t < a \\ 1 & \text{si } t > a \end{cases}$$

$$H(t) = \begin{cases} 0 \text{ si } t < 0 \\ 1 \text{ si } t > 0 \end{cases}$$

$$\mathcal{L}(A)(s) = \frac{1}{s} e^{-as}$$

$$A(t) = H(t-a)$$

$$\mathcal{L}(A)(s-3) = \frac{1}{s-3} e^{-a(s-3)} = \frac{1}{s-3} \overset{a=-2}{e^{-as}} \underbrace{e^{3a}}_{e^{-6}}$$

$$F(s) = \frac{1}{s-3} e^{2s} = \left[\frac{1}{s-3} e^{2s} e^{-6}\right] \cdot \frac{1}{e^{-6}} = \mathcal{L}(A)(s-3) \cdot e^{6}$$

$$= \mathcal{L}(e^{6} A)(s-3) = \mathcal{L}(e^{3t+6} A)(s)$$

IV) $F(s) = \dfrac{e^{-3s}}{(s-1)(s-2)} = e^{-3s}\left[\dfrac{1}{(s-1)(s-2)}\right]$

$$= e^{-3s}\left[\frac{A}{s-1} + \frac{B}{s-2}\right] = e^{-3s}\left[\frac{-1}{s-1} + \frac{1}{s-2}\right]$$

$A=-1 \qquad B=1$

$= -\dfrac{e^{-3s}}{s-1} + \dfrac{e^{-3s}}{s-2}$ = seguimos como en el ej III.

Ejemplo de las notas

$$F(s) = \frac{2}{s^3(s+1)} = \frac{A}{s^3} + \frac{B}{s^2} + \frac{C}{s} + \frac{D}{s+1}$$

$A = 2$ $\qquad$ $D = -2$

$$= \frac{2}{s^3} + \frac{B}{s^2} + \frac{C}{s} - \frac{2}{s+1} = \frac{2(s+1) + B(s+1)s + C(s+1)s^2 - 2s^3}{s^3(s+1)}$$

$C = 2$ $\qquad$ $B = -2$

$$= \frac{\underbrace{(C-2)}_{=0}s^3 + \underbrace{(C+B)}_{=0}s^2 + \underbrace{(2+B)}_{=0}s + 2}{s^3(s+1)}$$

$$\frac{2}{s^3} - \frac{2}{s^2} + \frac{2}{s} - \frac{2}{s+1}$$

$$F(s) = \frac{2}{s^3} - \frac{2}{s^2} + \frac{2}{s} - \frac{2}{s+1}$$

$$t^2 \qquad -2t \qquad 2 \qquad -2e^{-t}$$

$$\mathcal{L}(t^2) = \frac{2!}{s^3} = \frac{2}{s^3}$$

$$\mathcal{L}(t) = \frac{1!}{s^2}$$

$$\boxed{\mathcal{L}^{-1}(F)(t) = t^2 - 2t + 2 - 2e^{-t}}$$

Convolución:

$f(t),\ g(t)$

$f,g : [0,+\infty) \to \mathbb{R}$

$$f * g(t) = \int_0^{+\infty} f(\overbrace{u}^{?})\, g(\overbrace{t-u}^{?})\, du$$

$$= \int_0^t f(u)\, g(t-u)\, du$$

Ejemplo:

$f(t) = t^2$

$g(t) = t$

$\mathcal{L}(f)(s) = \overbrace{2/s^3}$ $\quad \mathcal{L}(g) = \overbrace{1/s^2}$ $\quad \boxed{\mathcal{L}(t^2 * t) = \mathcal{L}(t^2)\mathcal{L}(t)}$

$$f * g(t) = \int_0^t u^2 (t-u)\, du = \left. \frac{u^3}{3} t - \frac{u^4}{4} \right|_0^t$$

$$= \frac{t^4}{3} - \frac{t^4}{4} = \frac{t^4}{12}$$

$$t^2 * t = \frac{t^4}{12}$$

$$\mathcal{L}(t^2 * t) = \frac{1}{12} \frac{4!}{s^5} = \frac{2}{s^5} = \mathcal{L}(t^2)\mathcal{L}(t)$$

$$\mathcal{L}(f*g) = \mathcal{L}(f)\,\mathcal{L}(g)$$

Sirve para calcular anti-transformadas:

$$F(s) = \underbrace{F_1(s)}\,\underbrace{F_2(s)} \Rightarrow \mathcal{L}^{-1}(F) = f_1 * f_2$$

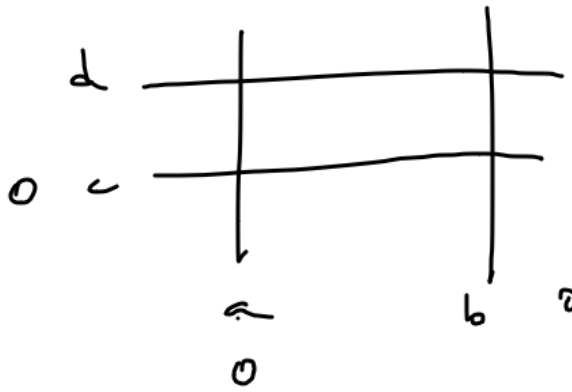
sé calcular $\mathcal{L}^{-1}$ sé calcular $\mathcal{L}^{-1}$

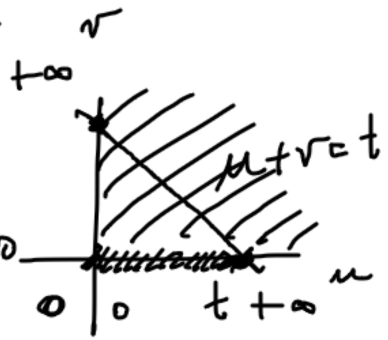
f_1 f_2

"Prueba":

$$\mathcal{L}(f)(s) = \int_0^{+\infty} e^{-su} f(u)\, du$$

$$\mathcal{L}(g)(s) = \int_0^{+\infty} e^{-sv} g(v)\, dv$$





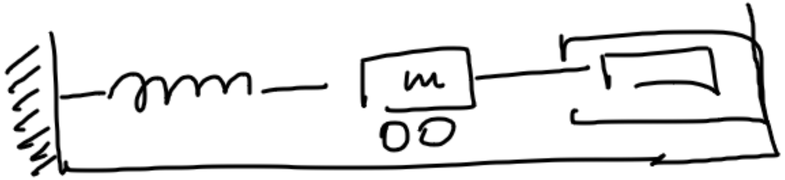
$$\overbrace{\mathcal{L}(f)(s)\,\mathcal{L}(g)(s)} = \left[\int_0^{+\infty} e^{-su} f(u)\, du\right]\left[\int_0^{+\infty} e^{-sv} g(v)\, dv\right] \quad \mathcal{L}(f*g)(s)$$

$$= \int_0^{+\infty}\int_0^{+\infty} e^{-su} e^{-sv} f(u) g(v)\, du\, dv = \int_0^{+\infty}\int_0^{+\infty} e^{-s\overbrace{(u+v)}^{t}} f(u) g(v)\, du\, dv$$

$v = t - u$

$$= \begin{bmatrix} t = u+v \\ u = u \end{bmatrix} \int_0^{+\infty}\int_0^{t} e^{-st} f(u)\, g(t-u)\, du\, dt = \int_0^{+\infty} e^{-st}\Big[\underbrace{\int_0^t f(u) g(t-u)\, du}_{\text{"} f*g(t) \text{"}}\Big] dt$$

Oscilador armónico:



$\omega_0 = \sqrt{k/m}$ frecuencia natural del resorte

$$m\ddot{x} + b\dot{x} + kx = F_{ext}(t)$$

Sin fricción $(b=0)$

$$m\ddot{x} + kx = F_{ext}(t) = F\cos(\omega t)$$

$$m\ddot{x} + kx = F\cos(\omega t)$$

Caso 1: $\omega \neq \omega_0$

Caso 2: $\omega = \omega_0$

Resonancia

Ecuación homogénea: $m\ddot{x} + kx = 0$

$x = e^{rt}$

$\dot{x} = re^{rt}$

$\ddot{x} = r^2 e^{rt}$

$mr^2\cancel{e^{rt}} + k\cancel{e^{rt}} = 0$

$mr^2 + k = 0$

$r^2 + \omega_0^2 = 0$

raíces complejas $\pm \omega_0 i$

$$x_h = C_1 \cos(\omega_0 t) + C_2 \operatorname{sen}(\omega_0 t)$$

Solución particular: $m\ddot{x} + kx = F\cos(\omega t)$

Caso 1 $\omega \neq \omega_0$

$$x_p = A\cos(\omega t) + B\operatorname{sen}(\omega t)$$

$$\dot{x}_p = -A\omega \operatorname{sen}(\omega t) + B\omega\cos(\omega t)$$

$$\ddot{x}_p = -A\omega^2\cos(\omega t) - B\omega^2 \operatorname{sen}(\omega t)$$

$$\overbrace{A(k - m\omega^2)}^{F}\cos(\omega t) + (k - m\omega^2)B\operatorname{sen}(\omega t) = F\cos(\omega t)$$

$$A = \frac{F}{k - m\omega^2} = \frac{F}{m(\omega_0^2 - \omega^2)}$$

$B = 0$

$$\boxed{x_p = \frac{F}{m(\omega_0^2 - \omega^2)}\cos(\omega t)}$$

Caso 2: $\omega = \omega_0$

$$m\omega_0^2 = k$$

$$x_p = \overbrace{At\cos(\omega_0 t) + Bt\,\mathrm{sen}(\omega_0 t)}^{\times k}$$

$$\dot{x}_p = A\cos(\omega_0 t) + B\,\mathrm{sen}(\omega_0 t) - A\omega_0 t\,\mathrm{sen}(\omega_0 t) + B\omega_0 t\cos(\omega_0 t)$$

$$\ddot{x}_p = -A\omega_0\,\mathrm{sen}(\omega_0 t) + B\omega_0\cos(\omega_0 t) - A\omega_0\,\mathrm{sen}(\omega_0 t) + B\omega_0\cos(\omega_0 t) \underbrace{- A\omega_0^2 t\cos(\omega_0 t) - B\omega_0^2 t\,\mathrm{sen}(\omega_0 t)}_{\times m}$$

$$m\ddot{x}_p + kx_p$$

$A = 0$

$$\cancel{-2mA\omega_0\,\mathrm{sen}(\omega_0 t)} + 2mB\omega_0\cos(\omega_0 t) = F\cos(\omega_0 t)$$

$$B = \frac{F}{2m\omega_0} \qquad x_p = \frac{Ft}{2m\omega_0}\,\mathrm{sen}(\omega_0 t)$$

$$x = C_1 \cos(w_0 t) + C_2 \operatorname{sen}(w_0 t) + \underbrace{\frac{F t}{2 m w_0}} \operatorname{sen}(w_0 t)$$

amplitud crece

linealmente con t