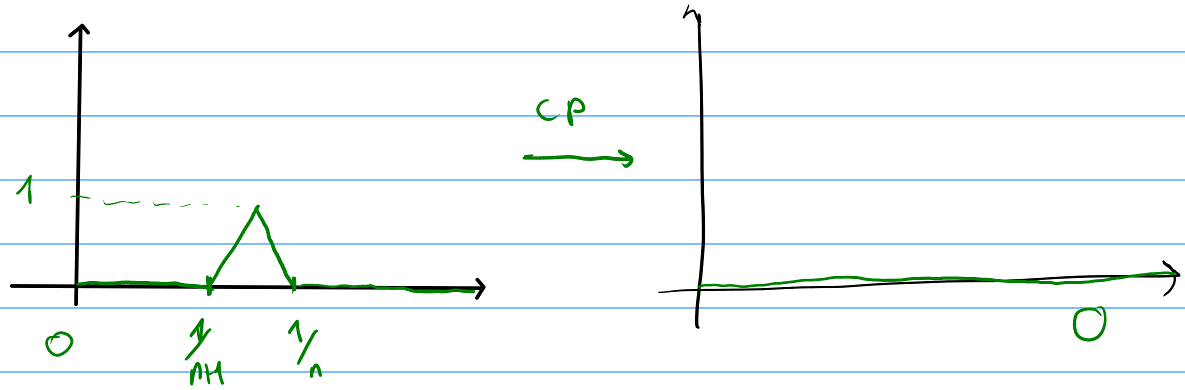
Criterio de Cauchy para convergencia uniforme: $f_n : X \to \mathbb{R}^n$

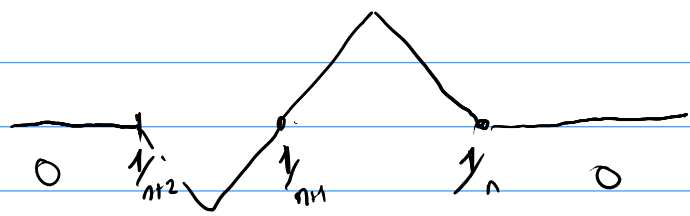
la sucesión converge uniformemente si y sólo si $\forall \varepsilon > 0, \exists N_0$ tq

Si $n, m \geq N_0 \Rightarrow \sup\limits_{x \in X} \| f_n(x) - f_m(x) \| < \varepsilon$.

$f_n(x) =$



$f_n(x) - f_{n+1}(x) =$



$\sup\limits_{x \in [0,1]} \| f_n(x) - f_{n+1}(x) \| = 1 \quad \forall n$.

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1. Sea $\{f_n\}$ una sucesión de funciones $(f_n : \mathbb{R} \to \mathbb{R})$ tal que

$$f_n''(x) = f_{n+2}(x)(n+2)(n+1) - f_n(x) \quad \forall x \in \mathbb{R}, \ \forall n \in \mathbb{N}$$

$$f_0(x) = e^x, \quad f_1(x) = 0$$

a) Probar que $f_{2n+1}(x) = 0 \ \forall x \in \mathbb{R}, \forall n \in \mathbb{N}$ y $f_{2n}(x) = \dfrac{2^n e^x}{(2n)!}$

*) $f_{2n+1}(x) = 0 \ \forall x, \ \forall n \in \mathbb{N}$

Caso base $f_1(x) = 0$ ✓

Paso inductivo : $f_{2(n+1)+1}(x) = f_{2n+3}(x) = \dfrac{\overset{0}{f''_{2n+1}} + \overset{0}{f_{2n+1}}}{(2n+3)(2n+2)} = 0$

$$f''_n(x) = f_{n+2}(x)\,(n+2)(n+1) - f_n(x)$$

$2n+1 \quad 2n+1 \quad 2n+1 \quad 2n+1 \quad 2n+1$

* $f_{2n}(x) = \dfrac{2^n e^x}{(2n)!}$

Paso base: $f_0 = e^x$ ✓

Paso inductivo: $f_{2(n+1)}(x) = f_{2n+2}(x) = \dfrac{(f''_{2n} + f_{2n})(x)}{(2n+2)(2n+1)} = \dfrac{\dfrac{2^n e^x}{(2n)!} + \dfrac{2^n e^x}{(2n)!}}{(2n+2)(2n+1)}$

$$f''_n(x) = f_{n+2}(x)\,(n+2)(n+1) - f_n(x)$$

$2n \quad 2n \quad 2n \quad 2n \quad 2n$

$$= \frac{2^{n+1} e^x}{(2n)!(2n+1)(2n+2)} = \frac{2^{n+1} e^x}{(2n+2)!} = \frac{2^{n+1} e^x}{(2(n+1))!} \quad ✓$$

b) Se considera $u_{tt}(x,t) = u_{xx}(x,t) + u(x,t)$, $-L < x < L$, $0 < t < 1$.

1) Buscar soluciones de la forma $u(x,t) = \displaystyle\sum_{n=0}^{+\infty} g_n(x)\, t^n$

Con $g_n(x)$ ciertas funciones.

Buscando soluciones: Asumo que la serie converge, que es C^2 y vale pasar las derivadas para adentro.

$$u_{tt} = \partial_t^2 \left(\sum_{n=0}^{+\infty} g_n(x) t^n \right) = \sum_{n=2}^{+\infty} g_n(x)\, n(n-1) t^{n-2}$$

$$u_{xx} = \partial_x^2 \left(\sum_{n=0}^{+\infty} g_n(x) t^n \right) = \sum_{n=0}^{+\infty} g_n''(x) t^n$$

$$u(x,t) = \sum_{n=0}^{+\infty} g_n(x) t^n$$

Igualando
$$\sum_{n=2}^{+\infty} g_n(x)\, n(n-1) t^{n-2} = \sum_{n=0}^{+\infty} g_n''(x) t^n + g_n(x) t^n$$

$$\parallel$$

$$\sum_{n=0}^{+\infty} g_{n+2}(x)(n+2)(n+1) t^n$$

Voy a exigir que $g_{n+2}(x)(n+2)(n+1) t^n = g_n''(x) t^n + g_n(x) t^n$

$$\Rightarrow \quad g_{n+2}(x)(n+2)(n+1) - g_n(x) = g_n''(x)$$

Es lo que aparecía en la parte a) ; Si $g_0(x) = e^x$, $g_1(x) = 0$

$$\Rightarrow \quad g_{2n+1}(x) = 0 \quad , \quad g_{2n}(x) = \frac{2^n e^x}{(2n)!}$$

$$\Rightarrow \quad u(x,t) = \sum_{n=0}^{+\infty} g_n(x) t^n = \sum_{k=0}^{+\infty} g_{2k}(x) t^{2k} = \sum_{k=0}^{+\infty} \frac{2^k e^x}{(2k)!} t^{2k}$$

2) Probar que la solución encontrada es efectivamente solución.

* $S_n(x,t) = \sum_{k=0}^{n} \frac{2^k e^x t^{2k}}{(2k)!}$ Converge uniformemente en $-L<x<L$, $0<t<1$.

$$\left|\frac{2^k e^x t^{2k}}{(2k)!}\right| \leq \frac{2^k}{(2k)!} e^L \quad y \quad \sum_{k=0}^{+\infty} \frac{2^k}{(2k)!} e^L < +\infty$$

$\Rightarrow$ Por el mayorante de Weierstrass $S_n(x,t)$ converge uniformemente a $\sum_{k=0}^{+\infty} \frac{2^k e^x t^{2k}}{(2k)!}$

y $u(x,t) := \sum_{k=0}^{+\infty} \frac{2^k e^x t^{2k}}{(2k)!}$

* $u(x,t)$ es continua, por la convergencia uniforme de una serie de funciones continuas

* $u(x,t)$ es derivable respecto a t

$$\partial_t S_n(x,t) = \sum_{k=1}^{n} \frac{2^k e^x 2k t^{2k-1}}{(2k)!}, \quad \left|\frac{2^k e^x 2k t^{2k-1}}{(2k)!}\right| \leq e^L \cdot \frac{2^k \cdot 2k}{(2k)!}$$

y $\sum_{k=1}^{+\infty} \frac{e^L \cdot 2^k \cdot 2k}{(2k)!} < +\infty \Rightarrow$ Por el mayorante de Weierstrass

$\partial_t S_n(x,t)$ converge uniformemente.

$$\left\{\begin{array}{l} * \ S_n(x,t) \rightrightarrows u(x,t) \\ * \ \underbrace{\partial_t S_n(x,t)}_{} \rightrightarrows h(x,t) \end{array}\right. \Rightarrow \begin{array}{l} u \text{ es derivable respecto a } t \\ y \ \partial_t u = h. \end{array}$$

Es decir, $\partial_t \left(\sum_{k=0}^{+\infty} \frac{2^k e^x t^{2k}}{(2k)!} \right) = \sum_{k=0}^{+\infty} \partial_t \left(\frac{2^k e^x t^{2k}}{(2k)!} \right)$

Para ∂_t^2: $\left\{\begin{array}{l} S_n(x,t) \rightrightarrows u \\ \partial_t S_n(x,t) \rightrightarrows \partial_t u \\ \underbrace{\partial_t^2 S_n(x,t)}_{\downarrow} \rightrightarrows j(x,t) \end{array}\right. \Rightarrow$ u es derivable dos veces respecto a t y $\partial_t^2(x,t) = j(x,t)$

es decir,

$$\sum_{k=0}^{n} \partial_t^2\left(\frac{2^k e^x t^{2k}}{(2k)!}\right) \rightrightarrows \partial_t^2\left(\sum_{k=0}^{+\infty} \frac{2^k e^x t^{2k}}{(2k)!}\right)$$

es decir,

$$\sum_{k=0}^{+\infty} \partial_t^2\left(\frac{2^k e^x t^{2k}}{(2k)!}\right) = \partial_t^2\left(\sum_{k=0}^{+\infty} \frac{2^k e^x t^{2k}}{(2k)!}\right)$$

Chequeo lo rojo: $\partial_t^2 S_n(x,t) = \sum_{k=0}^{n} \partial_t^2\left(\frac{2^k e^x t^{2k}}{(2k)!}\right) = \sum_{k=2}^{n} \frac{2^k e^x t^{2k-2}}{(2k)!} \cdot (2k)(2k-1)$

$$\left|\frac{2^k e^x t^{2k-2}(2k)(2k-1)}{(2k)!}\right| \leq \frac{e^L \cdot 2^k}{(2k-2)!}, \quad \sum_{k=2}^{+\infty} \frac{e^L 2^k}{(2k-2)!} < +\infty$$

$\Rightarrow$ Por mayorante de Weierstrass $\partial_t^2 S_n$ converge uniformemente

$*$ $S_n(x,t) \rightrightarrows u$

pero $\partial_x^2 S_n(x,t) = \partial_x S_n(x,t) = S_n(x,t) \Rightarrow \left\{\begin{array}{l} S_n(x,t) \rightrightarrows u \\ \partial_x S_n(x,t) \rightrightarrows h \\ \partial_x^2 S_n(x,t) \rightrightarrows j \end{array}\right.$

$\Rightarrow$ u es derivable 2 veces respecto a x y $h = \partial_x u$, $j = \partial_x^2 u$

es decir, vale pasar la derivada para adentro.

$\cancel{e^x} + \frac{2e^x t^2}{2!}$

Paso derivada para adentro.

$$\partial_t^2 u = \partial_t^2\left(\sum_{k=0}^{+\infty} \frac{2^k e^x t^{2k}}{(2k)!}\right) \stackrel{!}{=} \sum_{k=1}^{+\infty} \frac{2^k e^x (2k)(2k-1)t^{2k-2}}{(2k)!} = \sum_{k=1}^{+\infty} \frac{2^k e^x t^{2k-2}}{(2k-2)!}$$

$$= \sum_{k=0}^{+\infty} \frac{2^{k+1} e^x t^{2k}}{(2k)!} = 2\cdot \sum_{k=0}^{+\infty} \frac{2^k e^x t^{2k}}{(2k)!} = 2u$$

$$\partial_x^2 u = \partial_x^2\left(\sum_{k=0}^{+\infty} \frac{2^k e^x t^{2k}}{(2k)!}\right) = \sum_{k=0}^{+\infty} \frac{2^k e^x t^{2k}}{(2k)!} = u$$

$$\underbrace{u_{tt}(x,t)}_{2u} = \underbrace{u_{xx}(x,t)}_{u} + \underbrace{u(x,t)}_{u} \quad ✓$$

Ejercicio 2 (1 punto)

Sea el problema

$$\begin{cases} u_t - u_{xx} = x - \pi + \sin(4x) & t > 0,\ 0 < x < \pi \\ u(0,t) = -\pi t \\ u(\pi,t) = \pi \\ u(x,0) = x + \sin(4x) & 0 < x < \pi \end{cases}$$

La solución es:

$$v_t - v_{xx} = (u_t - u_{xx}) + (x g'(t) + h'(t)) - 0$$
$$= x - \pi + \sin 4x + x g'(t) + h'(t) \Rightarrow x g'(t) + h'(t) = \pi - x$$

Tomo $g'(t) = -1 \ \forall t$

$h'(t) = \pi \ \forall t$

Sugerencia: considere $v(x,t) = u(x,t) + xg(t) + h(t)$ con h y g elegidas de modo tal que simplifiquen las condiciones de borde y que v satisfaga la ecuación $v_t - v_{xx} = \sin(4x)$. Suponga que existe solución $v(x,t) = T_n(t)\sin(nx)$ para este nuevo problema.

$\Rightarrow \ V(x,t) = u(x,t) + x\cdot(-t + \underbrace{b}_{?}) + \pi t + \underbrace{c}_{?}$

$V(0,t) = \underbrace{u(0,t)}_{-\pi t} + \pi t + c = 0 \Rightarrow c = 0.$

$V(\pi,t) = \underbrace{u(\pi,t)}_{\pi} + \pi(-t+b) + \pi t = \pi + \pi b = 0$

$\Rightarrow b = -1$

$$v(x,t) = u(x,t) + x(-t-1) + \pi t$$

$$\Rightarrow \begin{cases} v_t - v_{xx} = \sin 4x \\ v(0,t) = 0, \quad v(\pi,t) = 0 \\ v(x,0) = u(x,0) - x = x + \sin(4x) - x = \sin 4x \end{cases}$$