12 (a) Dado A matriz, probar que si α no es vap entonces la ecuación $\dot{x} = Ax + e^{\alpha t} b$, $b \in \mathbb{R}^n$, $A \in M_{n\times n}$. tiene una única solución de la forma $x(t) = e^{\alpha t} \cdot u$, $u \in \mathbb{R}^n$.

Debo mostrar que $\exists ! u$ tal que $x(t) = e^{\alpha t} \cdot u$ satisface "$\dot{x} = Ax + e^{\alpha t} b$"

$x(t) = e^{\alpha t} \cdot u$ es solución $\Leftrightarrow$ $\dot{(e^{\alpha t} \cdot u)} = A e^{\alpha t} u + e^{\alpha t} b$

$\Leftrightarrow \alpha e^{\alpha t} u = e^{\alpha t}(Au + b) \Leftrightarrow \alpha I u = Au + b \Leftrightarrow \overbrace{(\alpha I - A)}^{\det \neq 0 \text{ (}\alpha\text{ no es vap)}} u = b$

$\Leftrightarrow u = (\alpha I - A)^{-1} b$

$x(t) = e^{\alpha t} (\alpha I - A)^{-1} b$ es la solución con la forma pedida.

(b) Resolver
$$\begin{aligned} \dot{x} &= 2x + y + e^{2t} \\ \dot{y} &= x + 2y - e^{2t} \end{aligned} \Rightarrow \dot{X} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} X + e^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

La solución general es la homogénea + la particular: donde $X_H(t)$ es la solución gral de $\dot{X} = AX \longrightarrow X_H(t) = e^{At} \cdot C$

$$X(t) = e^{At} \cdot C + e^{2t} (2I - A)^{-1} b, \quad \text{con } A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \; b = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

- 2 no es valor propio: $A - 2I = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ invertible $\Rightarrow$ NO $\exists$ VAP.

$$X_0 = X(0) = e^{A0} \cdot C + (2I - A)^{-1} b = C + (2I - A)^{-1} b$$

$\Rightarrow C = X_0 - (2I - A)^{-1} b$

$$X(t) = e^{At} (X_0 - (2I - A)^{-1} b) + e^{2t} \cdot (2I - A)^{-1} b, \text{ solución con } X(0) = X_0.$$

$X(t_0) = X_0 \Rightarrow X_0 = e^{At_0} C + e^{2t_0} (2I - A)^{-1} b \Rightarrow C = (e^{At_0})^{-1} (X_0 - e^{2t_0} (2I - A)^{-1} b)$

$\Rightarrow C = e^{-At_0} \cdot (X_0 - e^{2t_0} (2I - A)^{-1} b)$

$e^{A} \cdot e^{-A} = e^{0} = I$

$$X(t) = e^{At}\left(e^{-At_0}\left(X_0 - e^{2t_0}(2I-A)^{-1}b\right)\right) + e^{2t}(2I-A)^{-1}b$$

$$= \underline{e^{A(t-t_0)} X_0} - e^{A(t-t_0)}e^{2t_0}(2I-A)^{-1}b + e^{2t}(2I-A)^{-1}b$$, solución que pasa por X_0 en $t=t_0$

$$\dot{X} = A(t)X(t) + b(t)$$

$$X(t_0) = X_0$$

$$X(t) = \underline{\varphi(t)\varphi^{-1}(t_0)X_0} + \varphi(t)\int_{t_0}^{t}\varphi^{-1}(t)b(t)\,dt$$

donde $\varphi(t)$ es la matriz fundamental, es decir, la solución a

$$\begin{cases} \dot{\varphi}(t) = A(t)\varphi(t) \\ \varphi(0) = I \end{cases}$$

$$X(t) = \varphi(t)\varphi^{-1}(t_0)X_0 + \varphi(t)\int_{t_0}^{t}\varphi^{-1}(x)b(x)\,dx$$

$$\dot{X}(t) = \underbrace{\dot{\varphi}(t)}_{A(t)\varphi(t)}\varphi^{-1}(t_0)X_0 + \underbrace{\dot{\varphi}(t)}_{A(t)\varphi(t)}\underbrace{\int_{t_0}^{t}\varphi^{-1}(x)b(x)\,dx} + \overbrace{\varphi(t)\cdot\varphi^{-1}(t)}^{I}b(t)$$

$$= A(t)\underbrace{\left(\varphi(t)\varphi^{-1}(t_0)X_0 + \varphi(t)\int_{t_0}^{t}\varphi^{-1}(x)b(x)\,dx\right)}_{X(t)} + b(t) = A(t)X(t) + b(t) \checkmark$$

$$X(t_0) = \varphi(t_0)\varphi^{-1}(t_0)X_0 + \varphi(t_0)\underbrace{\int_{t_0}^{t_0}\_\,dx}_{0} = X_0 \checkmark$$

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Recordar: Si $A(t) = A \ \forall t \Rightarrow \varphi(t) = e^{At}$.

Consultas: - Si v_λ es vector propio de valor $\lambda = \alpha + i\beta$.

$$\Rightarrow A v_\lambda = (\alpha + i\beta) v_\lambda$$

$$v_\lambda = Re(v_\lambda) + i Im(v_\lambda)$$

$$\Rightarrow A v_\lambda = \overbrace{A Re(v_\lambda)}^{\in \mathbb{R}^2} + i \overbrace{A Im(v_\lambda)}^{\in \mathbb{R}^2}$$

$$A v_\lambda = (\alpha + i\beta) v_\lambda = (\alpha + i\beta)(Re(v_\lambda) + i Im(v_\lambda)) = \overbrace{\alpha Re(v_\lambda) - \beta Im(v_\lambda)}^{\in \mathbb{R}^2}$$

$$+ i \underbrace{(\beta Re(v_\lambda) + \alpha Im(v_\lambda))}_{\in \mathbb{R}^2}.$$

$$\Rightarrow A Re(v_\lambda) = \alpha Re(v_\lambda) - \beta Im(v_\lambda)$$

$$(A)_{\mathcal{B}}^{\mathcal{B}} = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$$

$$\mathcal{B} = \{Re(v_\lambda), Im(v_\lambda)\}$$