Exponencial de una matriz: $A \in M_{n\times n}(\mathbb{R})$, quiero definir $e^A \in M_{n\times n}(\mathbb{R})$

Razones

1) $\dot{x} = ax \rightsquigarrow x(t) = Ce^{at}$, $a \in M_{1\times 1}(\mathbb{R})$

2) $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \rightsquigarrow \begin{matrix} x(t) = x_0 e^{at} \\ y(t) = y_0 e^{dt} \end{matrix} \rightsquigarrow X(t) = \begin{pmatrix} e^{at} & 0 \\ 0 & e^{dt} \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$

Me gustaría que esta matriz sea $= e^{\begin{pmatrix} at & 0 \\ 0 & dt \end{pmatrix}}$

$\rightsquigarrow \dot{X} = AX \rightsquigarrow X(t) = e^{At} \cdot X_0$

$x \in \mathbb{R}$ $\quad e^x = \sum_{k=0}^{+\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \rightsquigarrow e^A = \sum_{k=0}^{+\infty} \frac{A^k}{k!}$?

$* \quad \sum_{k=0}^{+\infty} \frac{\|A^k\|}{k!} \leq \sum_{k=0}^{+\infty} \frac{\|A\|^k}{k!} = e^{\|A\|} < \infty. \rightsquigarrow \sum_{k=0}^{+\infty} \frac{A^k}{k!}$ Converge "absolutamente" $\Rightarrow$ Converge.

$\|A^k\| \leq \|A\|^k$

Def: Sea $A \in M_{n\times n}$, definimos $e^A := \sum_{k=0}^{+\infty} \frac{A^k}{k!}$

Prop: Sea $\varphi(t) = e^{At}$ entonces es derivable y $\dot{\varphi}(t) = Ae^{At}$

Corolario: $X(t) = e^{At} \cdot X_0$ es la solución de $\begin{cases} \dot{X} = AX \\ X(0) = X_0 \end{cases}$

Dem: $\dot{X}(t) = \dot{(e^{At} \cdot X_0)} = \dot{(e^{At})} \cdot X_0 \underset{\text{Prop.}}{=} A e^{At} X_0 = A(e^{At} X_0) = A X(t)$ //

Prop 1: Si $A = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \Rightarrow e^{At} = \begin{pmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n t} \end{pmatrix}$.

Prop 2: Si $A = \begin{pmatrix} \lambda & & & 0 \\ 1 & \lambda & & \\ & 1 & \lambda & \\ 0 & & 1 \ddots & 1\,\lambda \end{pmatrix}$

$$\Rightarrow e^{At} = e^{\lambda t} \begin{pmatrix} 1 & & & & & & \\ t & 1 & & & & 0 & \\ t^2/2! & t & 1 & & & & \\ t^3/3! & t^2/2! & t & 1 & & & \\ \vdots & t^3/3! & \vdots & t & 1 & & \\ \frac{t^{n-1}}{(n-1)!} & \frac{t^{n-2}}{(n-2)!} & & \vdots & t & 1 & \\ & & \cdots & & \frac{t^2}{2} & t & 1 \end{pmatrix}$$

Prop 3: Si $A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \rightsquigarrow e^{At} = e^{at} \cdot \begin{pmatrix} \cos(bt) & \sin(bt) \\ -\sin(bt) & \cos(bt) \end{pmatrix}$

Prop 4: Si $A = \begin{pmatrix} \boxed{B} & 0 \\ 0 & \boxed{C} \end{pmatrix} \Rightarrow e^{At} = \begin{pmatrix} \boxed{e^{Bt}} & 0 \\ 0 & \boxed{e^{Ct}} \end{pmatrix}$

Prop 5: Si $A = P \cdot C \cdot P^{-1} \rightsquigarrow e^{At} = P e^{Ct} \cdot P^{-1}$

$$\left(\begin{aligned} e^{At} &= I + At + \frac{(At)^2}{2} + \frac{A^3 t^3}{3!} + \dots \\ &= PIP^{-1} + PCtP^{-1} + \frac{(PCtP^{-1})^2}{2} + \frac{(PCtP^{-1})^3}{3!} + \dots \\ &= PIP^{-1} + PCtP^{-1} + P\frac{(Ct)^2}{2}P^{-1} + P\frac{(Ct)^3}{3!}P^{-1} + \dots \end{aligned} \right)$$

9.5) Hallar e^{At} para $A = \begin{pmatrix} 2 & 1 & -1 \\ 0 & 2 & -1 \\ -3 & -2 & 3 \end{pmatrix}$.

- $\lambda = 1$ doble y $\lambda = 5$ simple.

$S_5 = Ker(A - 5I)$: $\left(\begin{array}{ccc|c} -3 & 1 & -1 & 0 \\ 0 & -3 & -1 & 0 \\ -3 & -2 & -2 & 0 \end{array}\right)$.

$\sim \left(\begin{array}{ccc|c} 3 & -1 & 1 & 0 \\ 0 & -3 & -1 & 0 \\ \cancel{0} & \cancel{-3} & \cancel{-1} & \cancel{0} \end{array}\right) \to$ $3x = y - z = y + 3y = 4y$
$-3y = +z$

$[(\frac{4}{3}, 1, -3)] = S_5$..

$S_1 = Ker(A - I)$, $\left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ -3 & -2 & 2 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ \cancel{0} & \cancel{1} & \cancel{-1} & \cancel{0} \end{array}\right)$

$y = z$ y $x = z - y = 0$ $\to$ $S_1 = [(0,1,1)]$.

$\to$ $A \sim \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix}$

$\begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ -3 & -2 & 2 \end{pmatrix}$

*) $Ker(A - I)^2$, $\begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ -3 & -2 & 2 \end{pmatrix} \begin{pmatrix} 4 & 4 & -4 \\ 3 & 3 & -3 \\ -9 & -9 & 9 \end{pmatrix}$

$\left(\begin{array}{ccc|c} 4 & 4 & -4 & 0 \\ \cancel{3} & \cancel{3} & \cancel{-3} & \cancel{0} \\ \cancel{-9} & \cancel{-9} & \cancel{9} & \cancel{0} \end{array}\right) \to$ $x + y = z$

No es propio

$Ker(A - I)^2 = [(1,0,1), (0,1,1)]$
$\cup\!|$
$S_1 = [(0,1,1)]$.

$$v_1 = (1,0,1), \quad v_2 := (A-I)\,v_1 = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ -3 & -2 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix} =: v_2 \in S_1.$$

$\Rightarrow \quad \mathcal{B} = \{ (1,0,1), (0,-1,-1), (4/3, 1, -3) \}$ base de Jordan.

$$P = (I)_{\mathcal{C}\,\mathcal{B}} = \begin{pmatrix} 1 & 0 & 4/3 \\ 0 & -1 & 1 \\ 1 & -1 & -3 \end{pmatrix}, \quad P^{-1} = (I)_{\mathcal{B}\,\mathcal{C}}$$

$$A = P \cdot \underbrace{\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 1 & 1 & 0 \\ \hline 0 & 0 & 5 \end{array}\right)}_{J} \cdot P^{-1} \Rightarrow e^{At} = P e^{Jt} P^{-1} = P \cdot \begin{pmatrix} e^t & 0 & 0 \\ te^t & e^t & 0 \\ 0 & 0 & e^{5t} \end{pmatrix} P^{-1}$$

Calcular la solución con condición inicial $X_0 = (1,2,3)$

$$\Rightarrow X(t) = P \begin{pmatrix} e^t & 0 & 0 \\ te^t & e^t & 0 \\ 0 & 0 & e^{5t} \end{pmatrix} P^{-1} \cdot X_0 \quad //$$