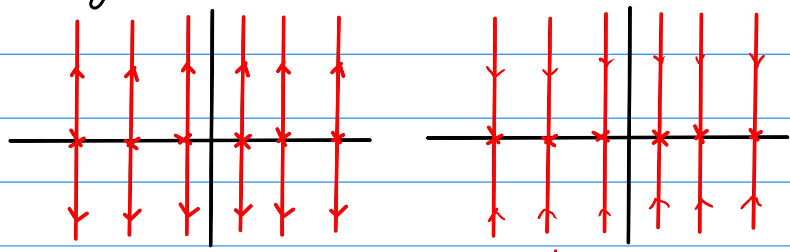
Resumen de diagrama de fase de sistemas 2x2:

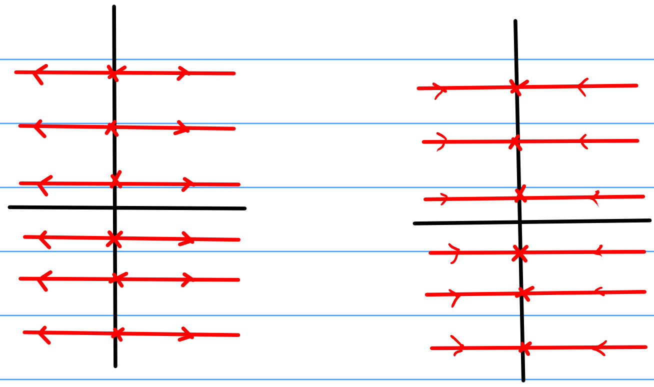
Caso $A = D = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$, diagonalizable

Caso $a=0, d\neq 0$



$d>0$ $d<0$

Caso $a\neq 0, d=0$

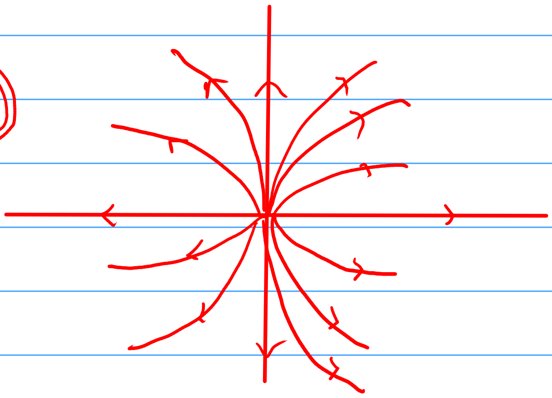


$a>0$ $a<0$

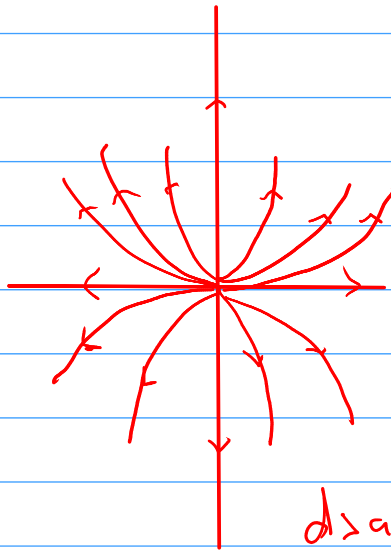
Caso $a\neq 0, d\neq 0$:

$y(t) = y_0 \left(\dfrac{x(t)}{x_0}\right)^{d/a}$

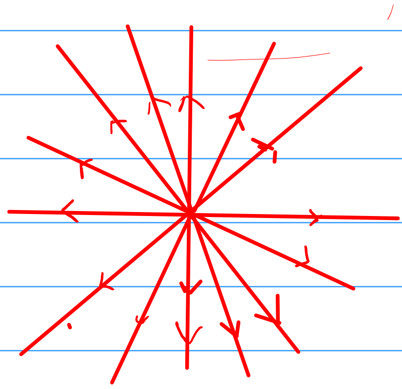
Igual pero convergiendo al origen.



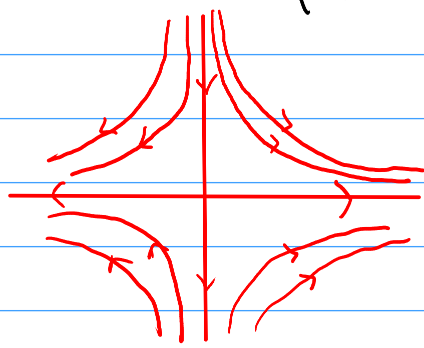
$a>d>0$ $(a<d<0)$



$d>a>0$ $(d<a<0)$



$a=d>0$ $(a=d<0)$



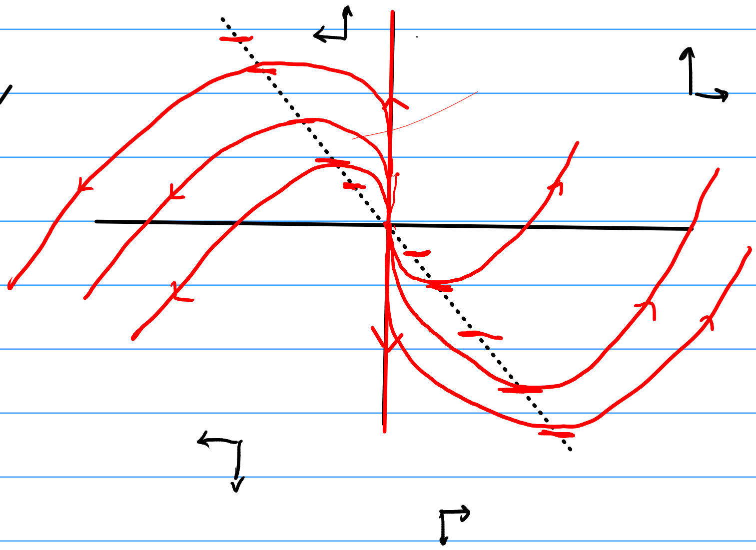
$a>0>d$ $(a<0<d)$

Caso $A = \begin{pmatrix} \lambda & 0 \\ 1 & \lambda \end{pmatrix} \rightsquigarrow$

$$\dot{x} = \lambda x \longrightarrow x(t) = x_0 e^{\lambda t}$$
$$\dot{y} = \underbrace{x + \lambda y}_{} \rightsquigarrow y(t) = e^{\lambda t}(x_0 t + y_0)$$

Caso $\lambda \neq 0$,

$\lambda > 0$



$\dot{x} > 0$ si $x > 0$

$\dot{y} > 0$ si $y > -\frac{x}{\lambda}$

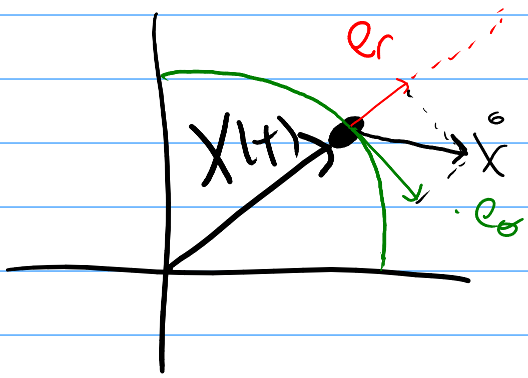
Caso $\lambda = 0$, rellenar.

Caso A con vaps $\lambda = a \pm ib$, es un teorema $A \sim \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$.

Entonces basta estudiar el caso $A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} = aI + b\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$x = r\cos\theta$

$y = r\,\mathrm{sen}\,\theta$



$\dot{x} = AX(t) = aX(t) + b\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} X(t)$

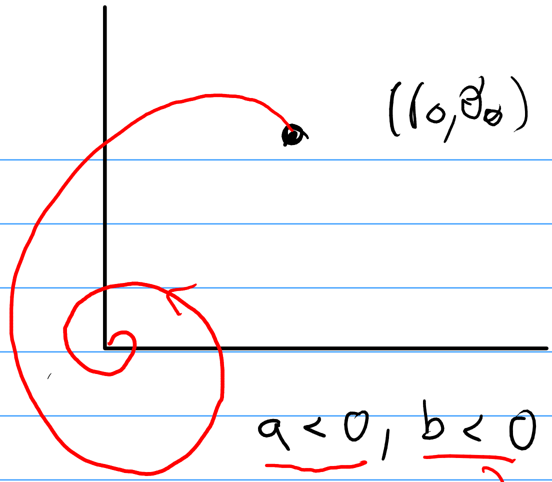
$= aX(t) + b\,\underbrace{\mathrm{rot}(X(t))}_{-90}$

$$r^2 = \underbrace{x^2 + y^2}_{} \rightarrow (r^2)^{\cdot} = 2r\dot{r} = 2x\dot{x} + 2y\dot{y} \rightarrow r\dot{r} = x(ax+by) + y(-bx+ay)$$

$= ax^2 + ay^2 = ar^2 \rightsquigarrow$

Análogamente se ve que $\begin{cases} \dot{r} = ar \longrightarrow r(t) = \underbrace{r_0 e^{at}}_{} \\ \dot{\theta} = -b \quad\quad \theta(t) = \underbrace{-bt}_{} + \theta_0 \end{cases}$

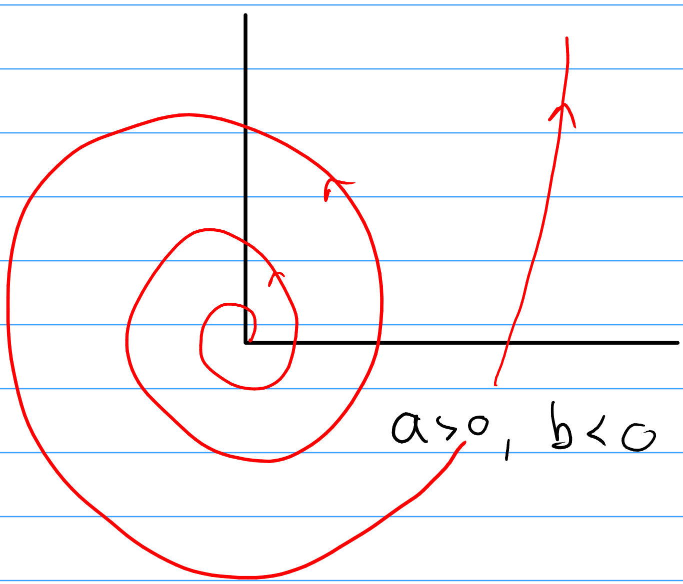
$$\rightsquigarrow x(t) = r(t)\cos\theta(t) = r_0 e^{at}\cos(-bt + \theta_0)$$
$$y(t) = r_0 e^{at}\sin(-bt + \theta_0).$$



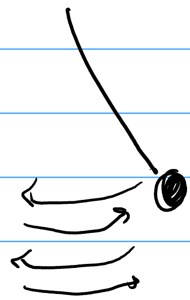
$a + ib$

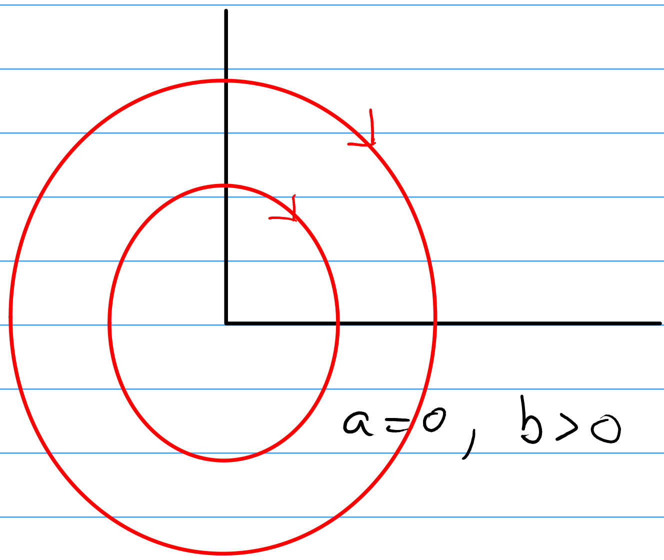
$\underbrace{a<0}_{b}$, $\underbrace{b<0}$ → Antihorario

La espiral converge a cero



$a>0$, $b<0$

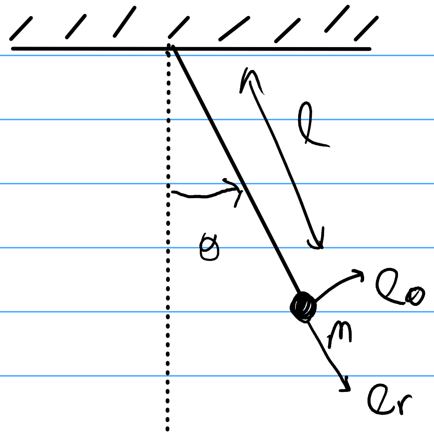




$a=0$, $b>0$

8 Escribir las siguientes ecuaciones diferenciales en la forma $\dot{x} = Ax$.

b) $x'' + x = 0$



$$ml\ddot{\theta} + mg\,\mathrm{sen}\,\theta + b\dot{\theta} = 0$$

$$\ddot{\theta} + \frac{g}{l}\,\mathrm{sen}\,\theta + \frac{b}{ml}\dot{\theta} = 0$$

$$x'' + \frac{b}{ml}x' + \frac{g}{l}\sin x = 0$$

*) $x'' + 2bx' + x = 0$ $\qquad \underbrace{0 < b < 1}$

$$X(t) = \begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix} \rightsquigarrow \dot{X} = \begin{pmatrix} \dot{x}(t) \\ \ddot{x}(t) \end{pmatrix} = \begin{pmatrix} \dot{x} \\ -2b\dot{x} - x \end{pmatrix}$$

$$\Rightarrow \dot{X} = \begin{pmatrix} 0 & 1 \\ -1 & -2b \end{pmatrix}\begin{pmatrix} x \\ \dot{x} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -2b \end{pmatrix} X$$

$$\rightsquigarrow A = \begin{pmatrix} 0 & 1 \\ -1 & -2b \end{pmatrix}, \quad \det(A - \lambda I) = -\lambda(-2b - \lambda) + 1$$

$$= \lambda^2 + \lambda 2b + 1$$

$$\lambda = \frac{-2b \pm \sqrt{\overbrace{4b^2 - 4}^{<0}}}{2} = \frac{-2b \pm i\sqrt{4 - 4b^2}}{2} = -b \pm i\sqrt{1 - b^2}.$$

Teorema: Si $A \in M_{2\times2}(\mathbb{R})$, tiene de valores propios a $\lambda = c \pm id$, y sea v_λ vector propio de $\lambda = c+id$.
Entonces $\mathcal{B} = \{Re(v_\lambda), Im(v_\lambda)\}$ es base de $\mathbb{R}^2$, y además

$$_{\mathcal{B}}(A)_{\mathcal{B}} = \begin{pmatrix} c & d \\ -d & c \end{pmatrix}$$

Hallo v_λ: $\quad A - \left(-b + i\sqrt{1-b^2}\right)I = \left(\begin{array}{cc|c} b - i\sqrt{1-b^2} & 1 & 0 \\ \cancel{-1} & \cancel{-b - i\sqrt{1-b^2}} & \cancel{0} \end{array}\right)$

$$v_\lambda = \left(1, -b + i\sqrt{1-b^2}\right) \rightsquigarrow \begin{array}{l} Re(v_\lambda) = (1, -b) \\ Im(v_\lambda) = \left(0, +\sqrt{1-b^2}\right) \end{array}$$

Revisamos: $\quad A\,Re(v_\lambda) = \begin{pmatrix} 0 & 1 \\ -1 & -2b \end{pmatrix}\begin{pmatrix} 1 \\ -b \end{pmatrix} = \begin{pmatrix} -b \\ -1+2b^2 \end{pmatrix}$

$$\overset{?}{=} -b\,Re(v_\lambda) - \sqrt{1-b^2}\,Im(v_\lambda)$$

$$-b\begin{pmatrix} 1 \\ -b \end{pmatrix} - \sqrt{1-b^2}\begin{pmatrix} 0 \\ \sqrt{1-b^2} \end{pmatrix} = \begin{pmatrix} -b \\ b^2 - (1-b^2) \end{pmatrix} = \begin{pmatrix} -b \\ -1+2b^2 \end{pmatrix} \checkmark$$

$$A\,Re(v_\lambda) = -b\,Re(v_\lambda) - \sqrt{1-b^2}\,Im(v_\lambda)$$

Quede verificado que si $\mathcal{B} = \left\{(1,-b), \left(0, \sqrt{1-b^2}\right)\right\}$

$$\Rightarrow \underline{\underline{_{\mathcal{B}}(A)_{\mathcal{B}}}} = \begin{pmatrix} -b & \sqrt{1-b^2} \\ -\sqrt{1-b^2} & -b \end{pmatrix} \rightsquigarrow \begin{array}{l} r(t) = r_0 e^{-bt} \\ \theta(t) = -\sqrt{1-b^2}\,t + \theta_0 \end{array}$$

$$X(t) = \begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix} = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, \quad Y(t) = P^{-1}X(t), \quad P = \begin{pmatrix} 1 & 0 \\ -b & \sqrt{1-b^2} \end{pmatrix}$$

$$Y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$$

$$\dot{y} = \begin{pmatrix} -b & \sqrt{1-b^2} \\ -\sqrt{1-b^2} & -b \end{pmatrix} y \quad , \quad \begin{array}{l} y_1(t) = r(t)\cos(\theta(t)) \\ y_2(t) = r(t)\,\text{sen}(\theta(t)) \end{array}$$

$$\rightsquigarrow \begin{cases} \dot{r} = -br \\ \dot{\theta} = -\sqrt{1-b^2} \end{cases} \rightsquigarrow \begin{array}{l} r(t) = r_0 e^{-bt} \\ \theta(t) = -\sqrt{1-b^2}\,t + \theta_0 \end{array}$$

$$\Rightarrow Y(t) = (y_1(t), y_2(t)) = \left(r_0 e^{-bt}\cos(-\sqrt{1-b^2}\,t+\theta_0), r_0 e^{-bt}\sin(-\sqrt{1-b^2}\,t+\theta_0) \right)$$

Recapitulando: 1) $\ddot{x} + 2b\dot{x} + x = 0$ $\quad x_1 = x$, $x_2 = \dot{x}$, $X = (x_1, x_2)$

2) $\dot{X} = \begin{pmatrix} 0 & 1 \\ 1 & -2b \end{pmatrix} X$

$\lambda = -b \pm i\sqrt{1-b^2}$, $0 < b < 1$

$Y = P^{-1}X$ $\quad \mathcal{B} = \{Re(v_\lambda), Im(v_\lambda)\}$

$P = (I)_{\varepsilon\,\mathcal{B}}$ $\quad = \{(1,-b), (0, \sqrt{1-b^2})\}$

3) $\dot{y} = \begin{pmatrix} -b & \sqrt{1-b^2} \\ -\sqrt{1-b^2} & -b \end{pmatrix} y$

$y_1 = r\cos\theta$
$y_2 = r\,\text{sen}\theta$

4) $\dot{r} = -br$
$\dot{\theta} = -\sqrt{1-b^2}$ $\rightsquigarrow$ $r(t) = r_0 e^{-bt}$, $\theta(t) = -\sqrt{1-b^2}\,t + \theta_0$

4) $r(t) = r_0 e^{-bt}$, $\theta(t) = -\sqrt{1-b^2}\,t + \theta_0$

3) $Y = \left(r_0 e^{-bt}\cos(-\sqrt{1-b^2}\,t+\theta_0), r_0 e^{-bt}\sin(-\sqrt{1-b^2}\,t+\theta_0) \right)$ $\left(= \text{Coord}_{\mathcal{B}}(X) \right)$

2) $X(t) = r_0 e^{-bt}\cos(-\sqrt{1-b^2}\,t+\theta_0)\,(1,-b) + r_0 e^{-bt}\sin(-\sqrt{1-b^2}\,t+\theta_0)\cdot(0, \sqrt{1-b^2})$

$$\doteq \left(r_0 e^{-bt} \cos(-\sqrt{1-b^2}\,t+\theta_0), -b r_0 e^{-bt} \cos(-\sqrt{1-b^2}\,t+\theta_0) + \sqrt{1-b^2}\, r_0 e^{-bt} \sin(-\sqrt{1-b^2}\,t+\theta_0) \right)$$

1) $X(t) = (x(t), x'(t))$, $\rightsquigarrow$ $x(t) = r_0 e^{-bt} \cos(-\sqrt{1-b^2}\,t+\theta_0)$. //

$\lambda = a \pm ib$, $\lambda = 2$

$\{v_1, Re(v_\lambda), Im(v_\lambda)\}$

$\Rightarrow (A)_{\beta\beta} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & a & b \\ 0 & -b & a \end{pmatrix}$, $y(t) = (y_1, y_2, y_3)$

$y_1 = y_1$

$y_2 = r\cos\theta$

$y_3 = r\sin\theta$

$\rightarrow y_1(t) = Ce^{2t}$, $y_2(t) = r_0 e^{at} \cos(-bt+\theta_0)$, $y_3(t) = r_0 e^{at} \text{sen}(-bt+\theta_0)$