Def Sea $A \in M_{n\times n}(\mathbb{R})$, llamamos e^A a una matriz $n\times n$ de la forma:

$$e^A := \sum_{k=0}^{+\infty} \frac{A^k}{k!}$$

Prop: Si $\varphi : \mathbb{R} \to M_{n\times n}(\mathbb{R})$, $\varphi(t) = e^{At}$ es derivable y $\dot{\varphi}(t) = A\varphi(t) = Ae^{At}$ (*)

Como $\varphi(0) = I$ (*) nos dice que $\varphi(t)$ es la matriz fundamental de $\dot{x} = Ax$.

Corolario: $X(t) = e^{At} \cdot \underbrace{X_0}_{\in \mathbb{R}^n}$ es la solución de $\begin{cases} \dot{x} = Ax \\ x(0) = x_0 \end{cases}$

Prop 1 Si $A = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \Rightarrow e^{At} = \begin{pmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{pmatrix}$.

Prop 2 Si $A = n \updownarrow \underbrace{\begin{pmatrix} \lambda & & & 0 \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ 0 & & 1 & \lambda \end{pmatrix}}_{n\times n}$

$$\Rightarrow e^{At} = e^{\lambda t} \cdot \begin{pmatrix} 1 & 0 & 0 & & & & \\ t & 1 & 0 & & & 0 & \\ \frac{t^2}{2!} & t & 1 & & & & \\ \frac{t^3}{3!} & \frac{t^2}{2!} & t & 1 & & & \\ \vdots & \vdots & \vdots & & \ddots & & \vdots \\ & & & \cdots & t & 1 & 0 \\ \frac{t^{n-1}}{(n-1)!} & \frac{t^{n-2}}{(n-2)!} & \frac{t^{n-3}}{(n-3)!} & \cdots & \frac{t^2}{2!} & t & 1 \end{pmatrix}$$

Prop 3: Si $A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \Rightarrow e^{At} = e^{at}\begin{pmatrix} \cos bt & \sin bt \\ -\sin bt & \cos bt \end{pmatrix}$

Prop 4: Si $A = \begin{pmatrix} \boxed{B} & 0 \\ 0 & \boxed{C} \end{pmatrix} \Rightarrow e^{At} = \begin{pmatrix} \boxed{e^{Bt}} & 0 \\ 0 & \boxed{e^{Ct}} \end{pmatrix}$

Prop 5: Si $A = PCP^{-1} \Rightarrow e^{At} = Pe^{Ct}P^{-1}$ | $(PCP^{-1})^k = PC^kP^{-1}$

9. Hallar e^{At} para las siguientes matrices

2) $A = \begin{pmatrix} 0 & 2 \\ -2 & 4 \end{pmatrix}$.

Vaps: $\det(A-\lambda I) = -\lambda(4-\lambda)+4 = \lambda^2-4\lambda+4 = (\lambda-2)^2$, $\lambda=2$, $ma(2)=2$.

Y esta claro que $mg(2)=1$ (sino $A=2I$).

$\Rightarrow \exists P$ invertible tal que $P^{-1}AP = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$ P es colgar la base de Jordan.

$\mathcal{B} = \{v_1, v_2\}$, necesito cumplir lo siguiente:

$$\begin{cases} \bullet\ v_1 \in \underbrace{Ker(A-2I)^2}_{\mathbb{R}^2} - \underbrace{Ker(A-2I)} \\ \bullet\ v_2 \in S_2 \text{ y } v_2 = (A-2I)v_1 \end{cases}$$

$Ker(A-2I)$: $\left(\begin{array}{cc|c} -2 & 2 & 0 \\ \cancel{-2} & \cancel{2} & \cancel{0} \end{array}\right) \Rightarrow v = (x,x) \Rightarrow S_2 = [(1,1)]$.

Elijo $v_1 = (-1,1)$ y $v_2 = \overbrace{\begin{pmatrix} -2 & 2 \\ -2 & 2 \end{pmatrix}}^{A-2I} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$

$\Rightarrow \mathcal{B} = \{(-1,1),(4,4)\}$, $P = \begin{pmatrix} -1 & 4 \\ 1 & 4 \end{pmatrix}$, $\det P = -8$

Sabemos que $A = P \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix} P^{-1} \Rightarrow e^{At} = P e^{\begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix} t} P^{-1}$

$\Rightarrow e^{At} = P \cdot e^{2t} \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} P^{-1}$ $P^{-1} = \frac{1}{-8} \begin{pmatrix} 4 & -4 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} -1/2 & 1/2 \\ 1/8 & 1/8 \end{pmatrix}$

$$P\begin{pmatrix}1&0\\t&1\end{pmatrix} = \begin{pmatrix}-1&4\\1&4\end{pmatrix}\begin{pmatrix}1&0\\t&1\end{pmatrix} = \begin{pmatrix}4t-1&4\\4t+1&4\end{pmatrix}.$$

$$\begin{pmatrix}4t-1&4\\4t+1&4\end{pmatrix}\begin{pmatrix}-1/2&1/2\\1/8&1/8\end{pmatrix} = \left(\begin{array}{c|c}-2t+1/2+1/2 & 2t-1/2+1/2\\ \hline -2t-1/2+1/2 & 2t+1/2+1/2\end{array}\right) = \begin{pmatrix}-2t+1&2t\\-2t&2t+1\end{pmatrix}$$

$$\Rightarrow e^{At} = e^{2t}P\begin{pmatrix}1&0\\t&1\end{pmatrix}P^{-1} = \begin{pmatrix}(-2t+1)e^{2t}&2te^{2t}\\-2te^{2t}&(2t+1)e^{2t}\end{pmatrix}$$

Agregado

a) Calcular la solución de $\begin{cases}\dot{X} = AX\\ X(0) = (1,0)\end{cases}$.

$$X(t) = e^{At}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}(-2t+1)e^{2t}\\-2te^{2t}\end{pmatrix}$$

b) Calcular la solución de $\begin{cases}\dot{X} = AX\\ X(0) = (1,1)\end{cases}$

$$X(t) = e^{At}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}e^{2t}\\e^{2t}\end{pmatrix} = e^{2t}\begin{pmatrix}1\\1\end{pmatrix}$$

, lógico $(1,1)$ es vector propio.

---

c) $x''' - 2x'' - x' + 2x = 0$

$x_1(t) = x(t)$

$x_2(t) = \dot{x}(t)$

$x_3(t) = \ddot{x}(t)$

$\dot{X} = AX$, $X(t) = (\underbrace{x_1(t)}, x_2(t), x_3(t))$

$$\dot{X} = \begin{pmatrix}\dot{x}_1\\\dot{x}_2\\\dot{x}_3\end{pmatrix} = \begin{pmatrix}\dot{x}\\\ddot{x}\\\dddot{x}\end{pmatrix} = \begin{pmatrix}\dot{x}\\\ddot{x}\\-2x+\dot{x}+2\ddot{x}\end{pmatrix} = \begin{pmatrix}0&1&0\\0&0&1\\-2&1&2\end{pmatrix}\overbrace{\begin{pmatrix}x\\\dot{x}\\\ddot{x}\end{pmatrix}}^{X}$$

$$\begin{cases}\dot x = x\\ \dot y = y\\ \dot z = x\end{cases} = \begin{cases}\dot y = y\\ \dot x = x\\ \dot z = x\end{cases}$$

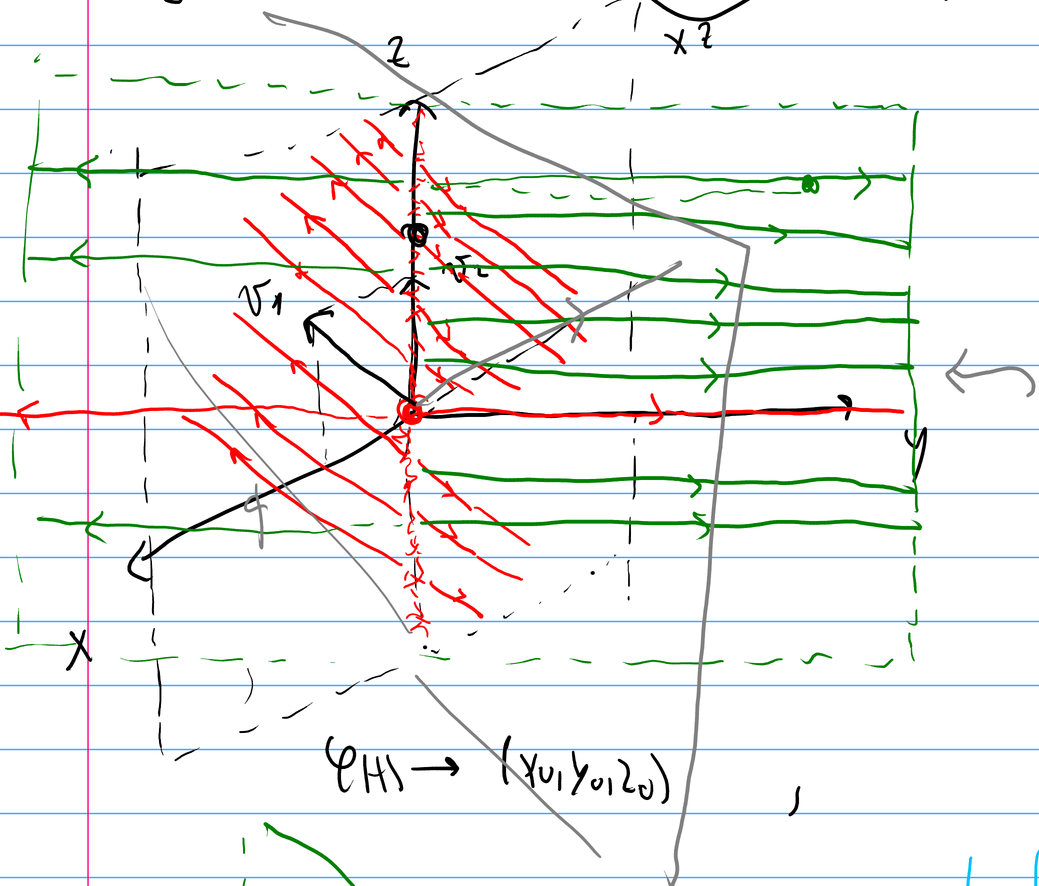
$y_0 = 0 \Rightarrow y(t) = 0 \ \forall t$

$$\begin{pmatrix}1\\1\end{pmatrix}$$

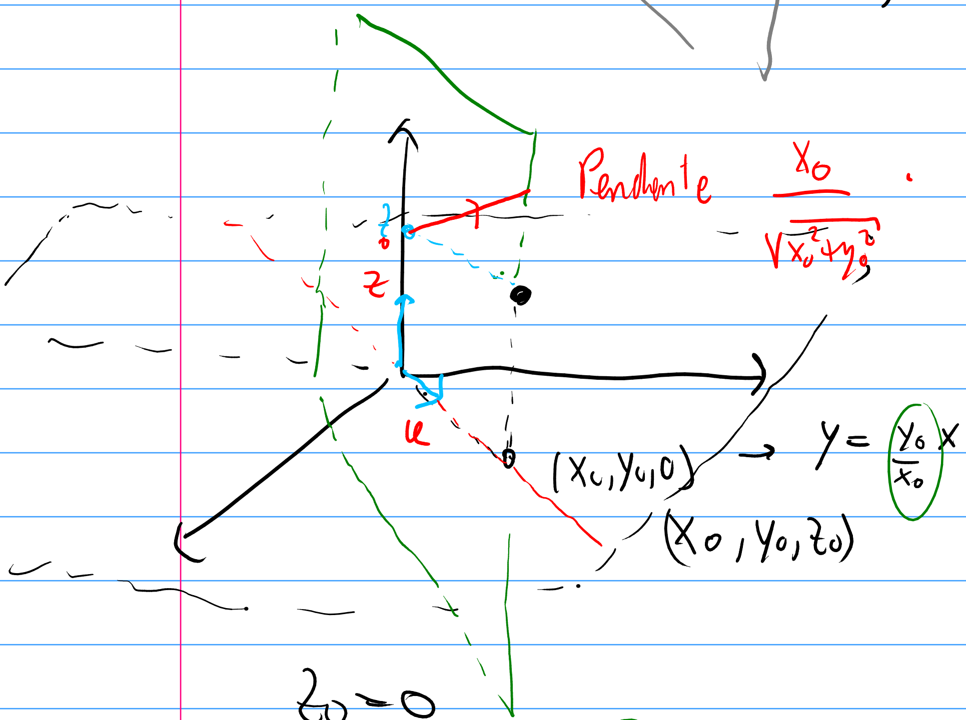
$$\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}$$

$v_1 = (1,0,1)$

$v_2 = (0,0,1)$



$\varphi(t) \to (x_0, y_0, z_0)$



$u(t)$

$$\left(\frac{x_0, y_0}{\sqrt{x_0^2+z_0^2}}, 0\right) e^{t\sqrt{x_0^2+y_0^2}}$$

$y(t) = y_0 e^t$

$z(t) = y_0 e^t + z_0$

$x(t) = x_0 e^t$

$z = \frac{x_0\, u}{\sqrt{x_0^2+y_0^2}} + z_0$

$\dot z(t) = x(t)$

$z(t) = x_0 e^t + z_0$

$$(x(t), y(t), z(t)) = (x_0 e^t, y_0 e^t, x_0 e^t + z_0) = (x_0, y_0, 0) e^t + (x_0 e^t + z_0)(0,0,1)$$

$z(t)$