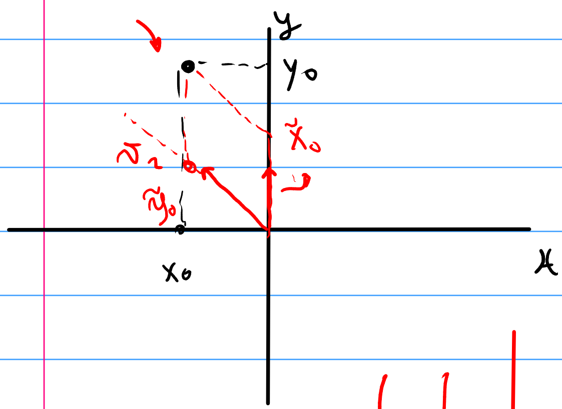
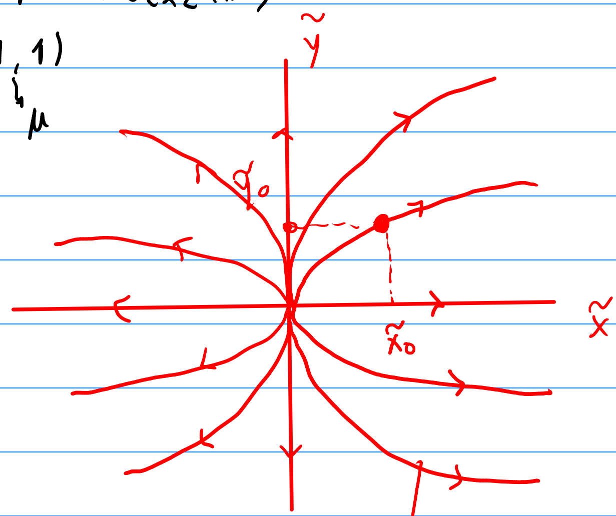
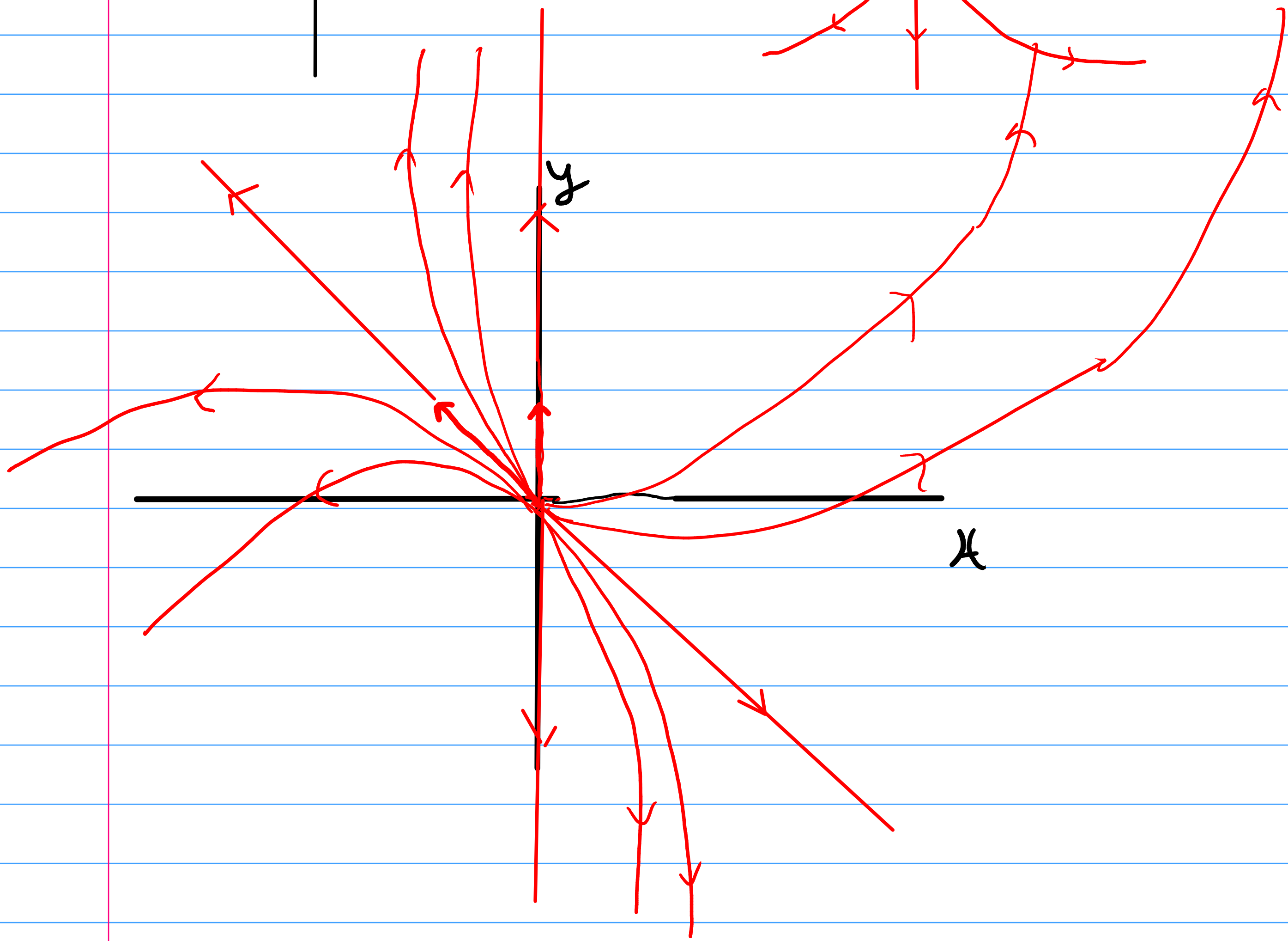
5b) $0<\mu<\lambda$ vaps de $A \in M_{2\times2}(\mathbb{R})$

$\lambda \rightsquigarrow v_1=(0,1)$, $\underbrace{v_2=(-1,1)}_{\mu}$



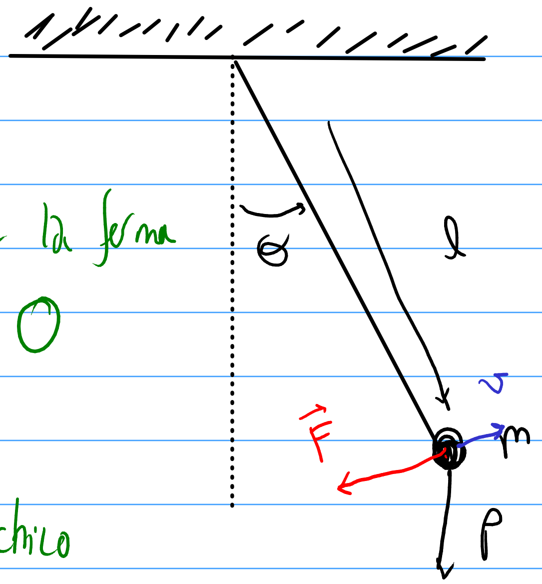




8. Escribir las siguientes ecuaciones diferenciales diferenciales en la forma $\dot{X} = AX$, y resolver.

b) $\ddot{\theta} + 2b\dot{\theta} + \theta = 0$

$\theta : (a,b) \subset \mathbb{R} \to \mathbb{R}$



y con una fuerza de fricción viscosa: $\vec{F} = -\tilde{b}\vec{v}$

Queda ecuación de la forma

$\ddot{\theta} + 2b\dot{\theta} + C\underbrace{\operatorname{sen}\theta}_{\simeq \theta} = 0$

Si $|\theta|$ chico

(*) $\ddot{\theta} + 2b\dot{\theta} + \theta = 0$, $0 < b < 1$

Defino $x = \theta$, $y = \dot{\theta}$, y considero $X(t) = (x, y) = (\theta, \dot{\theta})$

$$\dot{X} = \begin{pmatrix} \dot{\theta} \\ \ddot{\theta} \end{pmatrix} = \begin{pmatrix} 1\dot{\theta} \\ -2b\dot{\theta} - \theta \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & -2b \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} \theta \\ \dot{\theta} \end{pmatrix}}_{X}$$

- $\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -1 & -2b-\lambda \end{vmatrix} = -\lambda(-2b-\lambda) + 1 = \lambda^2 + 2b\lambda + 1$

(*) El polinomio característico de la ec diferencial es = al polinomio característico de A.

$$\lambda = \frac{-2b \pm \sqrt{4b^2 - 4}}{2} = \frac{-2b \pm 2\sqrt{b^2 - 1}}{2} = -b \pm i\sqrt{1 - b^2}$$

< 0, pues $0 < b < 1$

Teo: Sea $A \in M_{2\times 2}(\mathbb{R})$, con $\lambda = c+id$, y $\bar{\lambda} = c-id$ valores propios, y sea $v_\lambda \in \mathbb{C}^2$ vector propio de $\lambda = c+id$. Entonces

1) $\mathcal{B} = \{ Re(v_\lambda), Im(v_\lambda) \} \xrightarrow{b} \mathbb{R}^2$

2) $P^{-1} \cdot A \cdot P = \begin{pmatrix} c & d \\ -d & c \end{pmatrix}$, P = Colgar los vectores.

Tomo $\lambda = \underbrace{-b}_{c} + i \underbrace{\sqrt{1-b^2}}_{d}$

$S_\lambda = Ker(A - \lambda I)$: $\left(\begin{array}{cc|c} b - i\sqrt{1-b^2} & 1 & 0 \\ \cancel{-1} & \cancel{-b-i\sqrt{1-b^2}} & \cancel{0} \end{array}\right)$

$\Rightarrow v_\lambda = (1, -b + i\sqrt{1-b^2})$

$\Rightarrow \mathcal{B} = \{ (1,-b), (0, \sqrt{1-b^2}) \}$

Chequeemos: $A\,Re(v_\lambda) = A \begin{pmatrix} 1 \\ -b \end{pmatrix} = \left(\begin{array}{cc} 0 & 1 \\ -1 & -2b \end{array}\right) \begin{pmatrix} 1 \\ -b \end{pmatrix} = \begin{pmatrix} -b \\ -1+2b^2 \end{pmatrix}$

[Recordar, debería dar $= -b\,Re(v_\lambda) - \sqrt{1-b^2}\,Im(v_\lambda)$]

$-b\,Re(v_\lambda) - \sqrt{1-b^2}\,Im(v_\lambda) = -b \begin{pmatrix} 1 \\ -b \end{pmatrix} - \sqrt{1-b^2} \begin{pmatrix} 0 \\ \sqrt{1-b^2} \end{pmatrix} = \begin{pmatrix} -b \\ b^2 \end{pmatrix} - \begin{pmatrix} 0 \\ 1-b^2 \end{pmatrix} = \begin{pmatrix} -b \\ 2b^2-1 \end{pmatrix}$

En resumen, $(A)_{\mathcal{B}\mathcal{B}} = \begin{pmatrix} -b & \sqrt{1-b^2} \\ -\sqrt{1-b^2} & -b \end{pmatrix}$, $P = \begin{pmatrix} 1 & 0 \\ -b & \sqrt{1-b^2} \end{pmatrix}$

$Y = (\tilde{x}, \tilde{y})$, $Y = P^{-1}X \Rightarrow \dot{Y} = \begin{pmatrix} -b & \sqrt{1-b^2} \\ -\sqrt{1-b^2} & -b \end{pmatrix} Y$

Es el caso canónico de valores propios complejos, cuya solución hallamos la clase pasada usando polares

$$Y(t) = \left(r_0 e^{-bt} \cos(-\sqrt{1-b^2}\,t + \theta_0),\ r_0 e^{-bt} \sin(-\sqrt{1-b^2}\,t + \theta_0) \right)$$

$$X(t) = PY(t) = r_0 e^{-bt} \cos(-\sqrt{(1-b^2)}\,t + \theta_0)\,(1, -b) + r_0 e^{-bt} \sin(-\sqrt{1-b^2}\,t + \theta_0)\,(0, \sqrt{1-b^2})$$

$$= \Big(\underbrace{r_0 e^{-bt} \cos(-\sqrt{1-b^2}\,t + \theta_0)}_{x},\ \underbrace{\text{Lo otro}}_{y} \Big).$$

Entonces como $\theta = x$ se tiene que la solución es

$$\theta(t) = r_0 e^{-bt} \cos(\sqrt{1-b^2}\,t - \theta_0).$$

Sin rozamiento, $b = 0$, $\Rightarrow \theta(t) = r_0 \cos(t - \theta_0)$

Con rozamiento, $0 < b < 1 \Rightarrow \theta(t) = r_0 e^{-bt} \cos(\underbrace{\sqrt{1-b^2}}\,t - \theta_0)$

Exponencial de una matriz: $A \in \mathcal{M}_{n\times n}(\mathbb{R})$ quiero definir $e^A \in \mathcal{M}_{n\times n}(\mathbb{R})$.

Razones 1) $\dot{x} = a x \rightsquigarrow x(t) = x_0 e^{at}$

2) $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \underbrace{\begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \end{pmatrix} \rightsquigarrow X(t) = (x_0 e^{at}, y_0 e^{ct}) = \begin{pmatrix} e^{at} & 0 \\ 0 & e^{ct} \end{pmatrix} X_0$

$X_0 = (x_0, y_0)$

Estaría bueno que

$$\begin{pmatrix} e^{at} & 0 \\ 0 & e^{ct} \end{pmatrix} = e^{\begin{pmatrix} at & 0 \\ 0 & ct \end{pmatrix}}$$

$$\rightsquigarrow X(t) = e^{At} \cdot X_0$$

Def: Sea $A \in \mathcal{M}_{n\times n}(\mathbb{R}) \Rightarrow e^A := \sum_{k=0}^{+\infty} \frac{A^k}{k!}$.

Recordar que $e^x = \sum_{k=0}^{+\infty} \frac{x^k}{k!}$, $x \in \mathbb{R}$.

$$\left[\sum_{k=0}^{+\infty} \frac{\|A^k\|}{k!} \leq C \sum_{k=0}^{+\infty} \frac{(\|A\|)^k}{k!} < \infty \right].$$ Converge absolutamente.

Prop: $\varphi(t) = e^{At}$ es derivable y $\dot{\varphi}(t) = A e^{At} = A\varphi(t)$.

Corolario: $X(t) = e^{At} \cdot X_0$ es la solución de $\begin{cases} \dot{X} = AX \\ X(0) = X_0 \end{cases}$ //