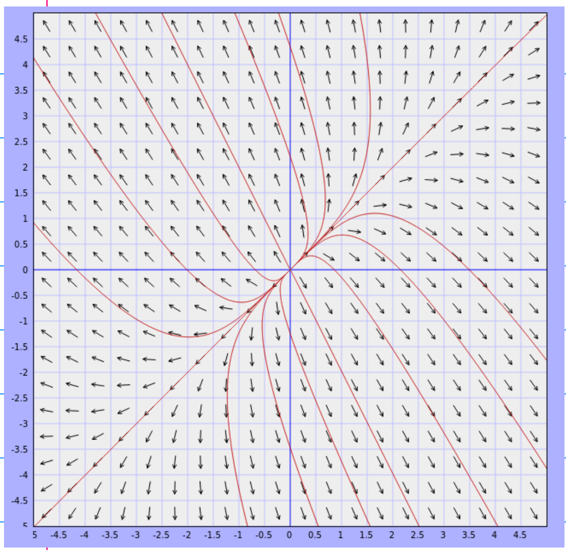
https://aeb019.hosted.uark.edu/pplane.html



6c) Hallar la solución general y dibujar el diagrama de fase

$$\dot{X} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} X$$

λ es valor propio de A si $\det(A-\lambda I)=0$

Vaps: $$\left| \begin{pmatrix} -1-\lambda & 1 \\ 1 & -1-\lambda \end{pmatrix} \right| = 0 \quad = (-1-\lambda)(-1-\lambda) - 1$$

$$= \lambda^2 + 2\lambda + 1 - 1 = \lambda^2 + 2\lambda = \lambda(\lambda+2) \rightarrow \underline{0},\ \underline{-2}$$

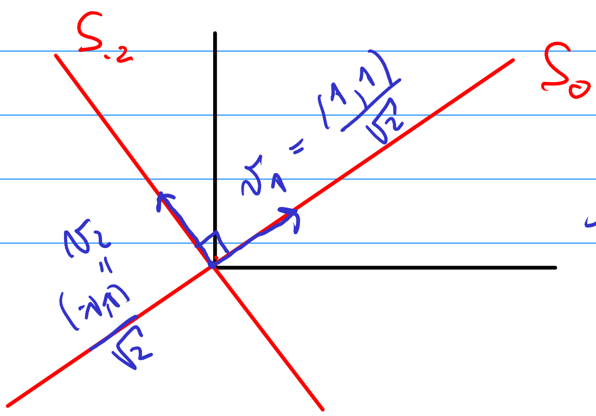
$S_0 = \mathrm{Ker}(A)$: $v = (x,y)$ $\quad$ $\underline{Av = 0}$

$$\left(\begin{array}{cc|c} -1 & 1 & 0 \\ \cancel{1} & \cancel{-1} & \cancel{0} \end{array} \right) \Rightarrow x = y \Rightarrow S_0 = \{ (x,x) : x \in \mathbb{R} \}$$

$S_{\underline{-2}} = \mathrm{Ker}(A+2I)$: $$\left(\begin{array}{cc|c} 1 & 1 & 0 \\ \cancel{1} & \cancel{1} & \cancel{0} \end{array} \right) \Rightarrow x = -y \quad S_{-2} = \{ (x,-x) : x \in \mathbb{R} \}$$

$v = (x,y) \Leftrightarrow Av = -2v$

$\Leftrightarrow \underbrace{(A+2I)}\, v = 0$



$B = \{ v_1, v_2 \}$

$(A)_{\mathcal{E}\,\mathcal{E}} = A$, Como $\mathcal{B}$ es base de vectores propios entonces

$$(A)_{\mathcal{B}\,\mathcal{B}} = \underline{P^{-1} A P} = D = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}$$

$$Y = P^{-1} X \Rightarrow \overset{\circ}{Y} = (\overset{\circ}{P^{-1} X}) = P^{-1} \overset{\circ}{X} = P^{-1} A X = P^{-1} A P Y$$
$$= D Y$$

$Y = (\tilde{x}, \tilde{y})$.

$$D = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix} \quad , \quad P = (I)_{\mathcal{E}\,\mathcal{B}} = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$ Matriz de rotación

$$Y = \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = P^{-1} \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \overset{\circ}{Y} = D Y$$ es decir

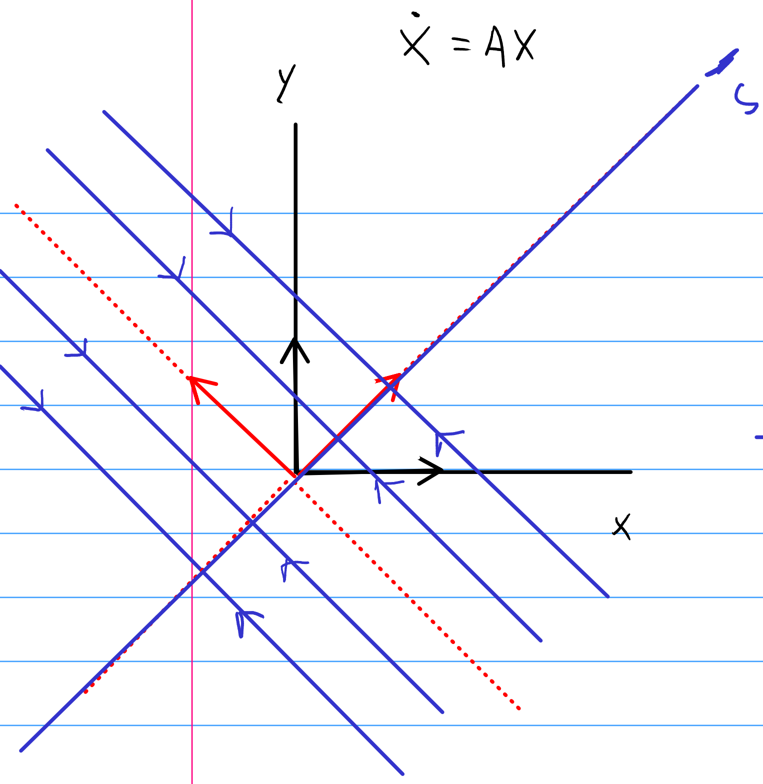
$$\Rightarrow \begin{cases} \overset{\circ}{\tilde{x}} = 0 \\ \overset{\circ}{\tilde{y}} = -2\tilde{y} \end{cases} \Rightarrow (\tilde{x}(t), \tilde{y}(t)) = (\tilde{x}_0 , \tilde{y}_0 e^{-2t})$$

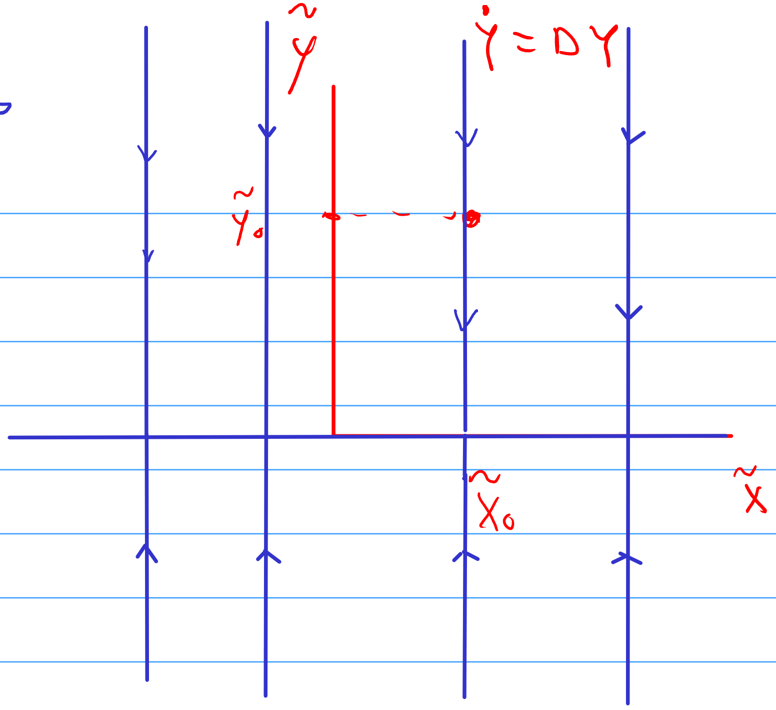
* ¿Como se relaciona la cond inicial de X, $X(0) = (x_0, y_0)$, con $(\tilde{x}_0, \tilde{y}_0) = Y(0)$?

$$Y(0) = P^{-1} X(0) \longrightarrow \begin{pmatrix} \tilde{x}_0 \\ \tilde{y}_0 \end{pmatrix} = P^{-1} \cdot \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

$$X(t) = P Y(t) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \tilde{x}_0 \\ \tilde{y}_0 e^{-2t} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \tilde{x}_0 - \tilde{y}_0 e^{-2t} \\ \tilde{x}_0 + \tilde{y}_0 e^{-2t} \end{pmatrix}$$

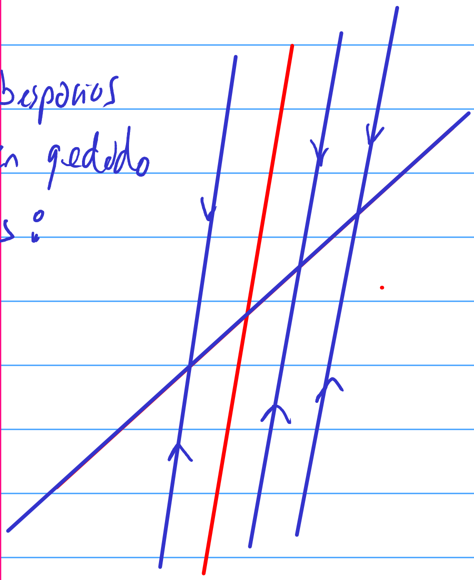
$$X(t) = \frac{1}{\sqrt{2}} (\tilde{x}_0 - \tilde{y}_0 e^{-2t}, \tilde{x}_0 + \tilde{y}_0 e^{-2t})$$





$$(\tilde x(t), \tilde y(t)) = (\tilde x_0 , \underbrace{\tilde y_0} e^{-2t})$$

Si los subespacios no hubiesen quedado ortogonales:



6(j) $\dot X = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} X$

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -1 \\ +1 & -1-\lambda \end{vmatrix} = (1-\lambda)(-1-\lambda) + 1 = \lambda^2 - 1 + 1 = \lambda^2$$

$\Rightarrow \lambda = 0$ valor propio con $ma(0) = 2$.

$S_0 = Ker(A)$: $\left(\begin{array}{cc|c} 1 & -1 & 0 \\ \cancel{1} & \cancel{-1} & \cancel{0} \end{array}\right) \Rightarrow S_0 = \{ (x,x) : x \in \mathbb{R} \}$

Obs: $mg(0) = \dim S_0 < ma(0) \Rightarrow$ No diagonalizable.

Como todas las raíces de $p(\lambda)=\det(A-\lambda I)$ están en $\mathbb{R}$

$\Rightarrow$ Jordan me dice que $\exists\ \beta=\{v_1,v_2\}\xrightarrow{b}\mathbb{R}^2$ tal que

$${}_{\beta}(A)_{\beta} = \begin{pmatrix}0&0\\1&0\end{pmatrix} \qquad \left[\begin{pmatrix}\lambda&0\\1&\lambda\end{pmatrix} \text{ con } \lambda=0\right]$$

En general:

$${}_{\beta}(A)_{\beta} = \begin{pmatrix}\boxed{J_1(\mu_1)} & & & 0\\ & \boxed{J_2(\mu_2)} & & \\ & & \boxed{J_3(\mu_3)} & \\ 0 & & & \ddots\end{pmatrix}$$

$$J_i(\mu_i) = \begin{bmatrix}\mu_i & & & 0\\ 1 & \mu_i & & \\ & 1 & \ddots & \\ 0 & & 1 & \mu_i\end{bmatrix}$$

donde μ_i es un valor propio (Cuidado, puede pasar que $\mu_i = \mu_j$)

Siguiendo con nuestro ejemplo: ${}_{\beta}(A)_{\beta} = \begin{pmatrix}0&0\\1&0\end{pmatrix}$

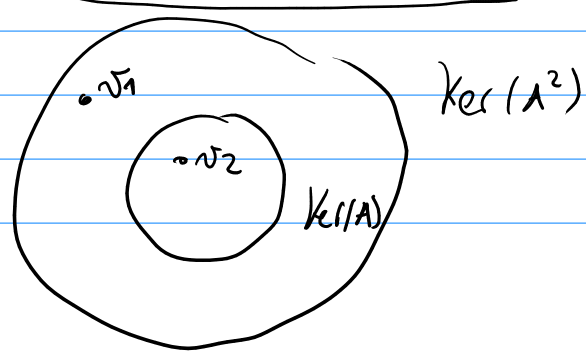
Si $\beta=\{v_1,v_2\} \Rightarrow \begin{cases} Av_2 = 0 \longrightarrow v_2 \text{ es vector propio: } v_2\in \mathrm{Ker}(A) \\ Av_1 = 0v_1 + 1v_2 \longrightarrow A^2v_1 = A(Av_1) = Av_2 = 0\end{cases}$

¿Cómo hallar v_1?

1) Primero elijo v_2 y después resuelvo el sistema $Av_1 = v_2$ donde v_1 es incógnita.

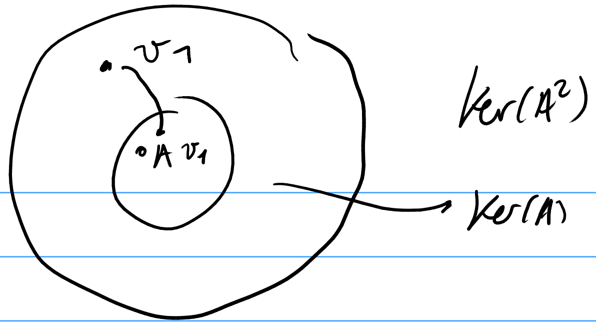
2) v_1 no es vector propio: $v_1 \notin \mathrm{Ker}(A)$ pero $v_1 \in \mathrm{Ker}(A^2)$

So $= \underbrace{\mathrm{Ker}(A) \subset \mathrm{Ker}(A^2)}$:



Además si $v_1 \in Ker(A^2) - Ker(A)$

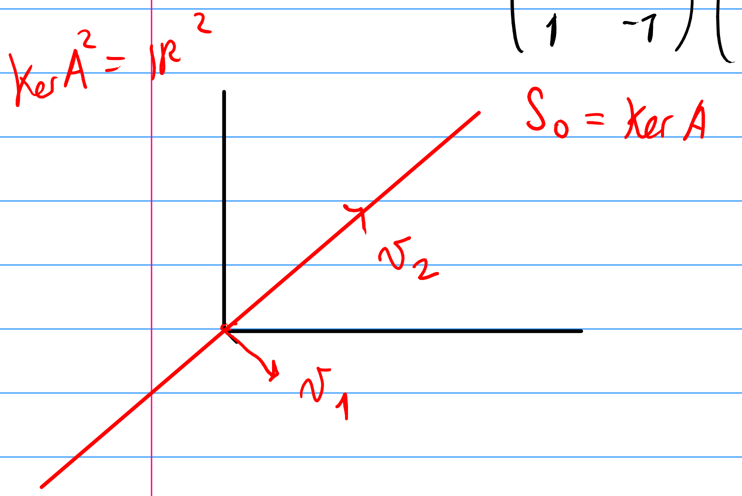
$\Rightarrow \underbrace{Av_1}_{\neq 0} \in Ker(A)$



$\Rightarrow$ Elijo $v_1 \in Ker(A^2) - Ker(A)$ y luego $v_2 := Av_1$.

En este ejemplo: $S_0 = Ker(A) = \{(x,x) : x \in \mathbb{R}\}$

$Ker(A^2)$: $A^2 = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow Ker(A^2) = \mathbb{R}^2$



$\Rightarrow$ De v_1 me sirve cualquier vector que no este en S_0.

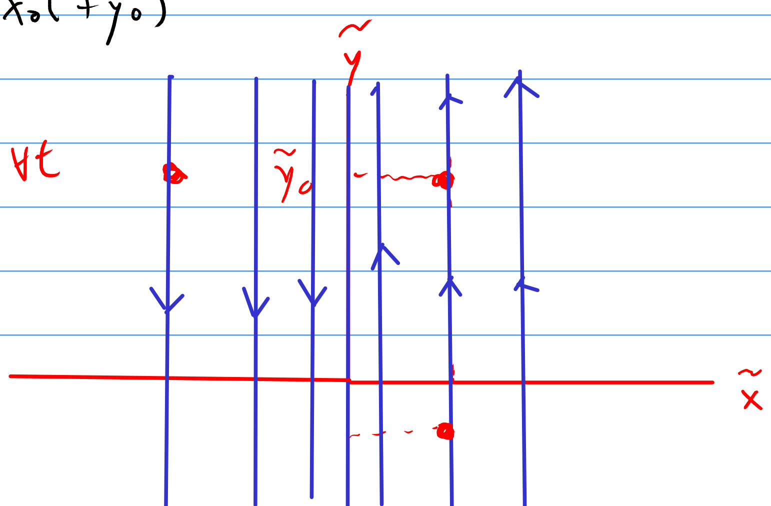
$v_1 = (1,-1)$, $v_2 = Av_1 = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$

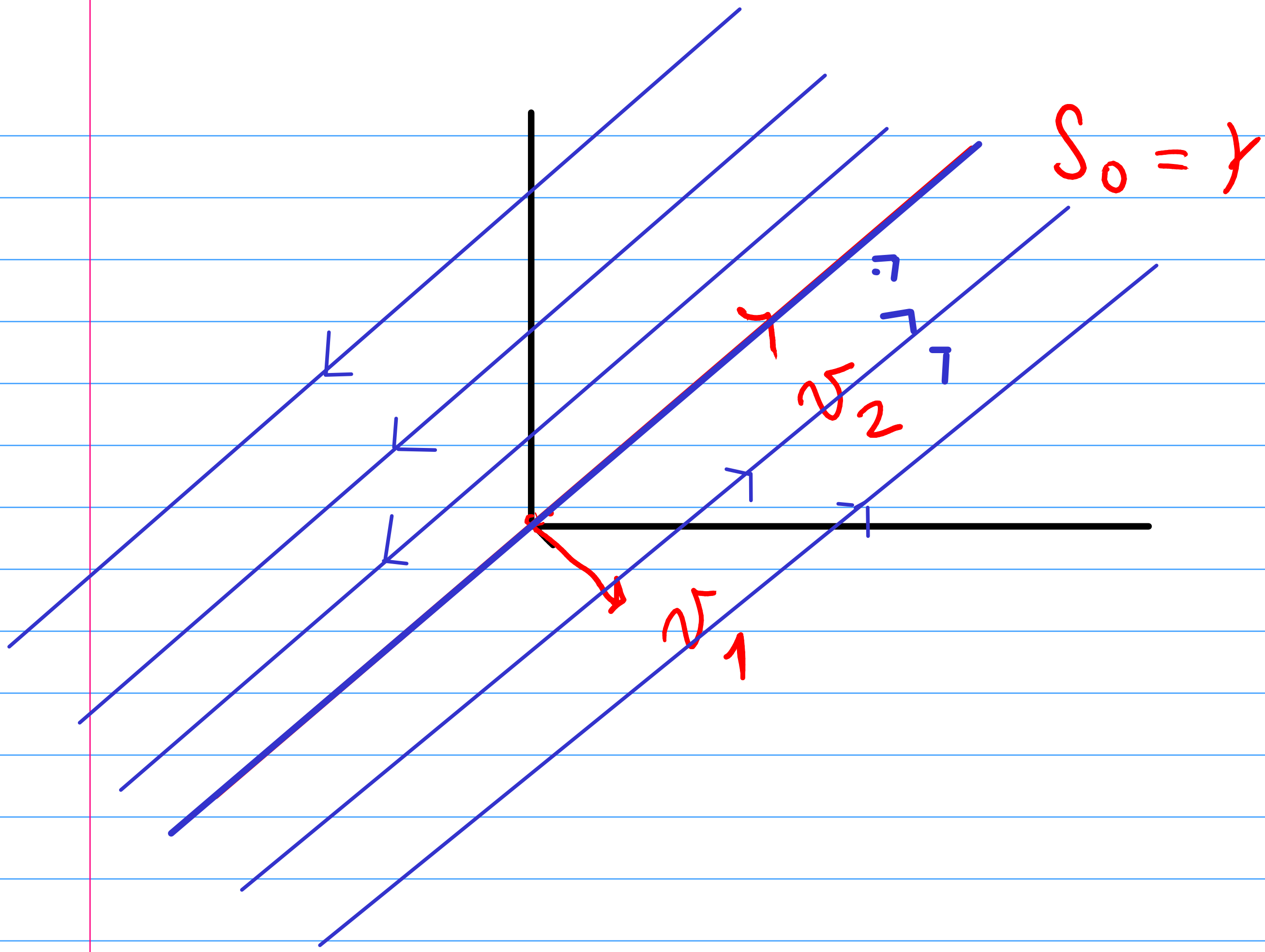
$\Rightarrow \mathcal{B} = \{(1,-1), (2,2)\}$, $P = \begin{pmatrix} 1 & 2 \\ -1 & 2 \end{pmatrix}$

C.V: $Y = P^{-1}X \Rightarrow \dot{Y} = JY$ con $J = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$\begin{cases} \dot{\tilde{x}} = 0 \\ \dot{\tilde{y}} = \tilde{x} \end{cases} \Rightarrow \begin{array}{l} \tilde{x} = \tilde{x}_0 \\ \dot{\tilde{y}} = \tilde{x}_0 \Rightarrow \tilde{y} = \tilde{x}_0 t + \tilde{y}_0 \end{array}$

$\Rightarrow (\tilde{x}(t), \tilde{y}(t)) = (\underbrace{\tilde{x}_0}_{\text{cte } \forall t}, \tilde{x}_0 t + \tilde{y}_0)$





$S_0 = \gamma$
γ
v_2
v_1