Resolver

1) a) $\begin{cases} x' = 2x - y \\ y' = -2x + 3y \end{cases}$, $(x_0, y_0) = (1,1)$

Contexto: Vamos a estudiar ecuaciones de la forma $\dot{x} = f(\overset{\downarrow\, n \text{ Componentes}}{x}, t)$ donde $x : (a,b) \to \mathbb{R}^n$, $x(t) = (x_1(t), \dots, x_n(t))$, $f : \underset{\mathbb{R}^n}{\underset{\cap}{\Omega}} \times \mathbb{R}^n \to \mathbb{R}^n$. En este práctico vamos a estudiar el caso $f(x,t) = Ax$ (Homogenea y lineal).

* Todas las ecuaciones lineales que vimos, por ejemplo $\dddot{x} + \ddot{x} + \dot{x} = 0$ con $x : (a,b) \to \mathbb{R}^1$, se pueden escribir en la forma $\dot{X} = A X$, donde $\underbrace{X}_{} : (a,b) \to \mathbb{R}^n$. esta relacionada con x.

Por ejemplo $\begin{cases} \ddot{x} + x = 0 \\ x(0) = x_0 \\ \dot{x}(0) = v_0 \end{cases}$, $x : (a,b) \to \mathbb{R}^1$

Defino $X : (a,b) \to \mathbb{R}^2$
$X(t) = (\underbrace{x}_{\text{Mi incognita}}(t), \dot{x}(t))$

$$\Rightarrow \quad \dot{X}(t) = (\dot{x}(t), \ddot{x}(t)) = (\dot{x}(t), -x(t)) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} X(t)$$

Es decir, $\dot{X}(t) = A X(t)$.

Resumen: estudiando ecuaciones de la forma $\dot{X} = AX$ con $X : (a,b) \to \mathbb{R}^n$ estudio las ecuaciones lineales de orden n que vimos en los prácticos anteriores, donde la incógnita era $x : (a,b) \to \mathbb{R}$.

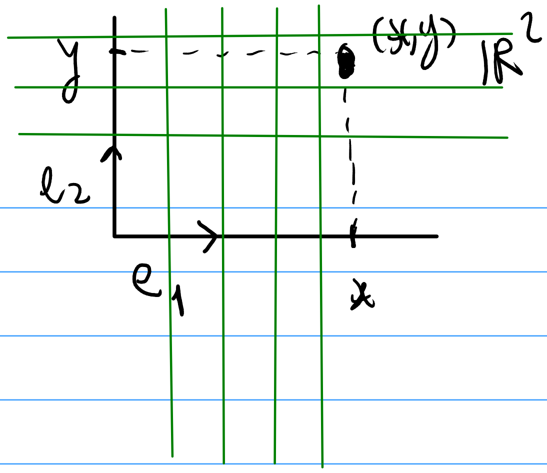
Resolver

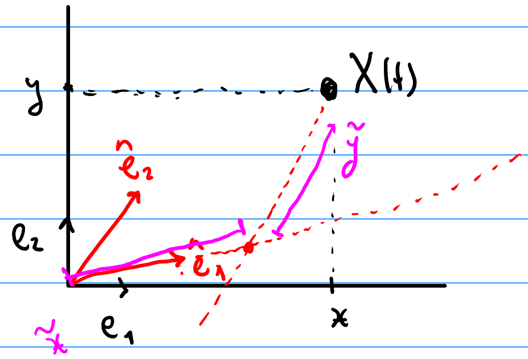
1) a) $\begin{cases} x' = 2x - y \\ y' = -2x + 3y \end{cases}$, $(x_0, y_0) = (1,1)$

$X(t) = (x(t), y(t))$

$$\begin{cases} \dot{X} = \begin{pmatrix} 2 & -1 \\ -2 & 3 \end{pmatrix} X \\ X(0) = (1,1) \end{cases}$$

$$X(t) = (x(t), y(t))$$
$$\shortparallel$$
$$\underbrace{x(t)}\, e_1 + \underbrace{y(t)}\, e_2$$
$$\shortparallel$$
$$\tilde{x}(t)\,\hat{e}_1 + \tilde{y}(t)\,\hat{e}_2$$





$$X(t) = \underbrace{\tilde{x}}\,\hat{e}_1 + \underbrace{\tilde{y}}\,\hat{e}_2 = x\, e_1 + y\, e_2$$

$$\text{Coord}_{\mathcal{B}}(X(t)) = (x, y)$$

$$\mathcal{B} = \{e_1, e_2\}$$

$$\text{Coord}_{\beta}(X(t)) = (\tilde{x}, \tilde{y})$$

$$\beta = \{\hat{e}_1, \hat{e}_2\}$$

La idea es hacer un cambio de variable de las "x, y" a las "$\tilde{x}, \tilde{y}$" donde la ec diferencial se vea más fácil.

¿Cómo se relaciona $\text{Coord}_{\mathcal{B}}(X)$, $\text{Coord}_{\beta}(X)$, $X \in \mathbb{R}^n$, $\mathcal{B}, \beta$ bases?

$$\text{Coord}_{\beta}(X) = \underbrace{{}_{\beta}(I)_{\mathcal{B}}} \cdot \text{Coord}_{\mathcal{B}}(X)$$

Matriz de cambio de base = la matriz asociada a la identidad en las bases β y $\mathcal{B}$ (en ese orden)

Equivalentemente: $\text{Coord}_{\mathcal{B}}(X) = {}_{\mathcal{B}}(I)_{\beta} \cdot \text{Coord}_{\beta}(X)$

Como se construye la matriz asociada a ${}_{\beta}(T)_{\mathcal{B}}$, con $T: V \to W$ lineal y $\mathcal{B} \xrightarrow{b} V$, $\beta \xrightarrow{b} W$.

$\mathcal{B} = \{v_1, \dots, v_n\}$, $\beta = \{w_1, \dots, w_m\}$: $\overbrace{T(v_1)}^{\in W} = \alpha_1 w_1 + \alpha_2 w_2 + \dots + \alpha_m w_m$

$T(v_2) = \alpha'_1 w_1 + \alpha'_2 w_2 + \dots + \alpha'_m w_m$

$$\Rightarrow {}_{\beta}(T)_{\mathcal{B}} = \left(\begin{array}{c|c|c} \alpha_1 & \alpha'_1 & \\ \alpha_2 & \alpha'_2 & \\ \vdots & \vdots & \\ \alpha_m & \alpha'_m & \end{array}\right)$$

Es decir, la columna i-ésima de $\underset{\mathcal{B}\ \mathcal{C}}{(T)}$ es $\operatorname{coord}_{\mathcal{B}}(T(v_i))$

En nuestro caso, $\mathcal{C} = \{(1,\dots,0), (0,1,\dots,0), \dots, (0,\dots,1)\}$

$\mathcal{B} = \{v_1, \dots, v_n\}$

$$\underset{\mathcal{C}\ \mathcal{B}}{(I)} = \begin{array}{c} \begin{matrix} v_1 & v_2 & & v_n \end{matrix} \\ \begin{pmatrix} \downarrow & \downarrow & \cdots & \downarrow \end{pmatrix} \end{array}, \quad \underset{\mathcal{B}\ \mathcal{C}}{(I)} = \left[\underset{\mathcal{C}\ \mathcal{B}}{(I)}\right]^{-1}$$

$$\begin{cases} \dot{X} = \overbrace{\begin{pmatrix} 2 & -1 \\ -2 & 3 \end{pmatrix}}^{A} X \\ X(0) = (1,1) \end{cases}$$

Voy a buscar $\mathcal{B}$ base donde la matriz asociada sea diagonal. Busco $\mathcal{B}$ base de vectores propios.

¿$\det(A - \lambda I) = 0$?

$$\left| \begin{pmatrix} 2-\lambda & -1 \\ -2 & 3-\lambda \end{pmatrix} \right| = (2-\lambda)(3-\lambda) - 2 = \lambda^2 - 5\lambda + 4 \begin{cases} \lambda = 1 \\ \lambda = 4 \end{cases}$$

$$= (\lambda - 1)(\lambda - 4)$$

$\Rightarrow$ es diagonalizable. $\left(\begin{array}{c} mg(\lambda_i) = ma(\lambda_i) \\ + \\ \sum ma(\lambda_i) = n \end{array} \right)$

$S_1 = \operatorname{Ker}(A - I)$

$$\begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \left(\begin{array}{cc|c} 1 & -1 & 0 \\ \cancel{-2} & \cancel{2} & \cancel{0} \end{array}\right) \rightarrow x = y$$

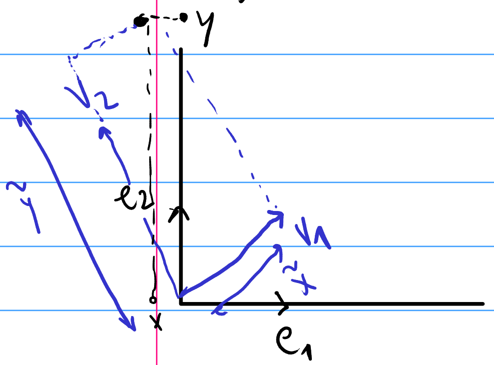
$\Rightarrow S_1 = [(1,1)]$

$$S_2 = \operatorname{Ker}(A - 4I) \rightsquigarrow \left(\begin{array}{cc|c} -2 & -1 & 0 \\ \cancel{-2} & \cancel{-1} & \cancel{0} \end{array}\right) \rightarrow -2x = y$$

$S_2 = [(-1,2)]$

$$\mathcal{B} = \{ \underset{v_1}{(1,1)}, \underset{v_2}{(-1,2)} \} \qquad P^{-1} = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix} \qquad P = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}$$

$$D := (A)_{\mathcal{B}\mathcal{B}} = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} = (I)_{\mathcal{B}\mathcal{C}} (A)_{\mathcal{C}\mathcal{C}} (I)_{\mathcal{C}\mathcal{B}} = P^{-1} A P$$



$$X(t) = (x,y) = \mathrm{Coord}_{\mathcal{C}}(X)$$

$$Y(t) = (\tilde{x}, \tilde{y}) = \mathrm{Coord}_{\mathcal{B}}(X)$$

Linealidad $\quad \dot{X} = AX \quad X = PY$

$$Y(t) = \underbrace{P^{-1} \cdot X(t)} \Rightarrow \dot{Y} = \dot{(P^{-1}X)} = P^{-1}\dot{X} = P^{-1}AX = P^{-1}APY = DY$$

Coordenadas ✓ en la base $\mathcal{B}$ de X

$$\Rightarrow \dot{Y} = DY.$$

Es decir, si $(\tilde{x}, \tilde{y}) = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \left(\frac{2}{3}x + \frac{1}{3}y, -\frac{x}{3} + \frac{y}{3}\right)$

$$\tilde{x} = \frac{2}{3}x + \frac{1}{3}y \qquad \tilde{y} = -\frac{x}{3} + y/3 \quad \Rightarrow \quad \begin{pmatrix} \dot{\tilde{x}} \\ \dot{\tilde{y}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} \tilde{x} \\ 4\tilde{y} \end{pmatrix}$$

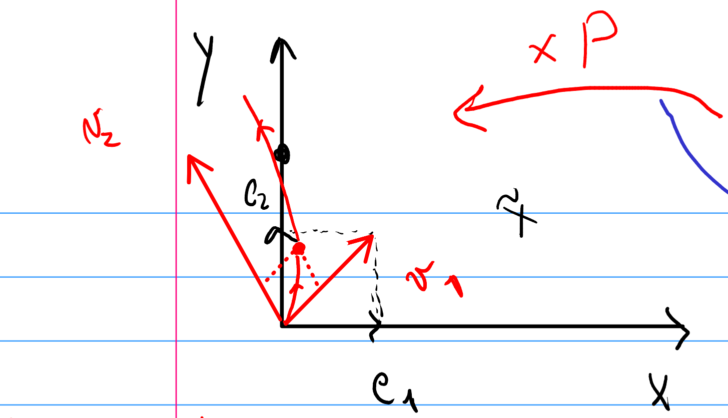
$$\Rightarrow \begin{cases} \dot{\tilde{x}} = \tilde{x} \\ \dot{\tilde{y}} = 4\tilde{y} \end{cases} \Rightarrow \begin{array}{l} \tilde{x}(t) = \tilde{x}_0 e^{t} \\ \tilde{y}(t) = \tilde{y}_0 e^{4t} \end{array} \Rightarrow Y(t) = (\tilde{x}_0 e^{t}, \tilde{y}_0 e^{4t})$$

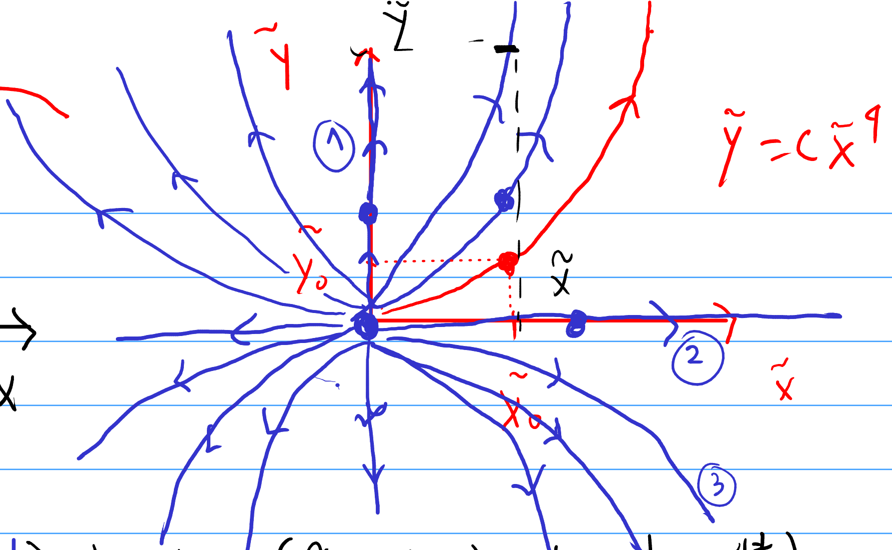
$$X(t) = PY(t) = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} \tilde{x}_0 e^{t} \\ \tilde{y}_0 e^{4t} \end{pmatrix} = \begin{pmatrix} \tilde{x}_0 e^{t} - \tilde{y}_0 e^{4t} \\ \tilde{x}_0 e^{t} + 2\tilde{y}_0 e^{4t} \end{pmatrix}$$

$$\Rightarrow X(t) = (\tilde{x}_0 e^{t} - \tilde{y}_0 e^{4t}, \tilde{x}_0 e^{t} + 2\tilde{y}_0 e^{4t}).$$

$$(\tilde{x}_0, \tilde{y}_0) = Y(0) = P^{-1}X(0) = P^{-1}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = P^{-1}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = (1,0)$$

Es decir, $X(0) = (1,1)$





$v_1 = (1,1)$, $v_2 = (-1,2)$

En azul otras soluciones

1) $\tilde{x}_0 = 0$, $\tilde{y}_0 > 0$

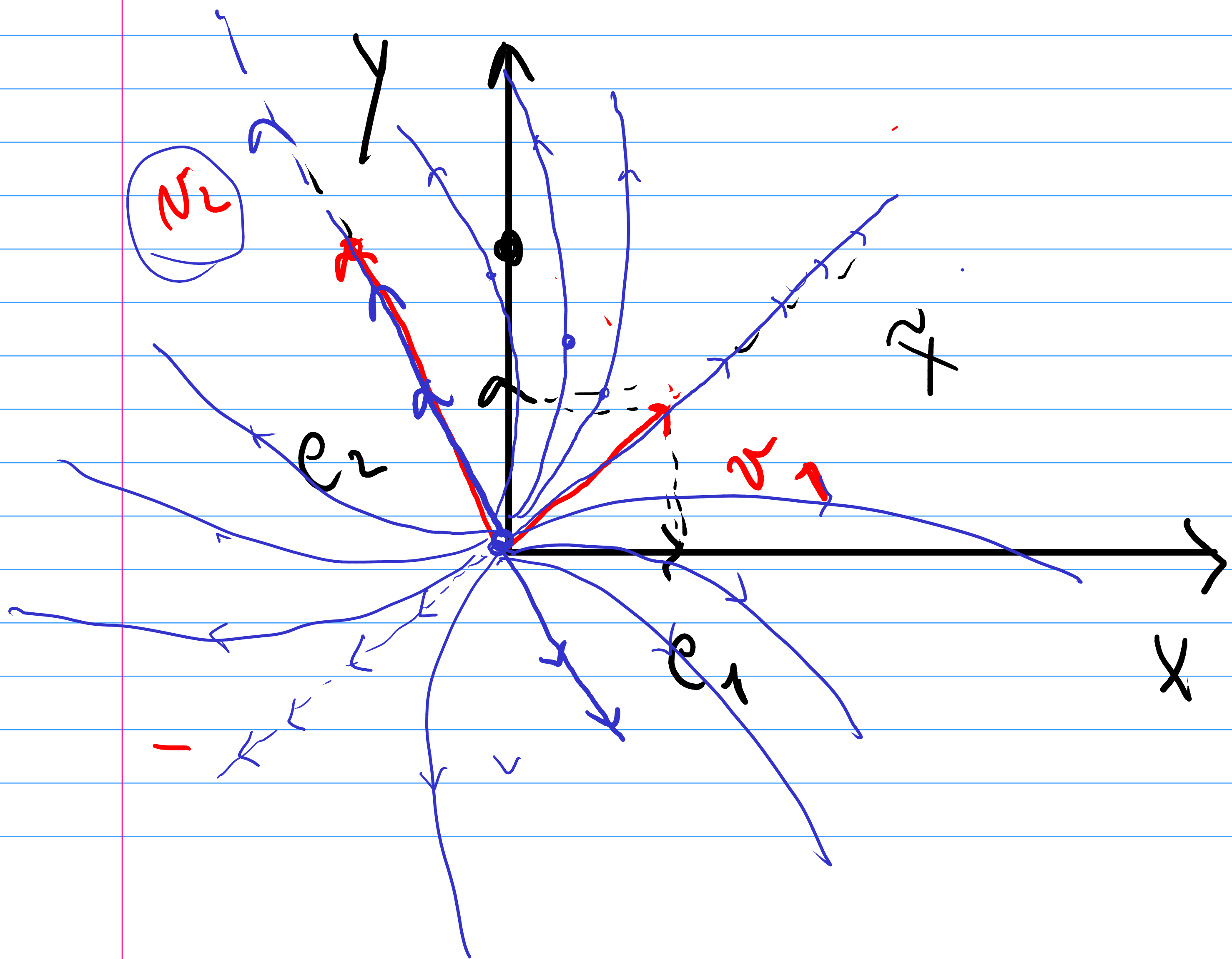
2) $\tilde{y}_0 = 0$, $\tilde{x}_0 > 0$

3) Cualquier otra.

De las cuentas: $(\tilde{x}(t), \tilde{y}(t)) = (\tilde{x}_0 e^{t}, \tilde{y}_0 e^{4t})$

$$\tilde{y}(t) = \tilde{y}_0 e^{4t} = \frac{\tilde{y}_0}{\tilde{x}_0^4} \cdot \tilde{x}_0^4 e^{4t}$$

$$= \frac{\tilde{y}_0}{\tilde{x}_0^4} \cdot (\underbrace{\tilde{x}_0 e^{t}}_{x(t)})^4 = \frac{\tilde{y}_0}{\tilde{x}_0^4} \cdot \tilde{x}(t)^4 \Rightarrow \tilde{y} = c\,\tilde{x}^4 \text{ !!}$$



Consulta: $f' + f = \delta(t-1)$

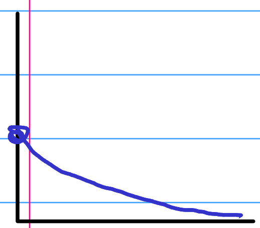
$f(0)=0$ $\quad sF(s) + F(s) = e^{-s} \Rightarrow F(s) = \dfrac{e^{-s}}{s+1} = e^{-s} \cdot \dfrac{1}{s+1}$

$\mathcal{L}^{-1}\left(\dfrac{1}{s+1}\right)(t) = e^{-t}$

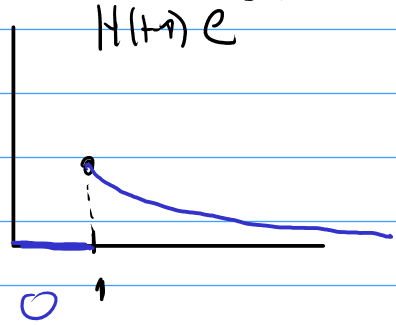
0 si $t<1$ y 1 luego

$\mathcal{L}^{-1}\left(e^{-s} \cdot \dfrac{1}{s+1}\right)(t) = H(t-1)e^{-(t-1)}$, $t \geq 0$. Dado $H(t) = \begin{cases} 1 & \text{si } t \geq 0 \\ 0 & \text{sino} \end{cases}$

$\mathcal{L}^{-1}\left(\dfrac{1}{s+1}\right)(t) = e^{-t}$ Propiedad de traslación en el tiempo



$H(t-1)e^{-(t-1)}$



$= \mathcal{L}^{-1}\left(e^{-s} \cdot \dfrac{1}{s+1}\right)(t)$