→ Ejercicio 1.

b. Sean $f: U \subset \mathbb{R}^2 \to \mathbb{R}$, $g: V \subset \mathbb{R} \to \mathbb{R}$ C^∞, $f(0,0) \in V$, $(0,0) \in U$

$g \circ f: U \to \mathbb{R}$ es una función de dos variables, entonces:

$$(g\circ f)(x,y) = (g\circ f)(0,0) + d_{(0,0)}(g\circ f)(x,y) + \frac{1}{2} d^2_{(0,0)}(g\circ f)(x,y) + \dots + \frac{1}{n!} d^n_{(0,0)}(g\circ f)(x,y)$$

$$+ r_n(x,y)$$

$$d_{(0,0)}(g\circ f)(x,y) = \frac{\partial}{\partial x}(g\circ f)(0,0)\cdot x + \frac{\partial}{\partial y}(g\circ f)(0,0)\, y =$$

$$= g'(0)\frac{\partial f}{\partial x}(0,0)\, x + g'(0)\frac{\partial f}{\partial y}(0,0)\, y = g'(0)\, d_{(0,0)} f(x,y)$$

$$d^2_{(0,0)}(g\circ f)(x,y) = \frac{\partial^2 (g\circ f)}{\partial x^2}(0,0)\cdot x^2 + 2\frac{\partial^2}{\partial x \partial y}(g\circ f)(0,0)\, xy + \frac{\partial^2 (g\circ f)}{\partial y^2}(0,0)\, y^2$$

$$= \left(g''(0)\cdot\left(\frac{\partial f}{\partial x}(0,0)\right)^2 + g'(0)\frac{\partial^2 f}{\partial x^2}(0,0)\right)x^2 + 2\left(g''(0)\cdot\frac{\partial f}{\partial y}(0,0)\frac{\partial f}{\partial x}(0,0) + g'(0)\cdot\frac{\partial^2 f}{\partial x \partial y}(0,0)\right)xy$$

$$+ \left(g''(0)\left(\frac{\partial f}{\partial y}(0,0)\right)^2 + g'(0)\cdot\frac{\partial^2 f}{\partial y^2}(0,0)\right)y^2 = g''(0)\cdot\left(d_{(0,0)} f(x,y)\right)^2 + g'(0)\cdot d^2_{(0,0)} f(x,y)$$

$$d^3_{(0,0)}(g\circ f)(x,y) = \Big(g'''(0)\left(\frac{\partial f}{\partial x}(0,0)\right)^3 + \underbrace{3g''(0)\cdot\frac{\partial f}{\partial x}(0,0)\cdot\frac{\partial^2 f}{\partial x^2}(0,0)} + g''(0)\cdot\frac{\partial f}{\partial x}(0,0)\frac{\partial^2 f}{\partial x^2}(0,0)$$

$$+ g'(0)\cdot\frac{\partial^3 f}{\partial x^3}(0,0)\Big)x^3 + 3\Big(g'''(0)\frac{\partial f}{\partial y}(0,0)\left(\frac{\partial f}{\partial x}(0,0)\right)^2 + g''(0)\cdot\frac{\partial f}{\partial x}(0,0)\cdot\frac{\partial^2 f}{\partial x \partial y}(0,0)$$

$$+ g''(0)\cdot\frac{\partial f}{\partial y}(0,0)\frac{\partial^2 f}{\partial x^2}(0,0) + g'(0)\frac{\partial^2 f}{\partial x \partial y}(0,0)\frac{\partial f}{\partial y}(0,0)\Big)x^2 y + 3\Big(g'''(0)\left(\frac{\partial f}{\partial y}(0,0)\right)^2\frac{\partial f}{\partial x}(0,0)$$

$$+ g''(0)\frac{\partial^2 f}{\partial y^2}(0,0)\frac{\partial f}{\partial x}(0,0) + g''(0)\frac{\partial^2 f}{\partial x \partial y}(0,0)\cdot\frac{\partial f}{\partial y}(0,0) + g'(0)\cdot\frac{\partial^3 f}{\partial x \partial y^2}(0,0)\Big)xy^2$$

$$+ \Big(g'''(0)\left(\frac{\partial f}{\partial y}(0,0)\right)^3 + g''(0)\cdot 2\frac{\partial f}{\partial y}(0,0)\cdot\frac{\partial^2 f}{\partial y^2}(0,0) + g''(0)\frac{\partial f}{\partial y}(0,0)\frac{\partial^2 f}{\partial y^2}(0,0) +$$

$$g'(0)\cdot\frac{\partial^3 f}{\partial y^3}(0,0)\Big)y^3 = g'''(0)\cdot\left(d_{(0,0)} f(x,y)\right)^3 + 3g''(0)\,(\quad) + g'(0)\, d^3_{(0,0)} f(x,y)$$