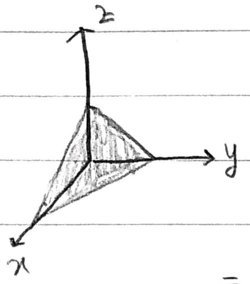
→ Ejercicio 8.

a. $f(x,y,z) = \dfrac{1}{(x+y+z+1)^2}$, $D = \{(x,y,z) \in \mathbb{R}^3 : 0 \leq y, 0 \leq x, 0 \leq z, x+y+z \leq 1\} =$
$= \{(x,y,z) \in \mathbb{R}^3 : 0 \leq x \leq 1, 0 \leq y \leq 1-x, z \leq 1-x-y\}$



$\Rightarrow \displaystyle\int_0^1 \int_0^{1-x} \int_0^{1-x-y} \frac{1}{(x+y+z+1)^2}\, dz\, dy\, dx = \int_0^1 \int_0^{1-x} \int_{x+y+1}^{2} u^{-2}\, du\, dy\, dx$

$u = x+y+z+1$
$du = dz$

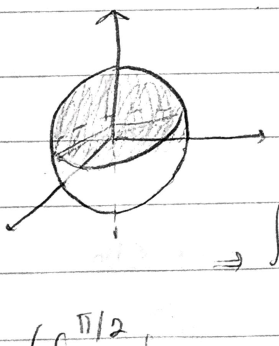
$= \displaystyle\int_0^1 \int_0^{1-x} -\frac{1}{u}\Big|_{x+y+1}^{2} dy\, dx = \int_0^1 \int_0^{1-x} -\frac{1}{2} + \frac{1}{x+y+1}\, dy\, dx$

$= \displaystyle\int_0^1 \left(-\frac{y}{2} + \ln(x+y+1)\right)\Big|_0^{1-x} dx = \int_0^1 \frac{x-1}{2} + \ln(2) - \ln(x+1)\, dx$

$= \dfrac{x^2}{4} - \dfrac{x}{2} + \ln(2).x - (x+1)\log(x+1) + x \Big|_0^1 = \dfrac{1}{4} - \dfrac{1}{2} + \ln(2) - 2\ln(2) + 1$

$= -\dfrac{1}{4} - \ln(2) + 1 = \dfrac{3}{4} - \ln(2)$

c. $f(x,y,z) = xyz$, $D = \{(x,y,z) \in \mathbb{R}^3 : 0 \leq y, 0 \leq x, 0 \leq z, x^2+y^2+z^2 \leq 1\}$
si $g(\rho,\theta,\varphi) = (\rho\cos\theta\,\mathrm{sen}\varphi, \rho\,\mathrm{sen}\theta\,\mathrm{sen}\varphi, \rho\cos\varphi) \Rightarrow \det J_g = \rho^2\,\mathrm{sen}\,\varphi$.



$g^{-1}(D) = \{(\rho,\theta,\varphi) : 0 \leq \rho \leq 1, 0 \leq \theta \leq \pi/2, 0 \leq \varphi \leq \pi/2\}$

$\Rightarrow \displaystyle\iiint_D f\, dx\,dy\,dz = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^1 \rho^3 \cos\theta\,\mathrm{sen}\theta\,\mathrm{sen}^2\varphi\cos\varphi \cdot \rho^2\mathrm{sen}\varphi\, d\rho\, d\theta\, d\varphi$

$= \displaystyle\left(\int_0^{\pi/2} \cos\theta\,\mathrm{sen}\theta\, d\theta\right)\left(\int_0^{\pi/2} \mathrm{sen}^3\varphi\cos\varphi\, d\varphi\right)\left(\int_0^1 \rho^5\, d\rho\right) = \frac{\mathrm{sen}^2\theta}{2}\Big|_0^{\pi/2} \cdot \frac{\mathrm{sen}^4\varphi}{4}\Big|_0^{\pi/2} \cdot \frac{\rho^6}{6}\Big|_0^1$

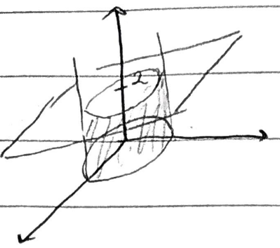
$= \dfrac{1}{2} \cdot \dfrac{1}{4} \cdot \dfrac{1}{6} = \dfrac{1}{48}$

e. $f(x,y,z)$

f $f(x,y,z) = x^2+y^2$, $D=$ región delimitada por $x^2+y^2=2x$, $z=0$ y $z=2$

$x^2-2x+y^2=0 \Rightarrow (x-1)^2+y^2-1=0$

$\Rightarrow (x-1)^2+y^2=1$



$\Rightarrow D=\{(x,y,z):(x-1)^2+y^2=1,\ 0\leq z\leq 2\}$

si $g(\rho,\theta,z)=(1+\rho\cos\theta,\ \rho\operatorname{sen}\theta,\ z) \rightarrow \det J_g=\rho$

$g^{-1}(D)=\{(\rho,\theta,z):\rho^2=1,\ 0\leq z\leq 2,\ 0\leq\theta\leq 2\pi\}$

$\rightarrow \iiint_D f\,dx\,dy\,dz = \int_0^{2\pi}\int_0^2\int_0^1 (1+\rho\cos\theta)^2+(\rho\operatorname{sen}\theta)^2\cdot\rho\,d\rho\,dz\,d\theta =$

$= \int_0^{2\pi}\int_0^2\int_0^1 (1+2\rho\cos\theta+\rho^2\cos^2\theta+\rho^2\operatorname{sen}^2\theta)\,\rho\,d\rho\,dz\,d\theta = \int_0^{}\int_0^{}\int_0^{}(1+2\rho\cos\theta+\rho^2)\rho\,d\rho\,dz\,d\theta$

$= \int_0^{2\pi}\int_0^2 \left(\frac{\rho^2}{2}+\frac{2\rho^3}{3}\cos\theta+\frac{\rho^4}{4}\Big|_0^1\right)dz\,d\theta = \int_0^{2\pi}\int_0^2 \frac{1}{2}+\frac{2}{3}\cos\theta+\frac{1}{4}\,dz\,d\theta$

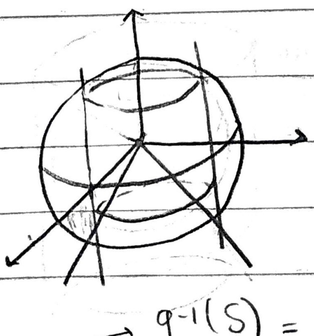
$= \int_0^{2\pi}\frac{z}{2}+\frac{2}{3}\cos\theta\cdot z+\frac{z}{4}\Big|_0^2\,d\theta = \int_0^{2\pi} 1+\frac{4\cos\theta}{3}+\frac{1}{2}\,d\theta =$

$= \frac{3}{2}\theta+\frac{4}{3}\operatorname{sen}\theta\Big|_0^{2\pi} = 3\pi$

$z\geq\sqrt{x^2+y^2}$

→ Ejercicio 9. $S=$ delimitado por $z\leq 0$, $x^2+y^2\leq 4$, $x^2+y^2\geq z^2$ y $x^2+y^2+z^2\leq 16$

$z\geq-\sqrt{x^2+y^2}$



$g(\rho,\theta,z)=(\rho\cos\theta,\ \rho\operatorname{sen}\theta,\ z) \rightarrow \det J_g=\rho$

$S=\{(x,y,z): z^2\leq x^2+y^2\leq 4,\ z\leq 0,\ x^2+y^2+z^2\leq 16\}$

$x^2+y^2+z^2\leq 2(x^2+y^2)\leq 8$

$\Rightarrow S=\{(x,y,z): z^2\leq x^2+y^2\leq 4,\ z\leq 0\}$

$\Rightarrow g^{-1}(S)=\{(\rho,\theta,z): z^2\leq\rho^2\leq 4,\ z\leq 0\}=\{(\rho,\theta,z): 0\leq\theta\leq 2\pi,\ -2\leq z\leq 0,\ -z\leq\rho\leq 2\}$

$\Rightarrow \int_0^{2\pi}\int_{-2}^0\int_{-z}^2 z\rho\,d\rho\,dz\,d\theta = 2\pi\int_{-2}^{0} z\cdot\frac{\rho^2}{2}\Big|_{-z}^2 dz = \pi\int_{-2}^0 4z-z^3\,dz =$

$= \pi\left(2z^2-\frac{z^4}{4}\right)\Big|_{-2}^0 = \pi\left(\frac{16}{4}-2\cdot 4\right)=\pi\cdot(4-8) = -4\pi$

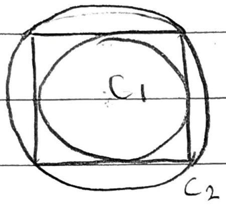
→ Ejercicio 10. Dado $r>0$, sean $I(r)=\int_{-r}^{r} e^{-u^2}\,du$, $R=[-r,r]\times[-r,r]$

a. $I(r)^2=\left(\int_{-r}^{r} e^{-u^2}\,du\right)^2$

$$\iint_R e^{-u^2}\,du=\int_{-r}^{r}\int_{-r}^{r} e^{-(x^2+y^2)}\,dx\,dy=\int_{-r}^{r}\int_{-r}^{r} e^{-x^2}e^{-y^2}\,dx\,dy=$$

$$=\int_{-r}^{r} e^{-x^2}\,dx\int_{-r}^{r} e^{-y^2}\,dy=I(r)I(r)$$

b. Sean C_1, C_2, los círculos inscrito y circunscrito a R resp.



Tenemos que $C_1\subset R\subset C_2$ y $e^{-(x^2+y^2)}>0\ \forall x,y$

⇒ por monotonía de la integral

$$\iint_{C_1} e^{-(x^2+y^2)}\,dx\,dy\le\iint_R e^{-(x^2+y^2)}\,dx\,dy\le\iint_{C_2} e^{-(x^2+y^2)}\,dx\,dy$$

c. Si $g(\rho,\theta)=(\rho\cos\theta,\rho\,\mathrm{sen}\,\theta)\rightarrow J_g(\rho,\theta)=\rho$. Sea $f(x,y)=e^{-(x^2+y^2)}$

Notar que el radio de C_1 es r y el de C_2 es $\sqrt{r^2+r^2}=\sqrt{2}\,r$.

$$\Rightarrow \iint_{C_1} f=\int_0^{2\pi}\int_0^{r} e^{-\rho^2}\rho\,d\rho\,d\theta=2\pi\cdot\left(-\frac{1}{2}\right)e^{-\rho^2}\Big|_0^r=-\pi\left(e^{-r^2}-1\right)=$$

$$=\pi\left(1-e^{-r^2}\right)$$

$$\Rightarrow \iint_{C_2} f=\pi\left(1-e^{-2r^2}\right)$$

d.
$$\left.\begin{array}{l}\lim\limits_{r\to 0}\iint_{C_1} f=\lim\limits_{r\to 0}\pi\left(1-e^{-r^2}\right)=\pi\\ \lim\limits_{r\to 0}\iint_{C_2} f=\lim\limits_{r\to 0}\pi\left(1-e^{-2r^2}\right)=\pi\end{array}\right\}\ \lim_{r\to\infty}\iint_R f=\pi$$

$$\rightarrow \int_{-\infty}^{+\infty} e^{-u^2}\,du=\lim_{r\to 0} I(r)=\lim_{r\to 0}\sqrt{\iint_R f}=\sqrt{\pi}$$