→ Ejercicio 5.

a. $f(x,y) = e^{-(x^2+y^2)}$, $D = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq r^2\}$

CDV: $g(\rho,\theta) = (\rho\cos\theta, \rho\,\text{sen}\theta)$, $g^{-1}(D) = \{(\rho,\theta): 0 \leq \rho \leq r,\ 0 \leq \theta \leq 2\pi\}$

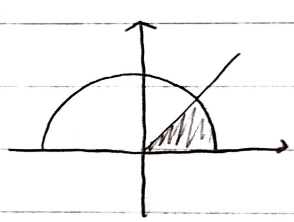
$$\iint_D f\,dx\,dy = \iint_{g^{-1}(D)} f(g(\rho,\theta)) \det J_g(\rho,\theta)\, d\rho\, d\theta$$

$$J_g(\rho,\theta) = \begin{pmatrix} \cos\theta & -\rho\,\text{sen}\theta \\ \text{sen}\theta & \rho\cos\theta \end{pmatrix} \rightarrow \det J_g(\rho,\theta) = \rho\cos^2\theta + \rho\,\text{sen}^2\theta = \rho$$

$$\Rightarrow \iint_D f\,dx\,dy = \int_0^{2\pi}\int_0^{r} e^{-\rho^2}\rho\, d\rho\, d\theta = \int_0^{2\pi}\left(-\frac{1}{2} e^{-\rho^2}\Big|_0^r\right) d\theta =$$

$$= 2\pi\left(\frac{1}{2}\left(1-e^{-r^2}\right)\right) = \pi(1-e^{-r^2})$$

b. $f(x,y) = x+y$, $D = \{(x,y): 0 \leq y \leq x,\ x^2+y^2 \leq 1\}$



CDV a polares nuevamente

$g^{-1}(D) = \{(\rho,\theta): 0 \leq \rho \leq 1,\ 0 \leq \theta \leq \pi/4\}$

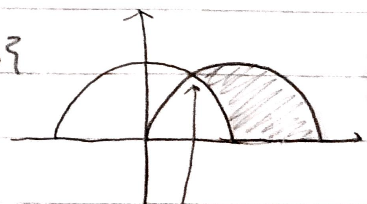
$$\iint_D f\,dx\,dy = \int_0^{\pi/4}\int_0^{1} \rho(\cos\theta+\text{sen}\theta)\rho\, d\rho\, d\theta = \int_0^{\pi/4}(\cos\theta+\text{sen}\theta)\,d\theta \int_0^1 \rho^2\, d\rho =$$

$$= \left(\text{sen}\theta - \cos\theta \Big|_0^{\pi/4}\right)\cdot \frac{\rho^3}{3}\Big|_0^1 = \frac{1}{3}$$

$\rho^2 \leq 2\rho\cos\theta$

c. $f(x,y) = \sqrt{x^2+y^2}$, $D = \{(x,y): 0 \leq y,\ x^2+y^2 \geq 1,\ x^2+y^2-2x \leq 0\}$

$(x-1)^2+y^2-1 \leq 0$



$g^{-1}(D) = \{(\rho,\theta): 1 \leq \rho \leq 2\cos\theta,\ 0 \leq \theta \leq \pi/3\}$

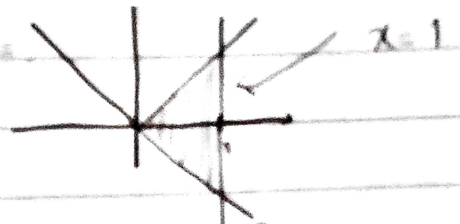
$1-x^2 = 2x-x^2 \Leftrightarrow x = 1/2$

$\Rightarrow y = \sqrt{1-x^2} = \frac{\sqrt{3}}{2}$

$$\Rightarrow \iint_D f\,dx\,dy = \int_0^{\pi/3}\int_1^{2\cos\theta} \rho^2\, d\rho\, d\theta = \int_0^{\pi/3} \frac{\rho^3}{3}\Big|_1^{2\cos\theta} d\theta$$

$$= \int_0^{\pi/3} \frac{2^3\cos^3\theta}{3} - \frac{1}{3}\, d\theta = \int_0^{\pi/3} \frac{2^3}{3}\cos\theta(1-\text{sen}^2\theta)\,d\theta - \frac{1}{3}\,d\theta = \frac{8}{3}\left(\text{sen}\theta - \frac{\text{sen}^3\theta}{3}\right)\Big|_0^{\pi/3}$$

$$-\frac{\theta}{3}\Big|_0^{\pi/3} = \frac{8}{3}\left(\text{sen}(\pi/3) - \frac{\text{sen}(\pi/3)^3}{3}\right) - \frac{\pi}{9} = \frac{8}{3}\left(\frac{\sqrt{3}}{2} - \frac{\sqrt{3}^3}{8\cdot 3}\right) - \frac{\pi}{9} = \sqrt{3} - \frac{\pi}{9}$$

d. $f(x,y) = \frac{x^2}{x^2+y^2}$, $D =$ 

$g^{-1}(D) = \{(\rho,\theta) : -\pi/4 \le \theta \le \pi/4 ,\ \rho\cos\theta \le 1\}$

$\to \iint_D f\,dx\,dy = \int_{-\pi/4}^{\pi/4} \int_0^{1/\cos\theta} \cos^2\theta\,\rho\,d\rho\,d\theta = \int_{-\pi/4}^{\pi/4} \cos^2\theta\,\frac{\rho^2}{2}\Big|_0^{1/\cos\theta} d\theta =$

$= \int_{-\pi/4}^{\pi/4} \frac{\cos^2\theta}{2}\left(\frac{1}{\cos^2\theta}\right) d\theta = \int_{-\pi/4}^{\pi/4} \frac{d\theta}{2} = \frac{1}{2}\left(\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right) = \frac{\pi}{4}$

e. $f(x,y) = (x-y)^2 \operatorname{sen}(x+y)$, $T(x,y) = (x-y;\, x+y) \Rightarrow F(u,v) = \left(\frac{u+v}{2}, \frac{v-u}{2}\right)$

$\Rightarrow |J_F(u,v)| = 1/2$. , $f(F(u,v)) = u^2 \operatorname{sen}(v) \rightarrow F^{-1}(0,\pi) = (-\pi,\pi)$, $F^{-1}(2\pi,\pi) = (\pi,3\pi)$

$F^{-1}(\pi,2\pi) = (-\pi,3\pi)$, $F^{-1}(\pi,0) = (\pi,\pi)$

$$\Rightarrow \iint_D f = \int_{-\pi}^{\pi} \int_{\pi}^{3\pi} u^2 \frac{\operatorname{sen}^2 v}{2}\, dv\, du = \frac{1}{2}\int_{-\pi}^{\pi} u^2 \left(\frac{1}{2}(v - \operatorname{sen}(v)\cos(v))\Big|_{\pi}^{3\pi}\right) du$$

$$= \frac{1}{4}\int_{-\pi}^{\pi} u^2 \cdot 2\pi\, du = \frac{\pi}{2} \frac{u^3}{3}\Big|_{-\pi}^{\pi} = \frac{\pi}{2 \cdot 3} \cdot 2\pi^3 = \frac{\pi^4}{3}$$

→ Ejercicio 1.

a. $D = \{(x,y,z) \in \mathbb{R}^3 \quad 0 \leq z \leq x^2/a^2 + y^2/b^2 \leq 1\}$

$g(\rho,\theta,z) = (a\rho\cos\theta,\ b\rho\sin\theta,\ z) \Rightarrow J_g(\rho,\theta,z) = \begin{pmatrix} a\cos\theta & -a\rho\sin\theta & 0 \\ b\sin\theta & b\rho\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\Rightarrow \det J_g(\rho,\theta,z) = ab\rho\cos^2\theta + ab\rho\sin^2\theta$

$= ab\rho$

$g^{-1}(D) = \{(\rho,\theta,z) \quad 0 \leq z \leq \rho^2 \leq 1\}$

$\iiint_D 1\, dx\,dy\,dz = \int_0^{2\pi}\int_0^1\int_0^{\rho^2} 1 .\ ab\rho\, dz\, d\rho\, d\theta = 2\pi \int_0^1 ab\rho . z\Big|_0^{\rho^2} d\rho =$

$= 2\pi \int_0^1 ab\ \rho^3\, d\rho = 2\pi ab \dfrac{\rho^4}{4}\Big|_0^1 = \dfrac{\pi ab}{2}$

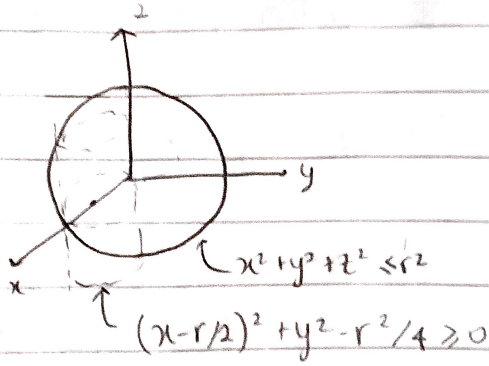
b. $D = \{(x,y,z) : \sqrt{x^2+y^2} \leq z \leq 1\} \to g(\rho,\theta,z) = (\rho\cos\theta, \rho\sin\theta, z) \to \det J_g = \rho$

$g^{-1}(D) = \{(\rho,\theta,z) \quad 0 \leq \rho \leq z \leq 1\}$

$\Rightarrow \text{vol}(D) = \iiint_D 1\, dx\,dy\,dz = \int_0^{2\pi}\int_0^1\int_\rho^1 \rho\, dz\, d\rho\, d\theta = 2\pi \int_0^1 \rho\ z\Big|_\rho^1 d\rho =$

$= 2\pi \int_0^1 \rho - \rho^2\, d\rho = 2\pi\left(\dfrac{\rho^2}{2} - \dfrac{\rho^3}{3}\Big|_0^1\right) = 2\pi\left(\dfrac{1}{2} - \dfrac{1}{3}\right) = 2\pi . \dfrac{1}{6} = \dfrac{\pi}{3}$

C. $D=\{(x,y,z): x^2+y^2+z^2 \leq r^2,\ x^2+y^2 \geq |rx|\}$



Alcanza calcular el volumen de $x^2+y^2+z^2 \leq r^2$ y sacarle el de $\{x^2+y^2 \geq |rx|\} \cap \{x^2+y^2+z^2 \leq r^2\}$

por simetría, podemos calcular $\{(x,y,z): x^2+y^2+z^2 \leq r^2,\ x^2+y^2 \geq rx,\ x,y \geq 0,\ z \leq 0\} = D''$ y multiplicar por 8.

Pasando a coordenadas esféricas: $x = \rho \cos\theta \operatorname{sen}\varphi,\ y = \rho \operatorname{sen}\theta \operatorname{sen}\varphi,\ z = \rho\cos\varphi$

$$J_g(\rho,\varphi,\theta) = \begin{pmatrix} \cos\theta \operatorname{sen}\varphi & \rho\cos\theta\cos\varphi & -\rho\operatorname{sen}\theta\operatorname{sen}\varphi \\ \operatorname{sen}\theta\operatorname{sen}\varphi & \rho\operatorname{sen}\theta\cos\varphi & \rho\cos\theta\operatorname{sen}\varphi \\ \cos\varphi & -\rho\operatorname{sen}\varphi & 0 \end{pmatrix} \Rightarrow |\det J_g(\rho,\varphi,\theta)| = \rho^2\operatorname{sen}\varphi$$

$g^{-1}(D') = \{\rho \leq r,\ \rho^2\operatorname{sen}^2\varphi \geq \rho r \cos\theta\operatorname{sen}\varphi,\ 0 \leq \theta \leq \pi/2,\ \pi/2 \leq \varphi \leq \pi\}$

$= \{r\cos\theta/\operatorname{sen}\varphi \leq \rho \leq r,\ 0 \leq \theta \leq \pi/2,\ \pi/2 \leq \varphi \leq \pi\} \Rightarrow \cos\theta/\operatorname{sen}\varphi \leq 1 \Rightarrow \operatorname{sen}(\theta+\pi/2) \leq \operatorname{sen}\varphi$

Como $\theta \in [0,\pi/2],\ \varphi \in [\pi/2,\pi]$, esto ocurre si $\theta+\pi/2 \geq \varphi$

$\Rightarrow g^{-1}(D) = \{(\rho,\theta,\varphi): r\cos\theta/\operatorname{sen}\theta \leq \rho \leq r,\ \theta \in [0,\pi/2],\ \varphi \in [\pi/2, \theta+\pi/2]\}$

$$\iiint_{D'} 1\, dx\,dy\,dz = \int_0^{\pi/2}\int_{\pi/2}^{\theta+\pi/2}\int_{r\cos\theta/\operatorname{sen}\varphi}^{r} 1.\rho^2\operatorname{sen}\varphi\, d\rho\, d\varphi\, d\theta = \int_0^{\pi/2}\int_{\pi/2}^{\theta+\pi/2} \operatorname{sen}\varphi . \frac{\rho^3}{3}\Big|_{\frac{r\cos\theta}{\operatorname{sen}\varphi}}^{r} d\varphi\, d\theta$$

$$= \int_0^{\pi/2}\int_{\pi/2}^{\theta+\pi/2} \operatorname{sen}\varphi\left(\frac{r^3}{3} - \frac{r^3}{3}\frac{\cos^3\theta}{\operatorname{sen}^3\varphi}\right) d\varphi\, d\theta = \frac{r^3}{3}\int_0^{\pi/2}\int_{\pi/2}^{\theta+\pi/2} \operatorname{sen}\varphi - \frac{\cos^3\theta}{\operatorname{sen}^2\varphi}\, d\varphi\, d\theta =$$

$$= \frac{r^3}{3}\int_0^{\pi/2} -\cos\varphi + \cos^3\theta . \frac{\cos\varphi}{\operatorname{sen}\varphi}\Big|_{\pi/2}^{\theta+\pi/2} d\theta = \frac{r^3}{3}\int_0^{\pi/2} -\cos(\theta+\pi/2) + \cos^3\theta . \frac{\cos(\theta+\pi/2)}{\operatorname{sen}(\theta+\pi/2)}\, d\theta$$

$$= \frac{r^3}{3}\int_0^{\pi/2} \operatorname{sen}\theta - \cos^3\theta \frac{\operatorname{sen}\theta}{\cos\theta}\, d\theta = \frac{r^3}{3}\int_0^{\pi/2} \operatorname{sen}\theta - \cos^2\theta\operatorname{sen}\theta\, d\theta =$$

$$= \frac{r^3}{3}\left(-\cos\theta + \frac{\cos^3\theta}{3}\Big|_0^{\pi/2}\right) = \frac{r^3}{3}\left(1 - \frac{1}{3}\right) = \frac{r^3}{3} . \frac{2}{3} = \frac{2r^3}{9}$$

$$\Rightarrow \mathrm{Vol}(D) = 8.\mathrm{Vol}(D') = 8.\frac{2r^3}{9} = \frac{16r^3}{9}$$