→ Ejercicio 6.

a. $\int_{-\infty}^{+\infty} e^{-x^2} dx = 2\int_{0}^{+\infty} e^{-x^2} dx$ y se cumple que $\frac{1}{e^{x^2}} < \frac{1}{x^2}$ $\forall x>1$

$\Rightarrow$ converge por comparación.

b. $\int_0^1 \frac{e^{-x}}{x} dx = \int_0^1 \frac{1}{xe^x} dx$, en $[0,1]$, e^x varía entre 1 y e

$\Rightarrow \frac{1}{xe^x} \geq \frac{1}{x \cdot 1} \Rightarrow \int_0^1 \frac{1}{xe^x} dx \geq \int_0^1 \frac{1}{x} dx = +\infty$

c. $\int_0^1 \frac{\log(x)}{\sqrt{x}} dx = 2\sqrt{x}\log(x)\Big|_0^1 - \int_0^1 \frac{1}{2x\sqrt{x}} dx = 2\sqrt{x}\log(x)\Big|_0^1 + \frac{1}{\sqrt{x}}\Big|_0^1 = +\infty$

$f = \log(x) \to f' = 1/x$

$g' = 1/\sqrt{x} \to g = 2\sqrt{x}$

d. $\int_1^{+\infty} \frac{\text{sen}(x)}{x^2} dx$, Notar que $\int_1^{+\infty} \left|\frac{\text{sen}(x)}{x^2}\right| dx = \int_1^{+\infty} \frac{|\text{sen}(x)|}{x^2} dx \leq \int_1^{+\infty} \frac{1}{x^2} dx < \infty$

e. $\int_0^{+\infty} \text{sen}(t)^2 dt$, notar que en los intervalos de la forma $[\pi/3 + k\pi, 2\pi/3 + k\pi]$

$\text{sen}(t)^2 \geq 1/4 \Rightarrow \int_0^{+\infty} \text{sen}(t)^2 dt \geq \sum_{k=0}^{+\infty} \frac{1}{4} \cdot \frac{\pi}{3} = +\infty$

↑ largo de los intervalos

g. $\int_{-\infty}^{+\infty} \frac{x}{\cosh(x)} dx$, como $\frac{x}{\cosh(x)}$ es impar y

$\int_0^{+\infty} \frac{x}{\cosh(x)} dx = \int_0^{+\infty} \frac{x}{e^x - e^{-x}} dx \sim \int_0^{+\infty} \frac{x}{e^x} dx < \infty \Rightarrow \int_{-\infty}^{+\infty} \frac{x}{\cosh(x)} dx$

$= \int_{-\infty}^{0} \frac{x}{\cosh(x)} dx + \int_0^{+\infty} \frac{x}{\cosh(x)} dx = -\int_0^{+\infty} \frac{x}{\cosh(x)} dx + \int_0^{+\infty} \frac{x}{\cosh(x)} dx = 0$

k. $\int_0^{+\infty} e^{x^2 - 1/x^2} dx = \int_0^1 e^{x^2 - 1/x^2} dx + \int_1^{+\infty} e^{x^2 - 1/x^2} dx \geq \int_0^1 e^{1 - 1/x^2} dx + \int_1^{+\infty} e^{x^2 - 1} dx$

m. $\int_0^{\pi/2} \frac{dx}{\text{sen}(x)^\alpha \cos(x)^\beta}$, tiene problema en los extremos pero en $x=0$, $\text{sen}(x) \sim x$ y $\cos(x) = 1$

$\Rightarrow \frac{1}{\text{sen}(x)^\alpha \cos(x)^\beta} \sim \frac{1}{x^\alpha}$ y en $x = \pi/2$, $\text{sen}(x) = 1$ y $\cos(x) \sim \pi/2 - x$

$\rightarrow \frac{1}{\text{sen}(x)^\alpha \cos(x)^\beta} \sim \frac{1}{(\pi/2 - x)^\beta}$. $\Rightarrow$ la integral converge para $\alpha, \beta > 1$

FECHA

→ Ejercicio 8.

a. $\sum_{n=1}^{+\infty} ne^{-n^2}$, tomando $f(x) = xe^{-x^2}$, $f: [1,+\infty) \to \mathbb{R}$, tenemos que $f'(x) = e^{-x^2} - 2x^2e^{-x^2} = 0 \Leftrightarrow e^{-x^2}(1-2x^2) = 0 \Leftrightarrow x^2 = \frac{1}{2}$ y no hay soluciones en $[1,+\infty)$,

$\Rightarrow$ estudiando el signo de f', vemos que f es decreciente estricta en $[1,+\infty)$

$\Rightarrow \sum_{n=1}^{+\infty} f(n)$ y $\int_1^{+\infty} f(x)\,dx$ son de la misma clase:

$$\int_1^{+\infty} f(x)dx = \int_1^{+\infty} xe^{-x^2}\,dx \underset{\substack{\uparrow \\ u = x^2 \\ du = 2xdx}}{=} \frac{1}{2}\int_1^{+\infty} e^{-u}\,du = -\frac{1}{2}e^{-u}\Big|_1^{+\infty}$$

$$= -\frac{1}{2}\lim_{y\to+\infty} e^{-y} - e^{-1} = \frac{1}{2e} < \infty$$

b. $\sum_{n=2}^{+\infty} \frac{1}{n\sqrt{\log(n)}}$, si $f(x) = \frac{1}{x\sqrt{\log(x)}} \longrightarrow f'(x) = \left(\sqrt{\log(x)} + \frac{x}{2\sqrt{\log(x)}}\cdot\frac{1}{x}\right)\cdot\frac{1}{x^2\log(x)}$

es decreciente pues $x\sqrt{\log(x)} \nearrow$

$$= \frac{1}{x^2\sqrt{\log(x)}} + \frac{1}{2x^2\log(x)^{3/2}}$$

$$\Rightarrow \int_2^{+\infty} f(x)dx = \int_2^{+\infty} \frac{1}{x\sqrt{\log(x)}}\,dx \underset{\substack{\uparrow \\ u = \log(x) \\ du = 1/x\, dx}}{=} \int_{\log(2)}^{+\infty} \frac{1}{\sqrt{u}}\,du = 2\sqrt{u}\Big|_{\log(2)}^{+\infty}$$

$$= \lim_{y\to+\infty} 2\sqrt{y} - 2\sqrt{\log(2)} = +\infty$$

→ Ejercicio 10. Probar que $\int_0^{+\infty} \log\left(1+\frac{a^2}{x^2}\right)dx$ converge y calcularla

Usando la sugerencia, calculamos la primitiva

$$\int_{y_1}^{y_2} \log\left(1+\frac{a^2}{x^2}\right) \underset{\uparrow}{=} x\log\left(1+\frac{a^2}{x^2}\right)\Big|_{x_1}^{x_2} + \int_{x_1}^{x_2} \frac{x\cdot 2a^2\,dx}{x^3(1+a^2/x^2)} =$$

$$f=\log(1+a^2/x^2) \rightarrow f' = \frac{-2a^2 x}{x^4(1+a^2/x^2)} = \frac{-2a^2}{x^3(1+a^2/x^2)}$$

$$g'=1 \longrightarrow g=x$$

$$= x\log\left(1+\frac{a^2}{x^2}\right)\Big|_{x_1}^{x_2} + \int_{x_1}^{x_2} \frac{2a^2}{x^2+a^2}\,dx = x\log\left(1+\frac{a^2}{x^2}\right)\Big|_{x_1}^{x_2} + \int_{x_1}^{x_2} \frac{2a^2}{\left(\left(\frac{x}{a}\right)^2+1\right)a^2}\,dx$$

$$\underset{\uparrow}{=} x\log\left(1+\frac{a^2}{x^2}\right)\Big|_{x_1}^{x_2} + 2a\int_{x_1/a}^{x_2/a} \frac{du}{u^2+1} = x\log\left(1+\frac{a^2}{x^2}\right)\Big|_{x_1}^{x_2} + 2a\,\mathrm{Arctg}\left(\frac{x}{a}\right)\Big|_{x_1}^{x_2}$$

$u=x/a$

$du = dx/a$

$$= x_2\log\left(1+\frac{a^2}{x_2^2}\right) - x_1\log\left(1+\frac{a^2}{x_1^2}\right) + 2a\,\mathrm{Arctg}\left(\frac{x_2}{a}\right) - 2a\,\mathrm{Arctg}\left(\frac{x_1}{a}\right)$$

Ahora, como $\lim\limits_{x_2\to+\infty} x_2\log\left(1+\frac{a^2}{x_2^2}\right) = \lim\limits_{x_2\to+\infty} x_2\frac{a^2}{x_2^2} = 0$

$$\lim_{x_1\to 0} x_1\log\left(1+\frac{a^2}{x_1^2}\right) = \lim_{x_1\to 0} \frac{\log\left(1+\frac{a^2}{x_1}\right)}{1/x_1}$$

$$= \lim_{x_1\to 0} \frac{-2a^2}{x_1^3(1+a^2/x_1^2)}\cdot(-x_1)^2 = \lim_{x_1\to 0} \frac{-2a^2}{x_1^2+a^2}\cdot x_1 = 0$$

$$\lim_{x_2\to+\infty} 2a\,\mathrm{Arctg}(x_2/a) = 2a\frac{\pi}{2} = a\pi$$

$$\lim_{x_1\to 0} 2a\,\mathrm{Arctg}\left(\frac{x_1}{a}\right) = 0$$

$$\Rightarrow \int_0^{+\infty} \log\left(1+\frac{a^2}{x^2}\right)dx = a\pi$$