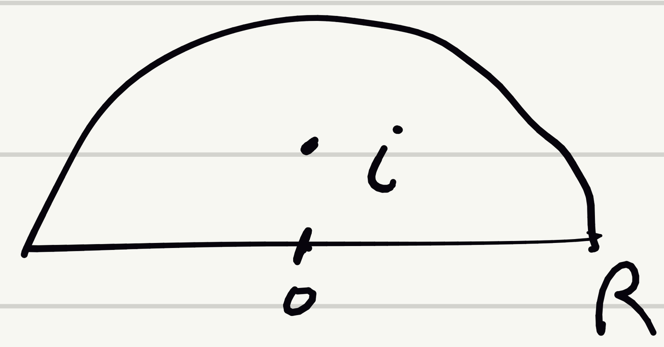
Problema 4.

(25 puntos)

Calcular justificando cada paso y enunciando resultados que se utilicen:

Par

No hay Polo en 0 $\rightarrow$ $$\int_0^{+\infty} \frac{x \operatorname{sen} x}{x^2+1} dx$$ $\rightarrow$ Polo de orden 1 en i



$$f(z) = \frac{z e^{iz}}{z^2+1} \Rightarrow \text{Si } x \in \mathbb{R},\ f(x) = \frac{x e^{ix}}{x^2+1}$$

$$\Rightarrow \operatorname{Im}(f(x)) = \frac{x \sin x}{x^2+1}$$

$$\gamma_R : [0,\pi] \to \mathbb{C} \ / \ \gamma_R(t) = Re^{it}$$

$$\Gamma_R := \gamma_R \cdot ([-R,R])$$

$$\int_{\Gamma_R} f(z)dz = 2\pi i \operatorname{Res}(f,i) = 2\pi i \lim_{z\to i} (z-i) f(z)$$

$$= 2\pi i \lim_{z \to i} \frac{z e^{iz}}{z+i} = \cancel{2}\pi i \frac{\cancel{i} e^{-1}}{\cancel{2}\cancel{i}} = \frac{\pi i}{e}$$

$$\lim_{R\to+\infty} \int_{\Gamma_R} f(z) = \lim_{R\to+\infty} \underbrace{\int_{\gamma_R} f(z)}_{\to 0} + \int_{-\infty}^{+\infty} f(x)dx = i \operatorname{Im}\left(\int_{-\infty}^{\infty} f(x)dx\right)$$

$$= i \operatorname{Im}\left(\overbrace{\int_{-\infty}^{\infty} \operatorname{Re}(f(x))dx}^{\in \mathbb{R}} + i \int_{-\infty}^{\infty} \operatorname{Im}(f(x))dx \right) = i \int_{-\infty}^{\infty} \operatorname{Im}(f(x))dx$$

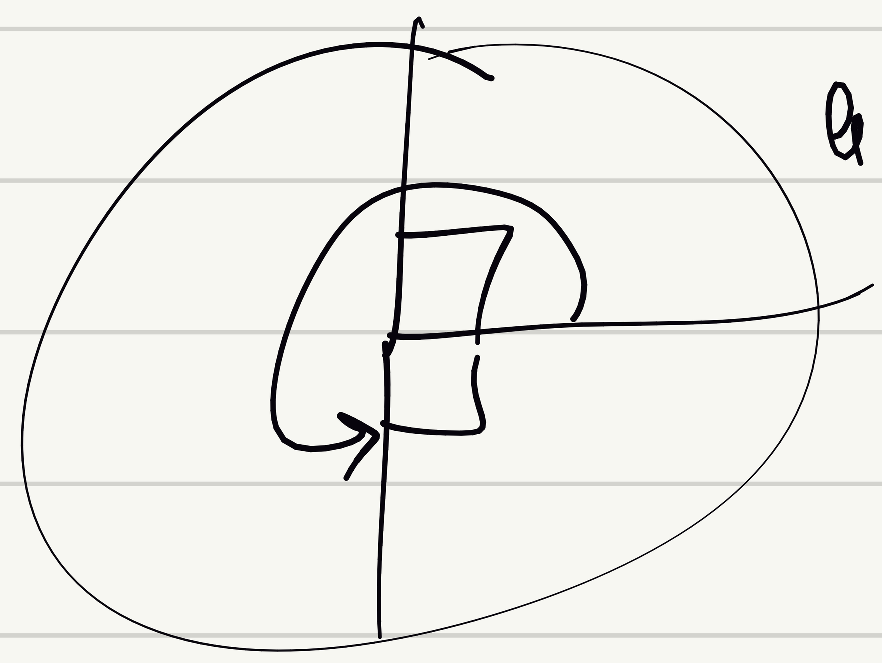
$$= i \int_{-\infty}^{\infty} \frac{x \sin x}{x^2+1} dx = 2i \int_0^{+\infty} \frac{x \sin x}{x^2+1} dx = \frac{\pi i}{e} \Rightarrow \int_0^{+\infty} \frac{x \sin x}{x^2+1} dx = \frac{\pi}{2e}$$

Lema 2 ***Lema de Jordan.***
Si $f(z)$ es una función compleja continua para todo z tal que $|z| \geq R_0$, que cumple $\lim_{z\to\infty} f(z) = 0$ entonces:

$$\lim_{R\to+\infty} \int_{\Gamma_R} e^{isz} f(z)dz = 0$$

donde $s > 0$ es constante y Γ_R es un arco contenido en la semicircunferencia: $z = Re^{it}, t \in [0,\pi]$.

$$\lim_{z\to+\infty} \frac{z}{z^2+1} = 0 \ \checkmark$$

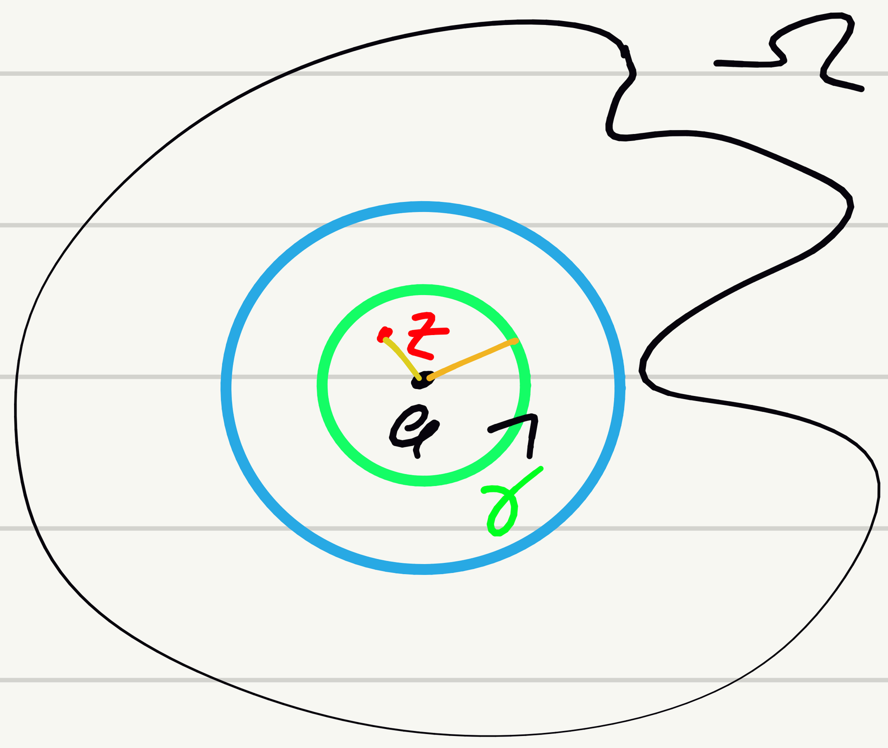


$\arg(\bar{z}) = 2\pi - \arg(z)$

Holomorfa $\Rightarrow$ Analítica

$f: \Omega \to \mathbb{C}$, Ω abierto , $a \in \Omega$, $r>0 \,/\, B(a,2r) \subseteq \Omega$

$z \in B(a,r)$, $\gamma := \partial B(a,r)$ antihorario



$\mathrm{Ind}(\gamma, a) = 1 = \mathrm{Ind}(\gamma, z)$

Fórmula de Cauchy:

$$2\pi i \, \overset{1}{\cancel{\mathrm{Ind}(\gamma, z)}} f(z) = \int_\gamma \frac{f(w)}{w-z}\,dw = \int_\gamma \frac{f(w)}{w-a+a-z}\,dw =$$

$$\int_\gamma \frac{f(w)}{(w-a)\left(1-\frac{z-a}{w-a}\right)}\,dw =$$

$|z-a| < r = |w-a| \Rightarrow \left|\frac{z-a}{w-a}\right| < 1$

$$\Rightarrow \sum_{n=0}^{\infty} \left(\frac{z-a}{w-a}\right)^n = \frac{1}{1-\frac{z-a}{w-a}}$$

$$\int_\gamma \frac{f(w)}{w-a} \sum_{n=0}^{\infty}\left(\frac{z-a}{w-a}\right)^n dw = \int_\gamma \sum_{n=0}^{\infty} \frac{f(w)(z-a)^n}{(w-a)^{n+1}}\,dw =$$

Por el teorema de Weirestrass, $|f(w)|\big|_\gamma$ tiene máximo $=M$

$$\left|\frac{f(w)}{(w-a)^{n+1}}(z-a)^n\right| = \frac{|f(w)|}{|w-a|}\frac{|z-a|^n}{|w-a|^n} \leq \frac{M}{r}\overbrace{\left|\frac{z-a}{w-a}\right|}^{\delta<1}{}^n = \frac{M}{r}\delta^n$$

Como $\sum_{n=0}^{\infty} \frac{M}{r}\delta^n = \frac{M}{r}\frac{1}{1-\delta}$ converge, por el criterio del Mayorante de Weirestrass, la serie conv. unif.

$\Rightarrow$ Podemos intercambiar serie con integral

$$\sum_{n=0}^{\infty} \int_\gamma \frac{f(w)}{(w-a)^{n+1}}(z-a)^n\,dw = \sum_{n=0}^{\infty} \int_\gamma \frac{f(w)}{(w-a)^{n+1}}\,dw\,(z-a)^n \Longrightarrow$$

$$f(z) = \sum_{n=0}^{\infty} \overbrace{\frac{1}{2\pi i}\int_\gamma \frac{f(w)}{(w-a)^{n+1}}\,dw}^{a_n}\,(z-a)^n$$

Además $f^{(n)}(a) = \dfrac{n!}{2\pi i}\displaystyle\int_\gamma \frac{f(w)}{(w-a)^{n+1}}\,dw$

$$\Rightarrow |f^{(n)}(a)| \leq \frac{n!}{2\pi} \int_\gamma \left| \frac{f(w)}{(w-a)^{n+1}} \right| |dw| \leq \frac{n!}{2\pi} \int_\gamma \frac{M}{r^{n+1}} |dw|$$

$$= \frac{n!}{\cancel{2\pi}} \frac{M}{r^{\cancel{n+1}}} \cancel{2\pi} \cancel{r} = \frac{n!\,M}{r^n} \leftarrow \text{Estimativas de Cauchy}$$

Si f es entera y acotada puedo tomar

$K > 0 / \forall z \in \mathbb{C}, |f(z)| \leq M$

$$|f'(a)| \leq \frac{M}{r} \xrightarrow{r \to +\infty} 0 \Rightarrow f'(a) = 0 \Rightarrow f' \equiv 0$$

$\Rightarrow f \equiv cte \leftarrow$ Liouville