# Graph Theory Review

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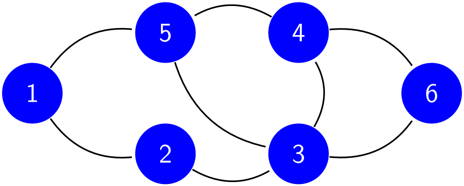
# Basic definitions and concepts




- Graph $G(\mathcal{V}, \mathcal{E})$ $\Rightarrow$ A set $\mathcal{V}$ of vertices or nodes
  $\Rightarrow$ Connected by a set $\mathcal{E}$ of edges or links
  $\Rightarrow$ Elements of $\mathcal{E}$ are unordered pairs $(u, v)$, $u, v \in \mathcal{V}$

- In figure $\Rightarrow$ Vertices are $\mathcal{V} = \{1, 2, 3, 4, 5, 6\}$
  $\Rightarrow$ Edges $\mathcal{E} = \{(1, 2), (1, 5), (2, 3), (3, 4), \ldots$
  $(3, 5), (3, 6), (4, 5), (4, 6)\}$

- Often we will say graph $G$ has order $N_v := |\mathcal{V}|$, and size $N_e := |\mathcal{E}|$

# From networks to graphs

- Networks are complex systems of inter-connected components
- Graphs are mathematical representations of these systems

  $\Rightarrow$ Formal language we use to talk about networks

- Components: nodes, vertices $\qquad \mathcal{V}$
- Inter-connections: links, edges $\qquad \mathcal{E}$
- Systems: networks, graphs $\qquad G(\mathcal{V}, \mathcal{E})$

# Vertices and edges in networks

| Network | Vertex | Edge |
|---|---|---|
| Internet | Computer/router | Cable or wireless link |
| Metabolic network | Metabolite | Metabolic reaction |
| WWW | Web page | Hyperlink |
| Food web | Species | Predation |
| Gene-regulatory network | Gene | Regulation of expression |
| Friendship network | Person | Friendship or acquaintance |
| Power grid | Substation | Transmission line |
| Affiliation network | Person and club | Membership |
| Protein interaction | Protein | Physical interaction |
| Citation network | Article/patent | Citation |
| Neural network | Neuron | Synapse |
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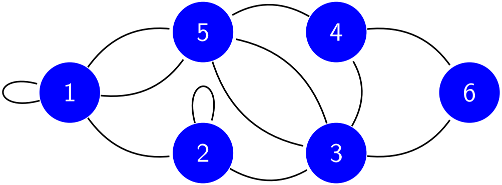
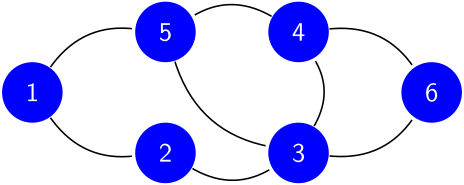
# Simple and multi-graphs

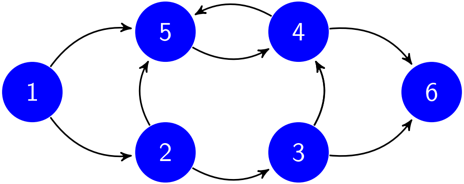
- In general, graphs may have self-loops and multi-edges
  - $\Rightarrow$ A graph with either is called a multi-graph




- Mostly work with simple graphs, with no self-loops or multi-edges



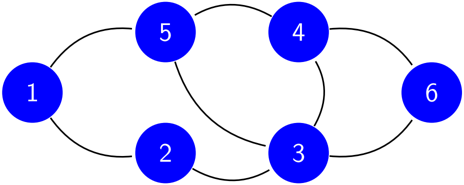
# Directed graphs



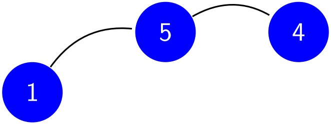

- In directed graphs, elements of $\mathcal{E}$ are ordered pairs $(u, v)$, $u, v \in \mathcal{V}$
  - $\Rightarrow$ Means $(u, v)$ distinct from $(v, u)$
  - $\Rightarrow$ Directed edges are called arcs
- Directed graphs often called digraphs
  - $\Rightarrow$ By convention arc $(u, v)$ points to $v$
  - $\Rightarrow$ If both $\{(u, v), (v, u)\} \subseteq \mathcal{E}$, the arcs are said to be mutual
- Ex: who-calls-whom phone networks, Twitter follower networks

# Subgraphs

- Consider a given graph $G(\mathcal{V}, \mathcal{E})$




- **Def:** Graph $G'(\mathcal{V}', \mathcal{E}')$ is an induced subgraph of $G$ if $\mathcal{V}' \subseteq \mathcal{V}$ and $\mathcal{E}' \subseteq \mathcal{E}$ is the collection of edges in G among that subset of vertices
- Ex: Graph induced by $\mathcal{V}' = \{1, 4, 5\}$

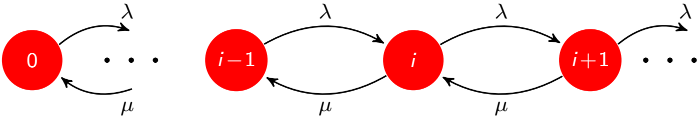


# Weighted graphs

- Oftentimes one labels vertices, edges or both with numerical values

  $\Rightarrow$ Such graphs are called weighted graphs

- Useful in modeling are e.g., Markov chain transition diagrams
- Ex: Single server queuing system (M/M/1 queue)


- Labels could correspond to measurements of network processes
- Ex: Node is infected or not with influenza, IP traffic carried by a link

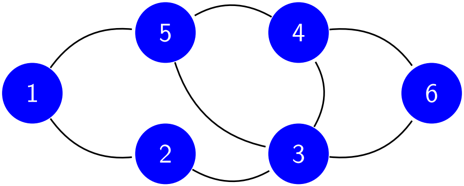
# Typical network representations

| Network | Graph representation |
|---|---|
| WWW | Directed multi-graph (with loops), unweighted |
| Facebook friendships | Undirected, unweighted |
| Citation network | Directed, unweighted, acyclic |
| Collaboration network | Undirected, unweighted |
| Mobile phone calls | Directed, weighted |
| Protein interaction | Undirected multi-graph (with loops), unweighted |
| ⋮ | ⋮ |

- Note that multi-edges are often encoded as edge weights (counts)

# Adjacency

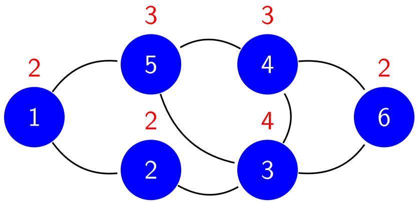
- Useful to develop a language to discuss the connectivity of a graph
- A simple and local notion is that of adjacency
  - $\Rightarrow$ Vertices $u, v \in \mathcal{V}$ are said adjacent if joined by an edge in $\mathcal{E}$
  - $\Rightarrow$ Edges $e_1, e_2 \in \mathcal{E}$ are adjacent if they share an endpoint in $\mathcal{V}$




- In figure $\Rightarrow$ Vertices 1 and 5 are adjacent; 2 and 4 are not
  - $\Rightarrow$ Edge $(1, 2)$ is adjacent to $(1, 5)$, but not to $(4, 6)$

# Degree

- An edge $(u, v)$ is incident with the vertices $u$ and $v$
- **Def:** The degree $d_v$ of vertex $v$ is its number of incident edges
  $\Rightarrow$ Degree sequence arranges degrees in non-decreasing order




- In figure $\Rightarrow$ Vertex degrees shown in red, e.g., $d_1 = 2$ and $d_5 = 3$
  $\Rightarrow$ Graph's degree sequence is 2,2,2,3,3,4
- High-degree vertices likely influential, central, prominent

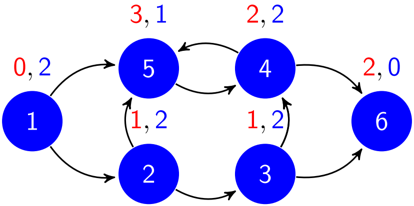
# Properties and observations about degrees

- Degree values range from 0 to $N_v - 1$
- The sum of the degree sequence is twice the size of the graph

$$\sum_{v=1}^{N_v} d_v = 2|\mathcal{E}| = 2N_e$$

$\Rightarrow$ The number of vertices with odd degree is even

- In digraphs, we have vertex in-degree $d_v^{in}$ and out-degree $d_v^{out}$




- In figure $\Rightarrow$ Vertex in-degrees shown in red, out-degrees in blue
  $\Rightarrow$ For example, $d_1^{in} = 0, d_1^{out} = 2$ and $d_5^{in} = 3, d_5^{out} = 1$

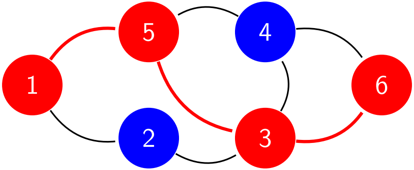
# Movement in a graph and connectivity

# Movement in a graph

- **Def:** A walk of length $l$ from $v_0$ to $v_l$ is an alternating sequence

$$\{v_0, e_1, v_1, \ldots, v_{l-1}, e_l, v_l\}, \text{ where } e_i \text{ is incident with } v_{i-1}, v_i$$

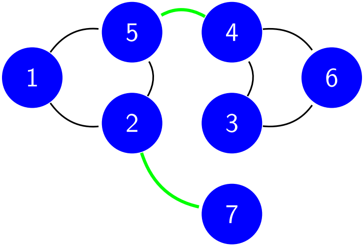
- A trail is a walk without repeated edges
- A path is a walk without repeated nodes (hence, also a trail)




- A walk or trail is closed when $v_0 = v_l$. A closed trail is a circuit
- A cycle is a closed walk with no repeated nodes except $v_0 = v_l$
- All these notions generalize naturally to directed graphs

# Connectivity

- Vertex $v$ is reachable from $u$ if there exists a $u - v$ walk
- **Def:** Graph is connected if every vertex is reachable from every other

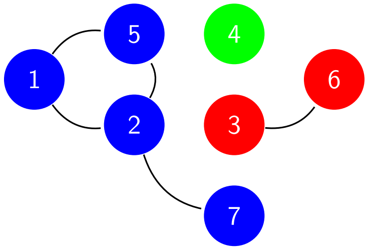



- If bridge edges are removed, the graph becomes disconnected

# Connected components

- **Def:** A component is a maximally connected subgraph

  $\Rightarrow$ Maximal means adding a vertex will ruin connectivity




- In figure $\Rightarrow$ Components are $\{1, 2, 5, 7\}$, $\{3, 6\}$ and $\{4\}$

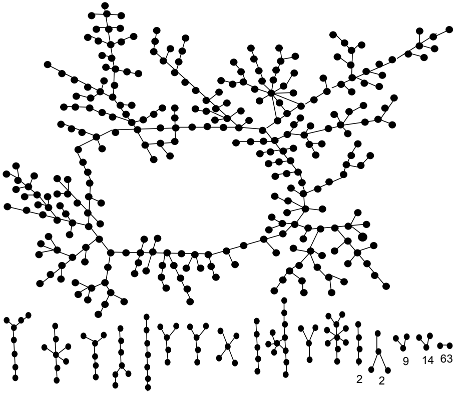
  $\Rightarrow$ Subgraph $\{3, 4, 6\}$ not connected, $\{1, 2, 5\}$ not maximal

- Disconnected graphs have 2 or more components

  $\Rightarrow$ Largest component often called giant component

# Giant connected components

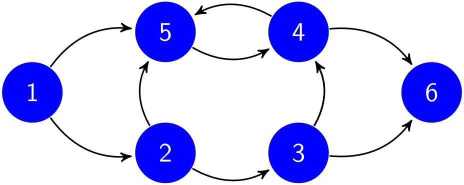
- Large real-world networks typically exhibit one giant component
- Ex: romantic relationships in a US high school [Bearman et al'04]




- Q: Why do we expect to find a single giant component?
- A: Well, it only takes one edge to merge two giant components

# Connectivity of directed graphs

- Connectivity is more subtle with directed graphs. Two notions
- **Def:** Digraph is strongly connected if for every pair $u, v \in \mathcal{V}$, $u$ is reachable from $v$ (via a directed walk) and vice versa
- **Def:** Digraph is weakly connected if connected after disregarding arc directions, i.e., the underlying undirected graph is connected

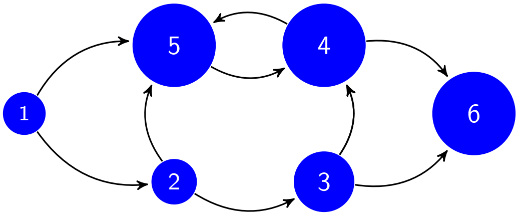



- Above graph is weakly connected but not strongly connected

  $\Rightarrow$ Strong connectivity obviously implies weak connectivity

# How well connected nodes are?

- Q: Which node is the most connected?
- A: Node rankings to measure website relevance, social influence
- There are two important connectivity indicators
  - $\Rightarrow$ How many links point to a node (outgoing links irrelevant)
  - $\Rightarrow$ How important are the links that point to a node




- Idea exploited by Google's PageRank© to rank webpages

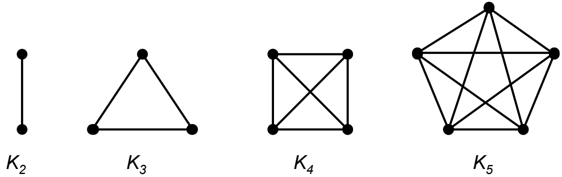
  ... by social scientists to study trust & reputation in social networks

  ... by ISI to rank scientific papers, journals ... More soon

# Families of graphs

# Complete graphs and cliques

- A complete graph $K_n$ of order $n$ has all possible edges

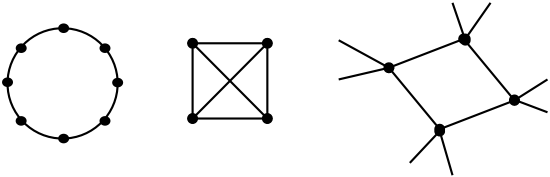



- Q: What is the size of $K_n$?
- A: Number of edges in $K_n$ = Number of vertex pairs $= \binom{n}{2} = \frac{n(n-1)}{2}$
- Of interest in network analysis are cliques, i.e., complete subgraphs

  $\Rightarrow$ Extreme notions of cohesive subgroups, communities

# Regular graphs

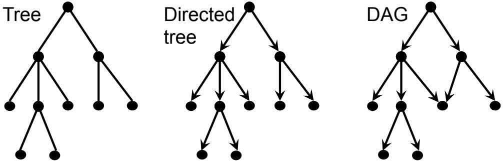
- A $d$-regular graph has vertices with equal degree $d$



- Naturally, the complete graph $K_n$ is $(n-1)$-regular
  $\Rightarrow$ Cycles are 2-regular (sub) graphs
- Regular graphs arise frequently in e.g.,
  - Physics and chemistry in the study of crystal structures
  - Geo-spatial settings as pixel adjacency models in image processing
  - Opinion formation, information cycles as regular subgraphs

# Trees and directed acyclic graphs

- A tree is a connected acyclic graph. An acyclic graph is forest
- Ex: river network, information cascades in Twitter, citation network



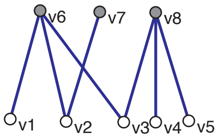

- A directed tree is a digraph whose underlying undirected graph is a tree
  $\Rightarrow$ Root is only vertex with paths to all other vertices
- Vertex terminology: parent, children, ancestor, descendant, leaf
- The underlying graph of a directed acyclic graph (DAG) is not a tree
  $\Rightarrow$ DAGs have a near-tree structure, also useful for algorithms

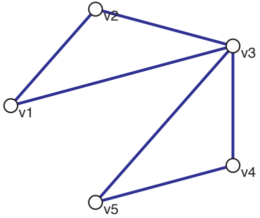
# Bipartite graphs

- A graph $G(\mathcal{V}, \mathcal{E})$ is called bipartite when

  $\Rightarrow$ $\mathcal{V}$ can be partitioned in two disjoint sets, say $\mathcal{V}_1$ and $\mathcal{V}_2$; and

  $\Rightarrow$ Each edge in $\mathcal{E}$ has one endpoint in $\mathcal{V}_1$, the other in $\mathcal{V}_2$



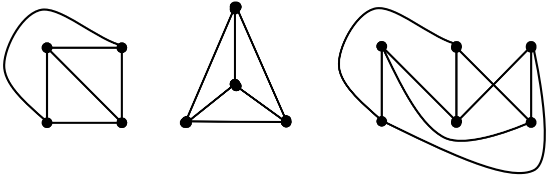

- Useful to represent e.g., membership or affiliation networks

  $\Rightarrow$ Nodes in $\mathcal{V}_1$ could be people, nodes in $\mathcal{V}_2$ clubs

  $\Rightarrow$ Induced graph $G(\mathcal{V}_1, \mathcal{E}_1)$ joins members of same club

# Planar graphs

- A graph $G(\mathcal{V}, \mathcal{E})$ is called planar if it can be drawn in the plane so that no two of its edges cross each other



- Planar graphs can be drawn in the plane using straight lines only
- Useful to represent or map networks with a spatial component

  $\Rightarrow$ Planar graphs are rare

  $\Rightarrow$ Some mapping tools minimize edge crossings

# Algebraic graph theory

# Adjacency matrix

- Algebraic graph theory deals with matrix representations of graphs
- Q: How can we capture the connectivity of $G(\mathcal{V}, \mathcal{E})$ in a matrix?
- A: Binary, symmetric adjacency matrix $\mathbf{A} \in \{0,1\}^{N_v \times N_v}$, with entries

$$A_{ij} = \begin{cases} 1, & \text{if } (i,j) \in \mathcal{E} \\ 0, & \text{otherwise} \end{cases}.$$

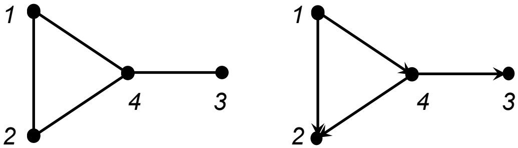
$\Rightarrow$ Note that vertices are indexed with integers $1, \ldots, N_v$

$\Rightarrow$ Binary and symmetric $\mathbf{A}$ for unweighted and undirected graph

- In words, $\mathbf{A}$ is one for those entries whose row-column indices denote vertices in $\mathcal{V}$ joined by an edge in $\mathcal{E}$, and is zero otherwise

# Adjacency matrix examples

- Examples for undirected graphs and digraphs




$$\mathbf{A}_u = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}, \quad \mathbf{A}_d = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

- If the graph is weighted, store the $(i, j)$ weight instead of 1

# Adjacency matrix properties

- Adjacency matrix useful to store graph structure. More soon
  - $\Rightarrow$ Also, operations on $\mathbf{A}$ yield useful information about $G$
- Degrees: Row-wise sums give vertex degrees, i.e., $\sum_{j=1}^{N_v} A_{ij} = d_i$
- For digraphs $\mathbf{A}$ is not symmetric and row-, colum-wise sums differ

$$\sum_{j=1}^{N_v} A_{ij} = d_i^{out}, \qquad \sum_{i=1}^{N_v} A_{ij} = d_j^{in}$$

- Walks: Let $\mathbf{A}^r$ denote the $r$-th power of $\mathbf{A}$, with entries $A_{ij}^{(r)}$
  - $\Rightarrow$ Then $A_{ij}^{(r)}$ yields the number of $i - j$ walks of length $r$ in $G$
- Corollary: $\text{trace}(\mathbf{A}^2)/2 = N_e$ and $\text{trace}(\mathbf{A}^3)/6 = \#\triangle$ in $G$
- Spectrum: $G$ is $d$-regular if and only if $\mathbf{1}$ is an eigenvector of $\mathbf{A}$, i.e.,

$$\mathbf{A}\mathbf{1} = d\mathbf{1}$$

# Incidence matrix

- A graph can be also represented by its $N_v \times N_e$ incidence matrix $\mathbf{B}$

  $\Rightarrow$ $\mathbf{B}$ is in general not a square matrix, unless $N_v = N_e$

- For undirected graphs, the entries of $\mathbf{B}$ are

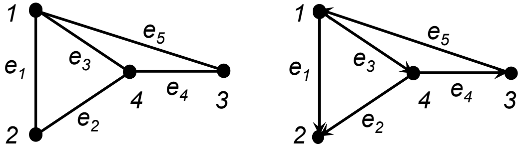
$$B_{ij} = \begin{cases} 1, & \text{if vertex } i \text{ incident to edge } j \\ 0, & \text{otherwise} \end{cases}.$$

- For digraphs we also encode the direction of the arc, namely

$$B_{ij} = \begin{cases} 1, & \text{if edge } j \text{ is } (k, i) \\ -1, & \text{if edge } j \text{ is } (i, k) \\ 0, & \text{otherwise} \end{cases}.$$

# Incidence matrix examples

- Examples for undirected graphs and digraphs

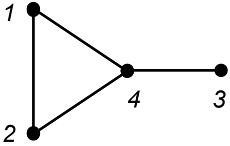



$$\mathbf{B}_u = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix}, \quad \mathbf{B}_d = \begin{pmatrix} -1 & 0 & -1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & -1 & 1 & -1 & 0 \end{pmatrix}$$

- If the graph is weighted, modify nonzero entries accordingly

- Vertex degrees often stored in the diagonal matrix $\mathbf{D}$, where $D_{ii} = d_i$

$$\mathbf{D} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$




- The $N_v \times N_v$ symmetric matrix $\mathbf{L} := \mathbf{D} - \mathbf{A}$ is called graph Laplacian

$$L_{ij} = \begin{cases} d_i, & \text{if } i = j \\ -1, & \text{if } (i,j) \in \mathcal{E} \\ 0, & \text{otherwise} \end{cases}, \quad \mathbf{L} = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix}$$

# Laplacian matrix properties

- Smoothness: For any vector $\mathbf{x} \in \mathbb{R}^{N_v}$ of "vertex values", one has

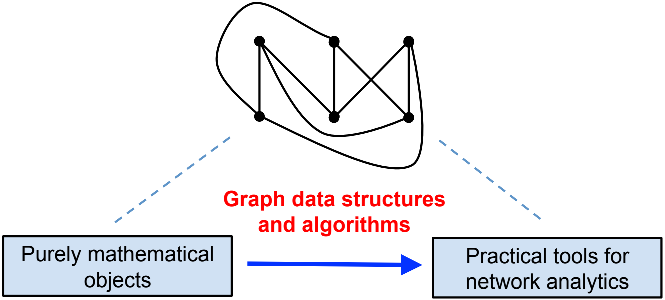
$$\mathbf{x}^\top \mathbf{L} \mathbf{x} = \sum_{(i,j) \in \mathcal{E}} (x_i - x_j)^2$$

  which can be minimized to enforce smoothness of functions on $G$

- Positive semi-definiteness: Follows since $\mathbf{x}^\top \mathbf{L} \mathbf{x} \geq 0$ for all $\mathbf{x} \in \mathbb{R}^{N_v}$
- Rank deficiency: Since $\mathbf{L}\mathbf{1} = \mathbf{0}$, $\mathbf{L}$ is rank deficient
- Spectrum and connectivity: The smallest eigenvalue $\lambda_1$ of $\mathbf{L}$ is 0
  - If the second-smallest eigenvalue $\lambda_2 \neq 0$, then $G$ is connected
  - If $\mathbf{L}$ has $n$ zero eigenvalues, $G$ has $n$ connected components

# Graph data structures and algorithms

- Q: How can we store and analyze a graph $G$ using a computer?




- Data structures: efficient storage and manipulation of a graph
- Algorithms: scalable computational methods for graph analytics

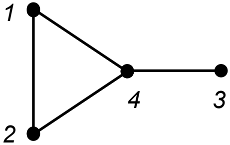
$\Rightarrow$ Contributions in this area primarily due to computer science

# Adjacency matrix as a data structure

- Q: How can we represent and store a graph $G$ in a computer?
- A: The $N_v \times N_v$ adjacency matrix $\mathbf{A}$ is a natural choice

$$A_{ij} = \begin{cases} 1, & \text{if } (i,j) \in \mathcal{E} \\ 0, & \text{otherwise} \end{cases} .$$

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$




- Matrices (arrays) are basic data objects in software environments
  - $\Rightarrow$ Naive memory requirement is $O(N_v^2)$
  - $\Rightarrow$ May be undesirable for large, sparse graphs

# Networks are sparse graphs

- Most real-world networks are sparse, meaning

$$N_e \ll \frac{N_v(N_v-1)}{2} \text{ or equivalently } \bar{d} := \frac{1}{N_v}\sum_{v=1}^{N_v} d_v \ll N_v - 1$$

- Figures from the study by Leskovec et al '09 are eloquent

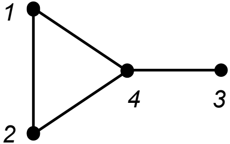
| **Network dataset** | **Order** $N_v$ | **Avg. degree** $\bar{d}$ |
|---|---|---|
| WWW (Stanford-Berkeley) | 319,717 | 9.65 |
| Social network (LinkedIn) | 6,946,668 | 8.87 |
| Communication (MSN IM) | 242,720,596 | 11.1 |
| Collaboration (DBLP) | 317,080 | 6.62 |
| Roads (California) | 1,957,027 | 2.82 |
| Proteins (S. Cerevisiae) | 1,870 | 2.39 |

- Graph density $\rho := \frac{N_e}{N_v^2} = \frac{\bar{d}}{2N_v}$ is another useful metric

# Adjacency and edge lists

- An adjacency-list representation of graph $G$ is an array of size $N_v$
  - $\Rightarrow$ The $i$-th array element is a list of the vertices adjacent to $i$

$$\begin{aligned} L_a[1] = &\quad \{2,4\} \\ L_a[2] = &\quad \{1,4\} \\ L_a[3] = &\quad \{4\} \\ L_a[4] = &\quad \{1,2,3\} \end{aligned}$$




- Similarly, an edge list stores the vertex pairs incident to each edge

$$\begin{aligned} L_e[1] = &\quad \{1,2\} \\ L_e[2] = &\quad \{1,4\} \\ L_e[3] = &\quad \{2,4\} \\ L_e[4] = &\quad \{3,4\} \end{aligned}$$

- In either case, the memory requirement is $O(N_e)$

# Graph algorithms and complexity

- Numerous interesting questions may be asked about a given graph
- For few simple ones, lookup in data structures suffices
  - Q1: Are vertices $u$ and $v$ linked by an edge?
  - Q2: What is the degree of vertex $u$?
- Some others require more work. Still can tackle them efficiently
  - Q1: What is the shortest path between vertices $u$ and $v$?
  - Q2: How many connected components does the graph have?
  - Q3: Is a given digraph acyclic?
- Unfortunately, in some cases there is likely no efficient algorithm
  - Q1: What is the maximal clique in a given graph?
- Algorithmic complexity key in the analysis of modern network data

# Testing for connectivity

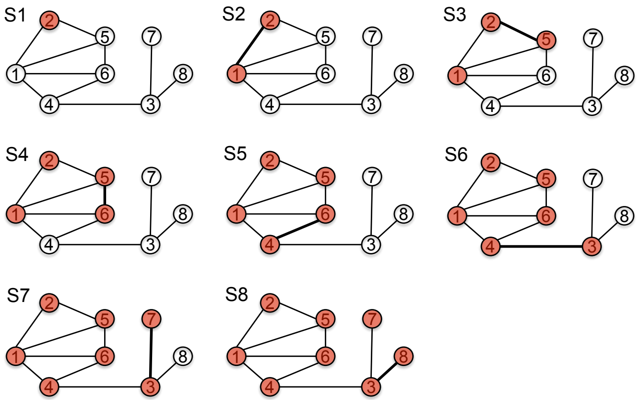
- Goal: verify connectivity of a graph based on its adjacency list
- Idea: start from vertex $s$, explore the graph, mark vertices you visit

Output : List $M$ of marked vertices in the component
Input : Graph $G$ (e.g., adjacency list)
Input : Starting vertex $s$

$L := \{s\}$; $M := \{s\}$; % Initialize exploration and marking lists
% Repeat while there are still nodes to explore
**while** $L \neq \emptyset$ **do**
  choose $u \in L$; % Pick arbitrary vertex to explore
  **if** $\exists\,(u, v) \in \mathcal{E}$ *such that* $v \notin M$ **then**
    choose $(u, v)$ with $v$ of smallest index;
    $L := L \cup \{v\}$; $M := M \cup \{v\}$; % Mark and augment
  **else**
    $L := L \setminus \{u\}$; % Prune
  **end**
**end**

- Below we indicate the chosen and marked nodes. Initialize $s = 2$

| $L$ | Mark |
|---|---|
| {2} | 2 |
| {2,1} | 1 |
| {2,1,5} | 5 |
| {2,1,5,6} | 6 |
| {1,5,6} | |
| {1,5,6,4} | 4 |
| {5,6,4} | |
| {5,4} | |
| {5,4,3} | 3 |
| {5,3} | |
| {5,3,7} | 7 |
| {5,3} | |
| {3} | |
| {3,8} | 8 |
| {3} | |
| {} | |




- Exploration takes $2N_v$ steps. Each node is added and removed once

# Breadth-first search

- Choices made arbitrarily in the exploration algorithm. Variants?
- Breadth-first search (BFS): choose for $u$ the first element of $L$

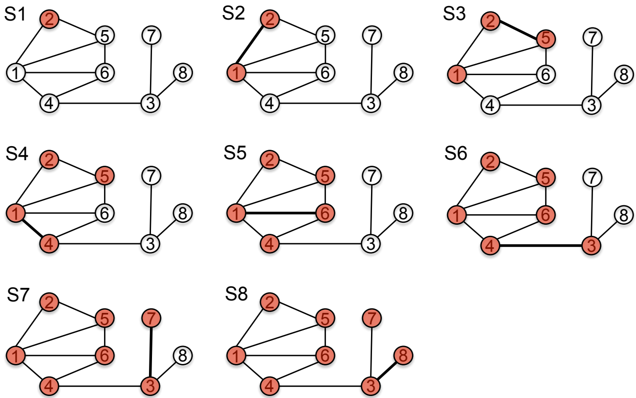
Output : List $M$ of marked vertices in the component
Input : Graph $G$ (e.g., adjacency list)
Input : Starting vertex $s$

$L := \{s\}$; $M := \{s\}$; % Initialize exploration and marking lists
% Repeat while there are still nodes to explore
**while** $L \neq \emptyset$ **do**
  $u := \text{first}(L)$; % Breadth first
  **if** $\exists\,(u, v) \in \mathcal{E}$ *such that* $v \notin M$ **then**
    choose $(u, v)$ with $v$ of smallest index;
    $L := L \cup \{v\}$; $M := M \cup \{v\}$; % Mark and augment
  **else**
    $L := L \setminus \{u\}$; % Prune
  **end**
**end**

# BFS example

- Below we indicate the chosen and marked nodes. Initialize $s = 2$

| $L$ | Mark |
|---|---|
| {2} | 2 |
| {2,1} | 1 |
| {2,1,5} | 5 |
| {1,5} | |
| {1,5,4} | 4 |
| {1,5,4,6} | 6 |
| {5,4,6} | |
| {4,6} | |
| {4,6,3} | 3 |
| {6,3} | |
| {3} | |
| {3,7} | 7 |
| {3,7,8} | 8 |
| {7,8} | |
| {8} | |
| {} | |




- The algorithm builds a wider tree (breadth first)

- Depth-first search (DFS): choose for $u$ the last element of $L$

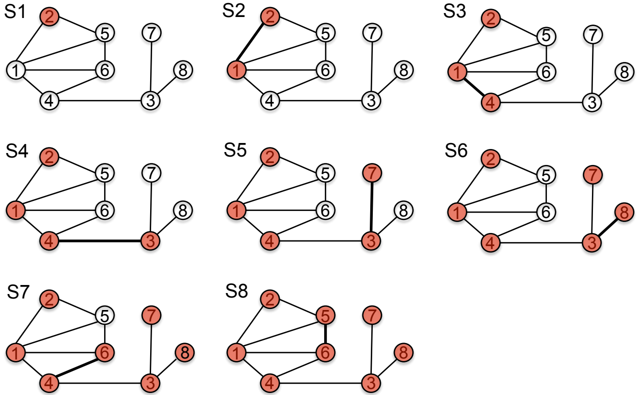
Output : List $M$ of marked vertices in the component
Input : Graph $G$ (e.g., adjacency list)
Input : Starting vertex $s$

$L := \{s\}$; $M := \{s\}$; % Initialize exploration and marking lists
% Repeat while there are still nodes to explore
**while** $L \neq \emptyset$ **do**
  $u := \text{last}(L)$; % Depth first
  **if** $\exists\,(u, v) \in \mathcal{E}$ *such that* $v \notin M$ **then**
    choose $(u, v)$ with $v$ of smallest index;
    $L := L \cup \{v\}$; $M := M \cup \{v\}$; % Mark and augment
  **else**
    $L := L \setminus \{u\}$; % Prune
  **end**
**end**

# DFS example

- Below we indicate the chosen and marked nodes. Initialize $s = 2$

| $L$ | Mark |
|---|---|
| {2} | 2 |
| {2,1} | 1 |
| {2,1,4} | 4 |
| {2,1,4,3} | 3 |
| {2,1,4,3,7} | 7 |
| {2,1,4,3} | |
| {2,1,4,3,8} | 8 |
| {2,1,4,3} | |
| {2,1,4} | |
| {2,1,4,6} | 6 |
| {2,1,4,6,5} | 5 |
| {2,1,4,6} | |
| {2,1,4} | |
| {2,1} | |
| {2} | |
| {} | |




- The algorithm builds longer paths (depth first)

# Distances in a graph

- Recall a path $\{v_0, e_1, v_1, \ldots, v_{l-1}, e_l, v_l\}$ has length $l$
  - $\Rightarrow$ Edges weights $\{w_e\}$, length of the walk is $w_{e_1} + \ldots + w_{e_l}$
- **Def:** The distance between vertices $u$ and $v$ is the length of the shortest $u - v$ path. Oftentimes referred to as geodesic distance
  - $\Rightarrow$ In the absence of a $u - v$ path, the distance is $\infty$
  - $\Rightarrow$ The diameter of a graph is the value of the largest distance
- Q: What are efficient algorithms to compute distances in a graph?
- A: BFS (for unit weights) and Dijkstra's algorithm

# Computing distances with BFS

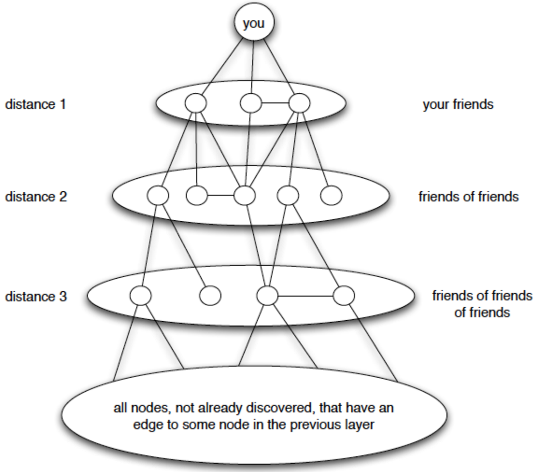
- Use BFS and keep track of path lengths during the exploration
- Increment distance by 1 every time a vertex is marked

**Output** : Vector $d$ of distances from reference vertex
**Input** : Graph $G$ (e.g., adjacency list)
**Input** : Reference vertex $s$

$L := \{s\}$; $M := \{s\}$; $d(s) = 0$; % Initialization
% Repeat while there are still nodes to explore
**while** $L \neq \emptyset$ **do**
    $u :=$ first$(L)$; % Breadth first
    **if** $\exists\,(u, v) \in \mathcal{E}$ *such that* $v \notin M$ **then**
        choose $(u, v)$ with $v$ of smallest index;
        $L := L \cup \{v\}$; $M := M \cup \{v\}$;% Mark and augment
        $d(v) := d(u) + 1$ % Increment distance
    **else**
        $L := L \setminus \{u\}$; % Prune
    **end**
**end**

# Example: Distances in a social network

- BFS tree output for your friendship network



# Glossary

- (Di) Graph
- Arc
- (Induced) Subgraph
- Incidence
- Degree sequence
- Walk, trail and path
- Connected graph
- Giant connected component
- Strongly connected digraph
- Clique
- Tree
- Bipartite graph
- Directed acyclic graph (DAG)
- Adjacency matrix
- Graph Laplacian
- Adjacency and edge lists
- Sparse graph
- Graph density
- Breadth-first search
- Depth-first search (DFS)
- Geodesic distance (BFS)
- Diameter