

Ejercicio 1.9

Sea $X = \left(\begin{array}{c|c} A & B \\ \hline 0 & C \end{array} \right)$ donde $A \in M_{n \times n}$, $B \in M_{n \times m}$
y $C \in M_{m \times m}$

Procedamos por inducción en n (m será fijo en toda la prueba)

Paso base, $n=1$: $A = (a) \in M_{1 \times 1}$

$$\begin{array}{c|c|c} |X| = & a & B \\ & 0 & \\ & \vdots & \\ & 0 & C \end{array} = (-1)^{1+1} \cdot a \cdot \det(C) = a \cdot \det(C) = \det(A) \cdot \det(C)$$

↓
Desarrollo por primer columna

Paso inductivo: (H_i) $\left(\begin{array}{c|c} A & B \\ \hline 0 & C \end{array} \right) = \det(A) \cdot \det(C)$
si $A \in M_{n-1 \times n-1}$
 $C \in M_{m \times m}$

(I_i) $\left(\begin{array}{c|c} A & B \\ \hline 0 & C \end{array} \right) = \det(A) \cdot \det(C)$
si $A \in M_{n \times n}$
 $C \in M_{m \times m}$

Dem. Sea $A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \in M_{n \times n}$

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$$\begin{array}{|c|} \hline |X| = \begin{array}{|c|} \hline a_{11} \quad a_{12} \quad \dots \quad a_{1n} \\ \hline a_{21} \quad a_{22} \quad \dots \quad a_{2n} \\ \vdots \quad \vdots \quad \quad \quad \vdots \\ \hline a_{n1} \quad a_{n2} \quad \dots \quad a_{nn} \\ \hline \end{array} \\ \hline \end{array} \quad \begin{array}{|c|} \hline B \\ \hline \end{array} \quad \begin{array}{|c|} \hline = \sum_{i=1}^n (-1)^{i+1} \cdot a_{i1} \cdot |X_{i1}| = * \\ \hline \end{array}$$

$$\begin{array}{|c|} \hline |X_{i1}| = \begin{array}{|c|} \hline a_{12} \quad a_{13} \quad \dots \quad a_{1n} \\ \vdots \quad \vdots \quad \quad \quad \vdots \\ \hline a_{i-1,2} \quad a_{i-1,3} \quad \dots \quad a_{i-1,n} \\ a_{i+1,2} \quad a_{i+1,3} \quad \dots \quad a_{i+1,n} \\ \vdots \quad \vdots \quad \quad \quad \vdots \\ \hline a_{n2} \quad a_{n3} \quad \dots \quad a_{nn} \\ \hline \end{array} \\ \hline \end{array} \quad \begin{array}{|c|} \hline B' \\ \hline \end{array} \quad \begin{array}{|c|} \hline = \begin{array}{|c|} \hline a_{12} \quad a_{13} \quad \dots \quad a_{1n} \\ \vdots \quad \vdots \quad \quad \quad \vdots \\ \hline a_{i-1,2} \quad a_{i-1,3} \quad \dots \quad a_{i-1,n} \\ a_{i+1,2} \quad a_{i+1,3} \quad \dots \quad a_{i+1,n} \\ \vdots \quad \vdots \quad \quad \quad \vdots \\ \hline a_{n2} \quad a_{n3} \quad \dots \quad a_{nn} \\ \hline \end{array} \cdot |C| \\ \hline \end{array}$$

$$= |A_{i1}| \cdot \det(C)$$

$$\Rightarrow * = \sum_{i=1}^n (-1)^{i+1} a_{i1} \cdot |A_{i1}| \cdot \det(C) = \det(C) \cdot \sum_{i=1}^n (-1)^{i+1} a_{i1} \cdot |A_{i1}|$$

$$= \det(C) \cdot \det(A)$$

↓ Desarrollo por primer columna de A