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always exhibit their greatest intensity near the ground and diminish with increasing height above the ground.

- ii. Some upper level lows do not extend downward to the ground, as indicated in Fig. 3.3b. It follows from the hypsometric equation that these lows must be *cold core* below the level at which they achieve their greatest intensity and *warm core* above that level, as shown in Fig. 3.3b.

### 3.2.4 Reduction of Pressure to Sea Level

In mountainous regions the difference in surface pressure from one observing station to another is largely due to differences in elevation. To isolate that part of the pressure field that is due to the passage of weather systems, it is necessary to reduce the pressures to a common reference level. For this purpose, sea level is normally used.

Let the subscripts  $g$  and  $0$  refer to conditions at the ground and at sea level ( $Z = 0$ ), respectively. Then, for the layer between the Earth's surface and sea level, the hypsometric equation (3.29) assumes the form

$$Z_g = \bar{H} \ln \frac{p_0}{p_g} \quad (3.30)$$

which can be solved to obtain the sea-level pressure

$$p_0 = p_g \exp\left(\frac{Z_g}{\bar{H}}\right) = p_g \exp\left(\frac{g_0 Z_g}{R_d \bar{T}_v}\right) \quad (3.31)$$

If  $Z_g$  is small, the scale height  $\bar{H}$  can be evaluated from the ground temperature. Also, if  $Z_g / \bar{H} \ll 1$ , the exponential in (3.31) can be approximated by  $1 + Z_g / \bar{H}$ , in which case (3.31) becomes

$$p_0 - p_g \approx p_g \frac{Z_g}{\bar{H}} = p_g \left(\frac{g_0 Z_g}{R_d \bar{T}_v}\right) \quad (3.32)$$

Because  $p_g \approx 1000$  hPa and  $\bar{H} \approx 8000$  m, the pressure correction (in hPa) is roughly equal to  $Z_g$  (in

meters) divided by 8. In other words, for altitudes up to a few hundred meters above (or below) sea level, the pressure decreases by about 1 hPa for every 8 m of vertical ascent.

## 3.3 The First Law of Thermodynamics<sup>15</sup>

In addition to the macroscopic kinetic and potential energy that a system as a whole may possess, it also contains *internal energy* due to the kinetic and potential energy of its molecules or atoms. Increases in internal kinetic energy in the form of molecular motions are manifested as increases in the temperature of the system, whereas changes in the potential energy of the molecules are caused by changes in their relative positions by virtue of any forces that act between the molecules.

Let us suppose that a closed system<sup>16</sup> of unit mass takes in a certain quantity of thermal energy  $q$  (measured in joules), which it can receive by thermal conduction and/or radiation. As a result the system may do a certain amount of *external work*  $w$  (also measured in joules). The excess of the energy supplied to the body over and above the external work done by the body is  $q - w$ . Therefore, if there is no change in the macroscopic kinetic and potential energy of the body, it follows from the principle of conservation of energy that the internal energy of the system must increase by  $q - w$ . That is,

$$q - w = u_2 - u_1 \quad (3.33)$$

where  $u_1$  and  $u_2$  are the internal energies of the system before and after the change. In differential form (3.33) becomes

$$dq - dw = du \quad (3.34)$$

where  $dq$  is the differential increment of heat added to the system,  $dw$  is the differential element

<sup>15</sup> The first law of thermodynamics is a statement of the conservation of energy, taking into account the conversions between the various forms that it can assume and the exchanges of energy between a system and its environment that can take place through the transfer of heat and the performance of mechanical work. A general formulation of the first law of thermodynamics is beyond the scope of this text because it requires consideration of conservation laws, not only for energy, but also for momentum and mass. This section presents a simplified formulation that ignores the macroscopic kinetic and potential energy (i.e., the energy that air molecules possess by virtue of their height above sea level and their organized fluid motions). As it turns out, the expression for the first law of thermodynamics that emerges in this simplified treatment is identical to the one recovered from a more complete treatment of the conservation laws, as is done in J. R. Holton, *Introduction to Dynamic Meteorology*, 4th Edition, Academic Press, New York, 2004, pp. 146–149.

<sup>16</sup> A closed system is one in which the total amount of matter, which may be in the form of gas, liquid, solid or a mixture of these phases, is kept constant.

of work done by the system, and  $du$  is the differential increase in internal energy of the system. Equations (3.33) and (3.34) are statements of the *first law of thermodynamics*. In fact (3.34) provides a definition of  $du$ . The change in internal energy  $du$  depends only on the initial and final states of the system and is therefore independent of the manner by which the system is transferred between these two states. Such parameters are referred to as *functions of state*.<sup>17</sup>

To visualize the work term  $dw$  in (3.34) in a simple case, consider a substance, often called the *working substance*, contained in a cylinder of fixed cross-sectional area that is fitted with a movable, frictionless piston (Fig. 3.4). The volume of the substance is proportional to the distance from the base of the cylinder to the face of the piston and can be represented on the horizontal axis of the graph shown in Fig. 3.4. The pressure of the substance in the cylinder can be represented on the vertical axis of this graph. Therefore, every state of the substance, corresponding to a given position of the piston, is represented by a point on this pressure–volume ( $p$ – $V$ ) diagram. When the substance is in equilibrium at a state represented by point  $P$  on the graph, its pressure is  $p$  and its volume is  $V$  (Fig. 3.4). If the piston moves outward through an incremental distance  $dx$  while its pressure remains essentially constant at  $p$ , the work  $dW$  done by the substance in pushing the external force  $F$  through a distance  $dx$  is

$$dW = Fdx$$

or, because  $F = pA$  where  $A$  is the cross-sectional area of the face of the piston,

$$dW = pA dx = pdV \quad (3.35)$$

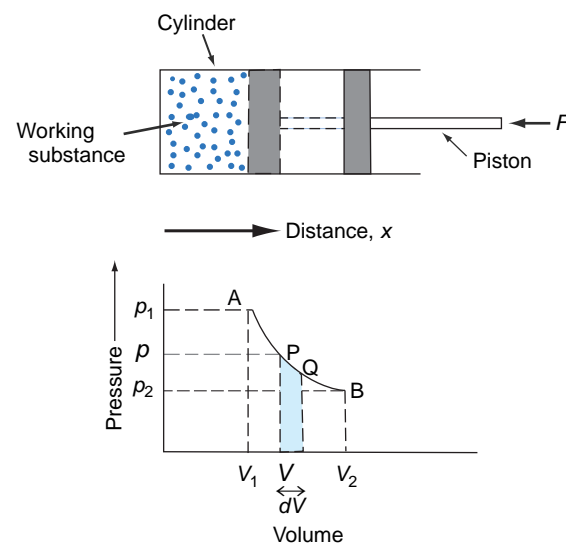
In other words, the work done by the substance when its volume increases by a small increment  $dV$  is equal to the pressure of the substance multiplied by its increase in volume, which is equal to the blue-shaded area in the graph shown in Fig. 3.4; that is, it is equal to the area under the curve  $PQ$ .

When the substance passes from state  $A$  with volume  $V_1$  to state  $B$  with volume  $V_2$  (Fig. 3.4), during which its pressure  $p$  changes, the work  $W$  done by the material is equal to the area under the curve  $AB$ . That is,

$$W = \int_{V_1}^{V_2} pdV \quad (3.36)$$

Equations (3.35) and (3.36) are quite general and represent work done by any substance (or system) due to a change in its volume. If  $V_2 > V_1$ ,  $W$  is positive, indicating that the *substance does work on its environment*. If  $V_2 < V_1$ ,  $W$  is negative, which indicates that the *environment does work on the substance*.

The  $p$ – $V$  diagram shown in Fig. 3.4 is an example of a *thermodynamic diagram* in which the physical state of a substance is represented by two thermodynamic variables. Such diagrams are very useful in meteorology; we will discuss other examples later in this chapter.



**Fig. 3.4** Representation of the state of a working substance in a cylinder on a  $p$ – $V$  diagram. The work done by the working substance in passing from  $P$  to  $Q$  is  $p dV$ , which is equal to the blue-shaded area. [Reprinted from *Atmospheric Science: An Introductory Survey*, 1st Edition, J. M. Wallace and P. V. Hobbs, p. 62, Copyright 1977, with permission from Elsevier.]

<sup>17</sup> Neither the heat  $q$  nor the work  $w$  are functions of state, since their values depend on *how* a system is transformed from one state to another. For example, a system may or may not receive heat and it may or may not do external work as it undergoes transitions between different states.

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If we are dealing with a unit mass of a substance, the volume  $V$  is replaced by the specific volume  $\alpha$ . Therefore, the work  $dw$  that is done when the specific volume increases by  $d\alpha$  is

$$dw = p d\alpha \quad (3.37)$$

Combination of (3.34) and (3.37) yields

$$dq = du + p d\alpha \quad (3.38)$$

which is an alternative statement of the first law of thermodynamics.<sup>18</sup>

### 3.3.1 Joule's Law

Following a series of laboratory experiments on air, Joule<sup>19</sup> concluded in 1848 that when a gas expands without doing external work, by expanding into a chamber that has been evacuated, and without taking in or giving out heat, the temperature of the gas does

not change. This statement, which is known as *Joule's law*, is strictly true only for an ideal gas, but air (and many other gases) behaves very similarly to an ideal gas over a wide range of conditions.

Joule's law leads to an important conclusion concerning the internal energy of an ideal gas. If a gas neither does external work nor takes in or gives out heat,  $dw = 0$  and  $dq = 0$  in (3.38), so that  $du = 0$ . Also, according to Joule's law, under these conditions the temperature of the gas does not change, which implies that the kinetic energy of the molecules remains constant. Therefore, because the total internal energy of the gas is constant, that part of the internal energy due to the potential energy must also remain unchanged, even though the volume of the gas changes. In other words, the internal energy of an ideal gas is independent of its volume if the temperature is kept constant. This can be the case only if the molecules of an ideal gas do not exert forces on each other. In this case, the internal energy of an ideal gas will depend only on its temperature.<sup>20</sup>

### 3.2 More Handball?

Box 3.1. showed that the gas laws can be illustrated by picturing the molecules of a gas as elastic balls bouncing around randomly in a handball court. Suppose now that the walls of the court are permitted to move outward when subjected to a force. The force on the walls is supplied by the impact of the balls, and the work required to move the walls outward comes from a decrease in the kinetic energy of the balls that rebound from the walls with lower velocities

than they struck them. This decrease in kinetic energy is in accordance with the first law of thermodynamics under adiabatic conditions. The work done by the system by pushing the walls outward is equal to the decrease in the internal energy of the system [see (3.38)]. Of course, if the outside of the walls of the court are bombarded by balls in a similar manner to the inside walls, there will be no net force on the walls and no work will be done.

<sup>18</sup> We have assumed here that the only work done by or on a system is due to a change in the volume of the system. However, there are other ways in which a system may do work, e.g., by the creation of new surface area between two phases (such as between liquid and air when a soap film is formed). Unless stated otherwise, we will assume that the work done by or on a system is due entirely to changes in the volume of the system.

<sup>19</sup> **James Prescott Joule** (1818–1889) Son of a wealthy English brewer; one of the great experimentalists of the 19th century. He started his scientific work (carried out in laboratories in his home and at his own expense) at age 19. He measured the mechanical equivalent of heat, recognized the dynamical nature of heat, and developed the principle of conservation of energy.

<sup>20</sup> Subsequent experiments carried out by Lord Kelvin<sup>21</sup> revealed the existence of small forces between the molecules of a gas.

<sup>21</sup> **Lord Kelvin 1st Baron (William Thomson)** (1824–1907) Scottish mathematician and physicist. Entered Glasgow University at age 11. At 22 became Professor of Natural Philosophy at the same university. Carried out incomparable work in thermodynamics, electricity, and hydrodynamics.

### 3.3.2 Specific Heats

Suppose a small quantity of heat  $dq$  is given to a unit mass of a material and, as a consequence, the temperature of the material increases from  $T$  to  $T + dT$  without any changes in phase occurring within the material. The ratio  $dq/dT$  is called the *specific heat* of the material. The specific heat defined in this way could have any number of values, depending on how the material changes as it receives the heat. If the volume of the material is kept constant, a *specific heat at constant volume*  $c_v$  is defined

$$c_v = \left( \frac{dq}{dT} \right)_{v \text{ const}} \quad (3.39)$$

However, if the volume of the material is constant (3.38) becomes  $dq = du$ . Therefore

$$c_v = \left( \frac{du}{dT} \right)_{v \text{ const}}$$

For an ideal gas, Joule's law applies and therefore  $u$  depends only on temperature. Therefore, regardless of whether the volume of a gas changes, we may write

$$c_v = \left( \frac{du}{dT} \right) \quad (3.40)$$

From (3.38) and (3.40), the first law of thermodynamics for an ideal gas can be written in the form<sup>22</sup>

$$dq = c_v dT + p d\alpha \quad (3.41)$$

Because  $u$  is a function of state, no matter how the material changes from state 1 to state 2, the change in its internal energy is, from (3.40),

$$u_2 - u_1 = \int_{T_1}^{T_2} c_v dT$$

We can also define a specific heat at constant pressure  $c_p$

$$c_p = \left( \frac{dq}{dT} \right)_{p \text{ const}} \quad (3.42)$$

where the material is allowed to expand as heat is added to it and its temperature rises, but its pressure remains constant. In this case, a certain amount of the heat added to the material will have to be expended to do work as the system expands against the constant pressure of its environment. Therefore, a larger quantity of heat must be added to the material to raise its temperature by a given amount than if the volume of the material were kept constant. For the case of an ideal gas, this inequality can be seen mathematically as follows. Equation (3.41) can be rewritten in the form

$$dq = c_v dT + d(p\alpha) - \alpha dp \quad (3.43)$$

From the ideal gas equation (3.3),  $d(p\alpha) = R dT$ . Therefore (3.43) becomes

$$dq = (c_v + R) dT - \alpha dp \quad (3.44)$$

At constant pressure, the last term in (3.44) vanishes; therefore, from (3.42) and (3.44),

$$c_p = c_v + R \quad (3.45)$$

The specific heats at constant volume and at constant pressure for dry air are 717 and 1004 J K<sup>-1</sup> kg<sup>-1</sup>, respectively, and the difference between them is 287 J K<sup>-1</sup> kg<sup>-1</sup>, which is the gas constant for dry air. It can be shown that for ideal monatomic gases  $c_p:c_v:R = 5:3:2$ , and for ideal diatomic gases  $c_p:c_v:R = 7:5:2$ .

By combining (3.44) and (3.45) we obtain an alternate form of the first law of thermodynamics:

$$dq = c_p dT - \alpha dp \quad (3.46)$$

### 3.3.3 Enthalpy

If heat is added to a material at constant pressure so that the specific volume of the material increases from  $\alpha_1$  to  $\alpha_2$ , the work done by a unit mass of the material is  $p(\alpha_2 - \alpha_1)$ . Therefore, from (3.38), the finite quantity of heat  $\Delta q$  added to

<sup>22</sup> The term  $dq$  is sometimes called the *diabatic* (or nonadiabatic) *heating* or *cooling*, where "diabatic" means involving the transfer of heat. The term "diabatic" would be redundant if "heating" and "cooling" were always taken to mean "the addition or removal of heat." However, "heating" and "cooling" are often used in the sense of "to raise or lower the temperature of," in which case it is meaningful to distinguish between that part of the temperature change  $dT$  due to diabatic effects ( $dq$ ) and that part due to adiabatic effects ( $p d\alpha$ ).

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a unit mass of the material at constant pressure is given by

$$\begin{aligned}\Delta q &= (u_2 - u_1) + p(\alpha_2 - \alpha_1) \\ &= (u_2 + p\alpha_2) - (u_1 + p\alpha_1)\end{aligned}$$

where  $u_1$  and  $u_2$  are, respectively, the initial and final internal energies for a unit mass of the material. Therefore, at constant pressure,

$$\Delta q = h_2 - h_1$$

where  $h$  is the *enthalpy* of a unit mass of the material, which is defined by

$$h \equiv u + p\alpha \quad (3.47)$$

Because  $u$ ,  $p$ , and  $\alpha$  are functions of state,  $h$  is a function of state. Differentiating (3.47), we obtain

$$dh = du + d(p\alpha)$$

Substituting for  $du$  from (3.40) and combining with (3.43), we obtain

$$dq = dh - \alpha dp \quad (3.48)$$

which is yet another form of the first law of thermodynamics.

By comparing (3.46) and (3.48) we see that

$$dh = c_p dT \quad (3.49)$$

or, in integrated form,

$$h = c_p T \quad (3.50)$$

where  $h$  is taken as zero when  $T = 0$ . In view of (3.50),  $h$  corresponds to the heat required to raise the temperature of a material from 0 to  $T$  K at constant pressure.

When a layer of air that is at rest and in hydrostatic balance is heated, for example, by radiative transfer, the weight of the overlying air pressing down on it

remains constant. Hence, the heating is at constant pressure. The energy added to the air is realized in the form of an increase in enthalpy (or *sensible heat*, as atmospheric scientists commonly refer to it) and

$$dq = dh = c_p dT$$

The air within the layer expands as it warms, doing work on the overlying air by lifting it against the Earth's gravitational attraction. Of the energy per unit mass imparted to the air by the heating, we see from (3.40) and (3.41) that  $du = c_v dT$  is reflected in an increase in internal energy and  $p d\alpha = R dT$  is expended doing work on the overlying air. Because the Earth's atmosphere is made up mainly of the diatomic gases  $N_2$  and  $O_2$ , the energy added by the heating  $dq$  is partitioned between the increase in internal energy  $du$  and the expansion work  $p d\alpha$  in the ratio 5:2.

We can write a more general expression that is applicable to a moving air parcel, the pressure of which changes as it rises or sinks relative to the surrounding air. By combining (3.20), (3.48), and (3.50) we obtain

$$dq = d(h + \Phi) = d(c_p T + \Phi) \quad (3.51)$$

Hence, if the material is a parcel of air with a fixed mass that is moving about in an hydrostatic atmosphere, the quantity  $(h + \Phi)$ , which is called the *dry static energy*, is constant provided the parcel neither gains nor loses heat (i.e.,  $dq = 0$ ).<sup>23</sup>

### 3.4 Adiabatic Processes

If a material undergoes a change in its physical state (e.g., its pressure, volume, or temperature) without any heat being added to it or withdrawn from it, the change is said to be *adiabatic*.

Suppose that the initial state of a material is represented by the point A on the  $p$ - $V$  diagram in Fig. 3.5 and that when the material undergoes an isothermal transformation it moves along the line AB. If the same material underwent a similar change in volume but under adiabatic conditions, the transformation would

<sup>23</sup> Strictly speaking, Eq. (3.51) holds only for an atmosphere in which there are no fluid motions. However, it is correct to within a few percent for the Earth's atmosphere where the kinetic energy of fluid motions represents only a very small fraction of the total energy. An exact relationship can be obtained by using Newton's second law of motion and the continuity equation in place of Eq. (3.20) in the derivation. See J. R. Holton, *An Introduction to Dynamic Meteorology*, 4th ed., Academic Press, pp. 46–49 (2004).



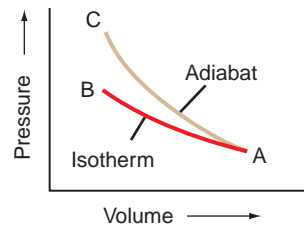


Fig. 3.5 An isotherm and an adiabat on a  $p$ - $V$  diagram.

be represented by a curve such as AC, which is called an *adiabat*. The reason why the adiabat AC is steeper than the isotherm AB on a  $p$ - $V$  diagram can be seen as follows. During adiabatic compression, the internal energy increases [because  $dq = 0$  and  $p d\alpha$  is negative in (3.38)] and therefore the temperature of the system rises. However, for isothermal compression, the temperature remains constant. Hence,  $T_C > T_B$  and therefore  $p_C > p_B$ .

### 3.4.1 Concept of an Air Parcel

In many fluid mechanics problems, mixing is viewed as a result of the random motions of individual molecules. In the atmosphere, molecular mixing is important only within a centimeter of the Earth's surface and at levels above the turbopause ( $\sim 105$  km). At intermediate levels, virtually all mixing in the vertical is accomplished by the exchange of macroscale "air parcels" with horizontal dimensions ranging from millimeters to the scale of the Earth itself.

To gain some insights into the nature of vertical mixing in the atmosphere, it is useful to consider the behavior of an air parcel of infinitesimal dimensions that is assumed to be

- i. thermally insulated from its environment so that its temperature changes adiabatically as it rises or sinks, always remaining at exactly the same pressure as the environmental air at the same level,<sup>24</sup> which is assumed to be in hydrostatic equilibrium; and
- ii. moving slowly enough that the macroscopic kinetic energy of the air parcel is a negligible fraction of its total energy.

Although in the case of real air parcels one or more of these assumptions is nearly always violated

to some extent, this simple, idealized model is helpful in understanding some of the physical processes that influence the distribution of vertical motions and vertical mixing in the atmosphere.

### 3.4.2 The Dry Adiabatic Lapse Rate

We will now derive an expression for the rate of change of temperature with height of a parcel of dry air that moves about in the Earth's atmosphere while always satisfying the conditions listed at the end of Section 3.4.1. Because the air parcel undergoes only adiabatic transformations ( $dq = 0$ ) and the atmosphere is in hydrostatic equilibrium, for a unit mass of air in the parcel we have, from (3.51),

$$d(c_p T + \Phi) = 0 \quad (3.52)$$

Dividing through by  $dz$  and making use of (3.20) we obtain

$$-\left(\frac{dT}{dz}\right)_{\text{dry parcel}} = \frac{g}{c_p} \equiv \Gamma_d \quad (3.53)$$

where  $\Gamma_d$  is called the *dry adiabatic lapse rate*. Because an air parcel expands as it rises in the atmosphere, its temperature will decrease with height so that  $\Gamma_d$  defined by (3.53) is a positive quantity. Substituting  $g = 9.81 \text{ m s}^{-2}$  and  $c_p = 1004 \text{ J K}^{-1} \text{ kg}^{-1}$  into (3.53) gives  $\Gamma_d = 0.0098 \text{ K m}^{-1}$  or  $9.8 \text{ K km}^{-1}$ , which is the numerical value of the dry adiabatic lapse rate.

It should be emphasized again that  $\Gamma_d$  is the rate of change of temperature following a parcel of dry air that is being raised or lowered adiabatically in the atmosphere. The actual lapse rate of temperature in a column of air, which we will indicate by  $\Gamma = \partial T / \partial z$ , as measured, for example, by a radiosonde, averages  $6\text{--}7 \text{ K km}^{-1}$  in the troposphere, but it takes on a wide range of values at individual locations.

### 3.4.3 Potential Temperature

The *potential temperature*  $\theta$  of an air parcel is defined as the temperature that the parcel of air would have if it were expanded or compressed adiabatically from its existing pressure and temperature to a standard pressure  $p_0$  (generally taken as 1000 hPa).

<sup>24</sup> Any pressure differences between the parcel and its environment give rise to sound waves that produce an almost instantaneous adjustment. Temperature differences, however, are eliminated by much slower processes.

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We can derive an expression for the potential temperature of an air parcel in terms of its pressure  $p$ , temperature  $T$ , and the standard pressure  $p_0$  as follows. For an adiabatic transformation ( $dq = 0$ ) (3.46) becomes

$$c_p dT - \alpha dp = 0$$

Substituting  $\alpha$  from (3.3) into this expression yields

$$\frac{c_p}{R} \frac{dT}{T} - \frac{dp}{p} = 0$$

Integrating upward from  $p_0$  (where, by definition,  $T = \theta$ ) to  $p$ , we obtain

$$\frac{c_p}{R} \int_{\theta}^T \frac{dT}{T} = \int_{p_0}^p \frac{dp}{p}$$

or

$$\frac{c_p}{R} \ln \frac{T}{\theta} = \ln \frac{p}{p_0}$$

Taking the antilog of both sides

$$\left(\frac{T}{\theta}\right)^{c_p/R} = \frac{p}{p_0}$$

or

$$\theta = T \left(\frac{p_0}{p}\right)^{R/c_p} \quad (3.54)$$

Equation (3.54) is called *Poisson's<sup>25</sup> equation*. It is usually assumed that  $R \approx R_d = 287 \text{ J K}^{-1} \text{ kg}^{-1}$  and  $c_p \approx c_{pd} = 1004 \text{ J K}^{-1} \text{ kg}^{-1}$ ; therefore,  $R/c_p \approx 0.286$ .

Parameters that remain constant during certain transformations are said to be *conserved*. Potential temperature is a conserved quantity for an air parcel that moves around in the atmosphere under adiabatic conditions (see Exercise 3.36). Potential temperature is an extremely useful parameter in atmospheric thermodynamics, since atmospheric processes are often close to adiabatic, and therefore  $\theta$  remains essentially constant, like density in an incompressible fluid.

### 3.4.4 Thermodynamic Diagrams

Poisson's equation may be conveniently solved in graphical form. If pressure is plotted on the ordinate on a distorted scale, in which the distance from the origin is proportional to  $p^{R_d/c_p}$ , or  $p^{0.286}$  is used, regardless of whether air is dry or moist, and temperature (in K) is plotted on the abscissa, then (3.54) becomes

$$p^{0.286} = \left(\frac{p_0^{0.286}}{\theta}\right) T \quad (3.55)$$

For a constant value of  $\theta$ , Eq. (3.55) is of the form  $y \propto x$  where  $y = p^{0.286}$ ,  $x = T$ , and the constant of proportionality is  $p_0^{0.286}/\theta$ . Each constant value of  $\theta$  represents a dry adiabat, which is defined by a straight line with a particular slope that passes through the point  $p = 0$ ,  $T = 0$ . If the pressure scale is inverted so that  $p$  increases downward, the relation takes the form shown in Fig. 3.6, which is the basis for the *pseudoadiabatic chart* that used to be widely used for meteorological computations. The region of the chart of greatest interest in the atmosphere is the portion shown within the dotted lines in Fig. 3.6, and this is generally the only portion of the chart that is printed.

In the pseudoadiabatic chart, isotherms are vertical and dry adiabats (constant  $\theta$ ) are oriented at an acute angle relative to isotherms (Fig. 3.6). Because changes in temperature with height in the atmosphere generally lie between isothermal and dry adiabatic, most temperature soundings lie within a narrow range of angles when plotted on a pseudoadiabatic chart. This restriction is overcome in the so-called *skew  $T - \ln p$  chart*, in which the ordinate ( $y$ ) is  $-\ln p$  (the minus sign ensures that lower pressure levels are located above higher pressure levels on the chart) and the abscissa ( $x$ ) is

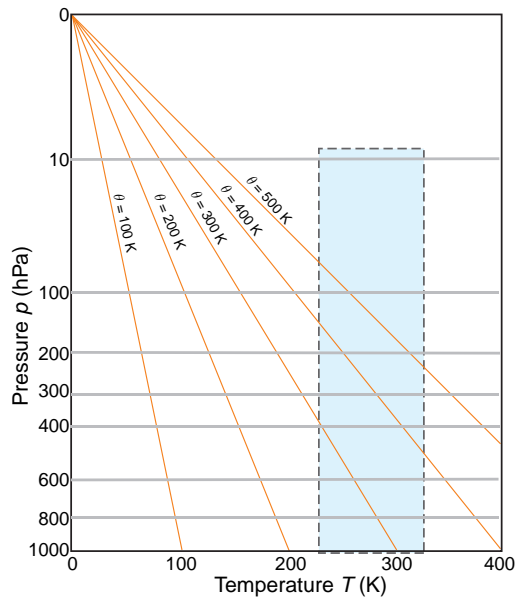
$$x = T + (\text{constant})y = T - (\text{constant}) \ln p \quad (3.56)$$

Since, from (3.56),

$$y = \frac{x - T}{(\text{constant})}$$

and for an isotherm  $T$  is constant, the relationship between  $y$  and  $x$  for an isotherm is of the form

<sup>25</sup> **Simeon Denis Poisson** (1781–1840) French mathematician. Studied medicine but turned to applied mathematics and became the first professor of mechanics at the Sorbonne in Paris.



**Fig. 3.6** The complete pseudoadiabatic chart. Note that  $p$  increases downward and is plotted on a distorted scale (representing  $p^{0.286}$ ). Only the blue-shaded area is generally printed for use in meteorological computations. The sloping lines, each labeled with a value of the potential temperature  $\theta$ , are dry adiabats. As required by the definition of  $\theta$ , the actual temperature of the air (given on the abscissa) at 1000 hPa is equal to its potential temperature.

$y = mx + c$ , where  $m$  is the same for all isotherms and  $c$  is a different constant for each isotherm. Therefore, on the skew  $T - \ln p$  chart, isotherms are straight parallel lines that slope upward from left to right. The scale for the  $x$  axis is generally chosen to make the angle between the isotherms and the isobars about  $45^\circ$ , as depicted schematically in Fig. 3.7. Note that the isotherms on a skew  $T - \ln p$  chart are intentionally “skewed” by about  $45^\circ$  from their vertical orientation in the pseudoadiabatic chart (hence the name *skew  $T - \ln p$  chart*). From (3.55), the equation for a dry adiabat ( $\theta$  constant) is

$$-\ln p = (\text{constant}) \ln T + \text{constant}$$

Hence, on a  $-\ln p$  versus  $\ln T$  chart, dry adiabats would be straight lines. Since  $-\ln p$  is the ordinate on the skew  $T - \ln p$  chart, but the abscissa is not  $\ln T$ , dry adiabats on this chart are slightly curved lines that run from the lower right to the upper left. The angle between the isotherms and the dry adiabats on a skew  $T - \ln p$  chart is approximately  $90^\circ$  (Fig. 3.7). Therefore, when atmospheric temperature soundings are plotted on this chart, small differences in slope



**Fig. 3.7** Schematic of a portion of the skew  $T - \ln p$  chart. (An accurate reproduction of a larger portion of the chart is available on the book web site that accompanies this book, from which it can be printed and used for solving exercises.)

are more apparent than they are on the pseudoadiabatic chart.

**Exercise 3.5** A parcel of air has a temperature of  $-51^\circ\text{C}$  at the 250-hPa level. What is its potential temperature? What temperature will the parcel have if it is brought into the cabin of a jet aircraft and compressed adiabatically to a cabin pressure of 850 hPa?

**Solution:** This exercise can be solved using the skew  $T - \ln p$  chart. Locate the original state of the air parcel on the chart at pressure 250 hPa and temperature  $-51^\circ\text{C}$ . The label on the dry adiabat that passes through this point is  $60^\circ\text{C}$ , which is therefore the potential temperature of the air.

The temperature acquired by the ambient air if it is compressed adiabatically to a pressure of 850 hPa can be found from the chart by following the dry adiabat that passes through the point located by 250 hPa and  $-51^\circ\text{C}$  down to a pressure of 850 hPa and reading off the temperature at that point. It is  $44.5^\circ\text{C}$ . (Note that this suggests that ambient air brought into the cabin of a jet aircraft at cruise altitude has to be *cooled* by about  $20^\circ\text{C}$  to provide a comfortable environment.) ■

### 3.5 Water Vapor in Air

So far we have indicated the presence of water vapor in the air through the vapor pressure  $e$  that it exerts, and we have quantified its effect on the density of air by introducing the concept of virtual temperature. However, the amount of water vapor present in a certain quantity of air may be expressed in many different ways, some of the more important



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of which are presented later. We must also discuss what happens when water vapor condenses in air.

### 3.5.1 Moisture Parameters

#### a. Mixing ratio and specific humidity

The amount of water vapor in a certain volume of air may be defined as the ratio of the mass  $m_v$  of water vapor to the mass of dry air; this is called the *mixing ratio*  $w$ . That is

$$w \equiv \frac{m_v}{m_d} \quad (3.57)$$

The mixing ratio is usually expressed in grams of water vapor per kilogram of dry air (but in solving numerical exercises  $w$  must be expressed as a dimensionless number, e.g., as kg of water vapor per kg of dry air). In the atmosphere, the magnitude of  $w$  typically ranges from a few grams per kilogram in middle latitudes to values of around 20 g kg<sup>-1</sup> in the tropics. If neither condensation nor evaporation takes place, the mixing ratio of an air parcel is constant (i.e., it is a conserved quantity).

The mass of water vapor  $m_v$  in a unit mass of air (dry air plus water vapor) is called the *specific humidity*  $q$ , that is

$$q \equiv \frac{m_v}{m_v + m_d} = \frac{w}{1 + w}$$

Because the magnitude of  $w$  is only a few percent, it follows that the numerical values of  $w$  and  $q$  are nearly equivalent.

**Exercise 3.6** If air contains water vapor with a mixing ratio of 5.5 g kg<sup>-1</sup> and the total pressure is 1026.8 hPa, calculate the vapor pressure  $e$ .

**Solution:** The partial pressure exerted by any constituent in a mixture of gases is proportional to the number of moles of the constituent in the mixture. Therefore, the pressure  $e$  due to water vapor in air is given by

$$e = \frac{n_v}{n_d + n_v} p = \frac{\frac{m_v}{M_w}}{\frac{m_d}{M_d} + \frac{m_v}{M_w}} p \quad (3.58)$$

$n_v$  and  $n_d$  are the number of moles of water vapor and dry air in the mixture, respectively,  $M_w$  is the molecular weight of water,  $M_d$  is the apparent molec-

ular weight of dry air, and  $p$  is the total pressure of the moist air. From (3.57) and (3.58) we obtain

$$e = \frac{w}{w + \varepsilon} p \quad (3.59)$$

where  $\varepsilon = 0.622$  is defined by (3.14). Substituting  $p = 1026.8$  hPa and  $w = 5.5 \times 10^{-3}$  kg kg<sup>-1</sup> into (3.59), we obtain  $e = 9.0$  hPa. ■

**Exercise 3.7** Calculate the virtual temperature correction for moist air at 30 °C that has a mixing ratio of 20 g kg<sup>-1</sup>.

**Solution:** Substituting  $e/p$  from (3.59) into (3.16) and simplifying

$$T_v = T \frac{w + \varepsilon}{\varepsilon(1 + w)}$$

Dividing the denominator into the numerator in this expression and neglecting terms in  $w^2$  and higher orders of  $w$ , we obtain

$$T_v - T \approx \frac{1 - \varepsilon}{\varepsilon} w T$$

or, substituting  $\varepsilon = 0.622$  and rearranging,

$$T_v \approx T(1 + 0.61w) \quad (3.60)$$

With  $T = 303$  K and  $w = 20 \times 10^{-3}$  kg kg<sup>-1</sup>, Eq. (3.60) gives  $T_v = 306.7$  K. Therefore, the virtual temperature correction is  $T_v - T = 3.7$  degrees (K or °C). Note that (3.60) is a useful expression for obtaining  $T_v$  from  $T$  and the moisture parameter  $w$ . ■

#### b. Saturation vapor pressures

Consider a small closed box, the floor of which is covered with pure water at temperature  $T$ . Initially assume that the air is completely dry. Water will begin to evaporate and, as it does, the number of water molecules in the box, and therefore the water vapor pressure, will increase. As the water vapor pressure increases, so will the rate at which the water molecules condense from the vapor phase back to the liquid phase. If the rate of condensation is less than the rate of evaporation, the box is said to be *unsaturated* at temperature  $T$  (Fig. 3.8a). When the water vapor pressure in the box increases to the point that the rate of condensation is equal to the rate of evaporation (Fig. 3.8b), the air is said to be *saturated with respect*

### 3.3 Can Air Be Saturated with Water Vapor?<sup>26</sup>

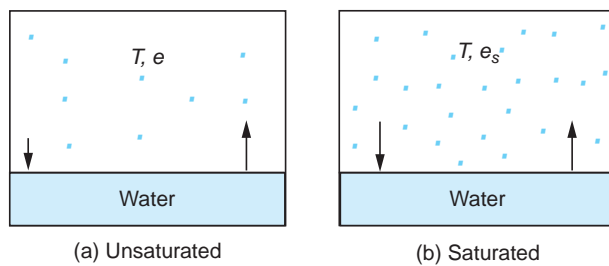
It is common to use phrases such as “the air is saturated with water vapor,” “the air can hold no more water vapor,” and “warm air can hold more water vapor than cold air.” These phrases, which suggest that air absorbs water vapor, rather like a sponge, are misleading. We have seen that the total pressure exerted by a mixture of gases is equal to the sum of the pressures that each gas would exert if it alone occupied the total volume of the mixture of gases (Dalton’s law of partial pressures). Hence, the exchange of water mole-

cules between its liquid and vapor phases is (essentially) independent of the presence of air. Strictly speaking, the pressure exerted by water vapor that is in equilibrium with water at a given temperature is referred more appropriately to as *equilibrium vapor pressure* rather than saturation vapor pressure at that temperature. However, the latter term, and the terms “unsaturated air” and “saturated air,” provide a convenient shorthand and are so deeply rooted that they will appear in this book.

to a plane surface of pure water at temperature  $T$ , and the pressure  $e_s$  that is then exerted by the water vapor is called the *saturation vapor pressure over a plane surface of pure water at temperature  $T$* .

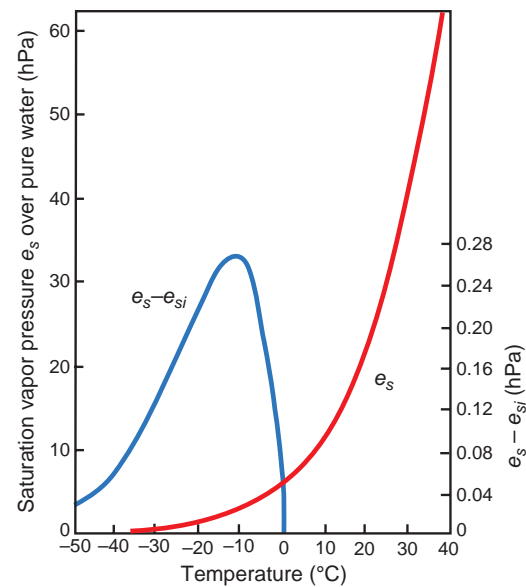
Similarly, if the water in Fig. 3.8 were replaced by a plane surface of pure ice at temperature  $T$  and the rate of condensation of water vapor were equal to the rate of evaporation of the ice, the pressure  $e_{si}$  exerted by the water vapor would be the saturation vapor pressure over a plane surface of pure ice at  $T$ . Because, at any given temperature, the rate of evaporation from ice is less than from water,  $e_s(T) > e_{si}(T)$ .

The rate at which water molecules evaporate from either water or ice increases with increasing temperature.<sup>27</sup> Consequently, both  $e_s$  and  $e_{si}$  increase with increasing temperature, and their magnitudes



**Fig. 3.8** A box (a) unsaturated and (b) saturated with respect to a plane surface of pure water at temperature  $T$ . Dots represent water molecules. Lengths of the arrows represent the relative rates of evaporation and condensation. The saturated (i.e., equilibrium) vapor pressure over a plane surface of pure water at temperature  $T$  is  $e_s$  as indicated in (b).

depend only on temperature. The variations with temperature of  $e_s$  and  $e_s - e_{si}$  are shown in Fig. 3.9, where it can be seen that the magnitude of  $e_s - e_{si}$  reaches a peak value at about  $-12^\circ\text{C}$ . It follows that if an ice particle is in water-saturated air it will grow due to the deposition of water vapor upon it. In Section 6.5.3 it is shown that this phenomenon



**Fig. 3.9** Variations with temperature of the saturation (i.e., equilibrium) vapor pressure  $e_s$  over a plane surface of pure water (red line, scale at left) and the difference between  $e_s$  and the saturation vapor pressure over a plane surface of ice  $e_{si}$  (blue line, scale at right).

<sup>26</sup> For further discussion of this and some other common misconceptions related to meteorology see C. F. Bohren’s *Clouds in a Glass of Beer*, Wiley and Sons, New York, 1987.

<sup>27</sup> As a rough rule of thumb, it is useful to bear in mind that the saturation vapor pressure roughly doubles for a  $10^\circ\text{C}$  increase in temperature.

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plays a role in the initial growth of precipitable particles in some clouds.

### c. Saturation mixing ratios

The *saturation mixing ratio*  $w_s$  with respect to water is defined as the ratio of the mass  $m_{vs}$  of water vapor in a given volume of air that is saturated with respect to a plane surface of pure water to the mass  $m_d$  of the dry air. That is

$$w_s \equiv \frac{m_{vs}}{m_d} \quad (3.61)$$

Because water vapor and dry air both obey the ideal gas equation

$$w_s = \frac{\rho'_{vs}}{\rho'_d} = \frac{e_s}{(R_v T)} \bigg/ \frac{(p - e_s)}{(R_d T)} \quad (3.62)$$

where  $\rho'_{vs}$  is the partial density of water vapor required to saturate air with respect to water at temperature  $T$ ,  $\rho'_d$  is the partial density of the dry air (see Section 3.1.1), and  $p$  is the total pressure. Combining (3.62) with (3.14), we obtain

$$w_s = 0.622 \frac{e_s}{p - e_s}$$

For the range of temperatures observed in the Earth's atmosphere,  $p \gg e_s$ ; therefore

$$w_s \approx 0.622 \frac{e_s}{p} \quad (3.63)$$

Hence, at a given temperature, the saturation mixing ratio is inversely proportional to the total pressure.

Because  $e_s$  depends only on temperature, it follows from (3.63) that  $w_s$  is a function of temperature and pressure. Lines of constant saturation mixing ratio are printed as dashed green lines on the skew  $T - \ln p$  chart and are labeled with the value of  $w_s$  in grams of water vapor per kilogram of dry air. It is apparent from the slope of these lines that at constant pressure  $w_s$  increases with increasing temperature, and at constant temperature  $w_s$  increases with decreasing pressure.

### d. Relative humidity; dew point and frost point

The relative humidity (RH) with respect to water is the ratio (expressed as a percentage) of the actual

mixing ratio  $w$  of the air to the saturation mixing ratio  $w_s$  with respect to a plane surface of pure water at the same temperature and pressure. That is

$$RH \equiv 100 \frac{w}{w_s} \approx 100 \frac{e}{e_s} \quad (3.64)$$

The *dew point*  $T_d$  is the temperature to which air must be cooled at constant pressure for it to become saturated with respect to a plane surface of pure water. In other words, the dew point is the temperature at which the saturation mixing ratio  $w_s$  with respect to liquid water becomes equal to the actual mixing ratio  $w$ . It follows that the relative humidity at temperature  $T$  and pressure  $p$  is given by

$$RH = 100 \frac{w_s(\text{at temperature } T_d \text{ and pressure } p)}{w_s(\text{at temperature } T \text{ and pressure } p)} \quad (3.65)$$

A simple rule of thumb for converting RH to a dew point depression ( $T - T_d$ ) for moist air ( $RH > 50\%$ ) is that  $T_d$  decreases by  $\sim 1^\circ\text{C}$  for every 5% decrease in RH (starting at  $T_d =$  dry bulb temperature ( $T$ ), where  $RH = 100\%$ ). For example, if the RH is 85%,  $T_d = T - \left(\frac{100 - 85}{5}\right)$  and the dew point depression is  $T - T_d = 3^\circ\text{C}$ .

The *frost point* is defined as the temperature to which air must be cooled at constant pressure to saturate it with respect to a plane surface of pure ice. Saturation mixing ratios and relative humidities with respect to ice may be defined in analogous ways to their definitions with respect to liquid water. When the terms mixing ratio and relative humidity are used without qualification they are with respect to liquid water.

**Exercise 3.8** Air at 1000 hPa and  $18^\circ\text{C}$  has a mixing ratio of  $6 \text{ g kg}^{-1}$ . What are the relative humidity and dew point of the air?

**Solution:** This exercise may be solved using a skew  $T - \ln p$  chart. The students should duplicate the following steps. First locate the point with pressure 1000 hPa and temperature  $18^\circ\text{C}$ . We see from the chart that the saturation mixing ratio for this state is  $\sim 13 \text{ g kg}^{-1}$ . Since the air specified in the problem has a mixing ratio of only  $6 \text{ g kg}^{-1}$ , it is unsaturated and its relative humidity is, from (3.64),  $100 \times 6/13 = 46\%$ . To find the dew point we move from right to

left along the 1000-hPa ordinate until we intercept the saturation mixing ratio line of magnitude  $6 \text{ g kg}^{-1}$ ; this occurs at a temperature of about  $6.5^\circ\text{C}$ . Therefore, if the air is cooled at constant pressure, the water vapor it contains will just saturate the air with respect to water at a temperature of  $6.5^\circ\text{C}$ . Therefore, by definition, the dew point of the air is  $6.5^\circ\text{C}$ . ■

At the Earth's surface, the pressure typically varies by only a few percent from place to place and from time to time. Therefore, the dew point is a good indicator of the moisture content of the air. In warm, humid weather the dew point is also a convenient indicator of the level of human discomfort. For example, most people begin to feel uncomfortable when the dew point rises above  $20^\circ\text{C}$ , and air with a dew point above about  $22^\circ\text{C}$  is generally regarded as extremely humid or "sticky." Fortunately, dew points much above this temperature are rarely observed even in the tropics. In contrast to the dew point, relative humidity depends as much upon the temperature of the air as upon its moisture content. On a sunny day the relative humidity may drop by as much as 50% from morning to afternoon, just because of a rise in air temperature. Neither is relative humidity a good indicator of the level of human discomfort. For example, a relative humidity of 70% may feel quite comfortable at a temperature of  $20^\circ\text{C}$ , but it would cause considerable discomfort to most people at a temperature of  $30^\circ\text{C}$ .

The highest dew points occur over warm bodies of water or vegetated surfaces from which water is evaporating. In the absence of vertical mixing, the air just above these surfaces would become saturated with water vapor, at which point the dew point would be the same as the temperature of the underlying surface. Complete saturation is rarely achieved over hot surfaces, but dew points in excess of  $25^\circ\text{C}$  are sometimes observed over the warmest regions of the oceans.

#### e. Lifting condensation level

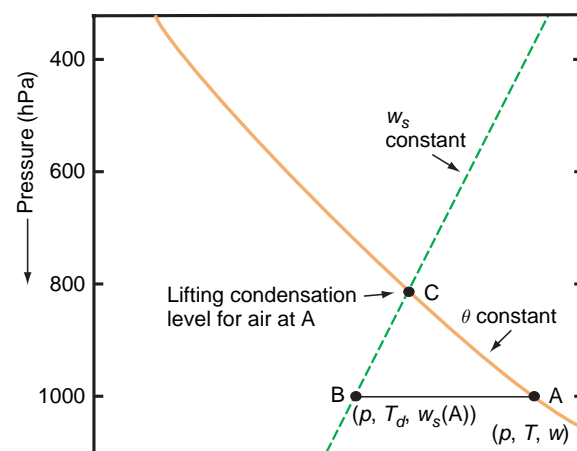
The *lifting condensation level* (LCL) is defined as the level to which an unsaturated (but moist) parcel of air can be lifted adiabatically before it becomes saturated with respect to a plane surface of pure water. During lifting the mixing ratio  $w$  and potential temperature  $\theta$  of the air parcel remain constant, but the saturation mixing ratio  $w_s$  decreases until it becomes equal to  $w$  at the LCL. Therefore, the LCL is located

at the intersection of the potential temperature line passing through the temperature  $T$  and pressure  $p$  of the air parcel, and the  $w_s$  line that passes through the pressure  $p$  and dew point  $T_d$  of the parcel (Fig. 3.10). Since the dew point and LCL are related in the manner indicated in Fig. 3.10, knowledge of either one is sufficient to determine the other. Similarly, a knowledge of  $T$ ,  $p$ , and any one moisture parameter is sufficient to determine all the other moisture parameters we have defined.

#### f. Wet-bulb temperature

The wet-bulb temperature is measured with a thermometer, the glass bulb of which is covered with a moist cloth over which ambient air is drawn. The heat required to evaporate water from the moist cloth to saturate the ambient air is supplied by the air as it comes into contact with the cloth. When the difference between the temperatures of the bulb and the ambient air is steady and sufficient to supply the heat needed to evaporate the water, the thermometer will read a steady temperature, which is called the *wet-bulb temperature*. If a raindrop falls through a layer of air that has a constant wet-bulb temperature, the raindrop will eventually reach a temperature equal to the wet-bulb temperature of the air.

The definition of wet-bulb temperature and dew point both involve cooling a hypothetical air parcel to saturation, but there is a distinct difference. If the unsaturated air approaching the wet bulb has a mixing ratio  $w$ , the dew point  $T_d$  is the temperature to which the air must be cooled at constant pressure



**Fig. 3.10** The lifting condensation level of a parcel of air at A, with pressure  $p$ , temperature  $T$ , and dew point  $T_d$ , is at C on the skew  $T - \ln p$  chart.

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to become saturated. The air that leaves the wet bulb has a mixing ratio  $w'$  that saturates it at temperature  $T_w$ . If the air approaching the wet bulb is unsaturated,  $w'$  is greater than  $w$ ; therefore,  $T_d \leq T_w \leq T$ , where the equality signs apply only to air saturated with respect to a plane surface of pure water. Usually  $T_w$  is close to the arithmetic mean of  $T$  and  $T_d$ .

### 3.5.2 Latent Heats

If heat is supplied to a system under certain conditions it may produce a change in phase rather than a change in temperature. In this case, the increase in internal energy is associated entirely with a change in molecular configurations in the presence of intermolecular forces rather than an increase in the kinetic energy of the molecules (and therefore the temperature of the system). For example, if heat is supplied to ice at 1 atm and 0 °C, the temperature remains constant until all of the ice has melted. The *latent heat of melting* ( $L_m$ ) is defined as the heat that has to be given to a unit mass of a material to convert it from the solid to the liquid phase without a change in temperature. The temperature at which this phase change occurs is called the *melting point*. At 1 atm and 0 °C the latent heat of melting of the water substance is  $3.34 \times 10^5 \text{ J kg}^{-1}$ . The *latent heat of freezing* has the same numerical value as the latent heat of melting, but heat is released as a result of the change in phase from liquid to solid.

Similarly, the *latent heat of vaporization* or *evaporation* ( $L_v$ ) is the heat that has to be given to a unit mass of material to convert it from the liquid to the vapor phase without a change in temperature. For the water substance at 1 atm and 100 °C (the *boiling point* of water at 1 atm), the latent heat of vaporization is  $2.25 \times 10^6 \text{ J kg}^{-1}$ . The *latent heat of condensation* has the same value as the latent heat of vaporization, but heat is released in the change in phase from vapor to liquid.<sup>28</sup>

As will be shown in Section 3.7.3, the melting point (and boiling point) of a material depends on pressure.

### 3.5.3 Saturated Adiabatic and Pseudoadiabatic Processes

When an air parcel rises in the atmosphere its temperature decreases with altitude at the dry adiabatic lapse rate (see Section 3.4.2) until it becomes saturated with water vapor. Further lifting results in the condensation of liquid water (or the deposition of ice), which releases latent heat. Consequently, the rate of decrease in the temperature of the rising parcel is reduced. If all of the condensation products remain in the rising parcel, the process may still be considered to be adiabatic (and reversible), even though latent heat is released in the system, provided that heat does not pass through the boundaries of the parcel. The air parcel is then said to undergo a *saturated adiabatic process*. However, if all of the condensation products immediately fall out of the air parcel, the process is irreversible, and not strictly adiabatic, because the condensation products carry some heat. The air parcel is then said to undergo a *pseudoadiabatic process*. As the reader is invited to verify in Exercise 3.44, the amount of heat carried by condensation products is small compared to that carried by the air itself. Therefore, the saturated-adiabatic and the pseudoadiabatic lapse rates are virtually identical.

### 3.5.4 The Saturated Adiabatic Lapse Rate

In contrast to the dry adiabatic lapse rate  $\Gamma_d$ , which is constant, the numerical value of the saturated adiabatic lapse rate  $\Gamma_s$  varies with pressure and temperature. (The reader is invited to derive an expression for  $\Gamma_s$  in Exercise 3.50; see the book Web site.) Because water vapor condenses when a saturated air parcel rises, it follows that  $\Gamma_s < \Gamma_d$ . Actual values of  $\Gamma_s$  range from about  $4 \text{ K km}^{-1}$  near the ground in warm, humid air masses to typical values of  $6\text{--}7 \text{ K km}^{-1}$  in the middle troposphere. For typical temperatures near the tropopause,  $\Gamma_s$  is only slightly less than  $\Gamma_d$  because the saturation vapor pressure of the air is so small that the effect of condensation is negligible.<sup>29</sup> Lines that show the rate of decrease in

<sup>28</sup> Normally, when heat is given to a substance, the temperature of the substance increases. This is called *sensible heat*. However, when heat is given to a substance that is melting or boiling, the temperature of the substance does not change until all of the substance is melted or vaporized. In this case, the heat appears to be *latent* (i.e., hidden). Hence the terms *latent heat of melting* and *latent heat of vaporization*.

<sup>29</sup> **William Thomson** (later Lord Kelvin) was the first (in 1862) to derive quantitative estimates of the dry and saturated adiabatic lapse rates based on theoretical arguments. For an interesting account of the contributions of other 19th-century scientists to the realization of the importance of latent heat in the atmosphere, see W. E. K. Middleton, *A History of the Theories of Rain*, Franklin Watts, Inc., New York, 1965, Chapter 8.



temperature with height of a parcel of air that is rising or sinking in the atmosphere under saturated adiabatic (or pseudoadiabatic) conditions are called *saturated adiabats* (or *pseudoadiabats*). On the skew  $T - \ln p$  chart these are the curved green lines that diverge upward and tend to become parallel to the dry adiabats.

**Exercise 3.9** A parcel of air with an initial temperature of  $15^\circ\text{C}$  and dew point  $2^\circ\text{C}$  is lifted adiabatically from the 1000-hPa level. Determine its LCL and temperature at that level. If the air parcel is lifted a further 200 hPa above its LCL, what is its final temperature and how much liquid water is condensed during this rise?

**Solution:** The student should duplicate the following steps on the skew  $T - \ln p$  chart (see the book Web site). First locate the initial state of the air on the chart at the intersection of the  $15^\circ\text{C}$  isotherm with the 1000-hPa isobar. Because the dew point of the air is  $2^\circ\text{C}$ , the magnitude of the saturation mixing ratio line that passes through the 1000-hPa pressure level at  $2^\circ\text{C}$  is the actual mixing ratio of the air at  $15^\circ\text{C}$  and 1000 hPa. From the chart this is found to be about  $4.4\text{ g kg}^{-1}$ . Because the saturation mixing ratio at 1000 hPa and  $15^\circ\text{C}$  is about  $10.7\text{ g kg}^{-1}$ , the air is initially unsaturated. Therefore, when it is lifted it will follow a dry adiabat (i.e., a line of constant potential temperature) until it intercepts the saturation mixing ratio line of magnitude  $4.4\text{ g kg}^{-1}$ . Following upward along the dry adiabat ( $\theta = 288\text{ K}$ ) that passes through 1000 hPa and  $15^\circ\text{C}$  isotherm, the saturation mixing ratio line of  $4.4\text{ g kg}^{-1}$  is intercepted at about the 820-hPa level. This is the LCL of the air parcel. The temperature of the air at this point is about  $-0.7^\circ\text{C}$ . For lifting above this level the air parcel will follow a saturated adiabat. Following the saturated adiabat that passes through 820 hPa and  $-0.7^\circ\text{C}$  up to the 620-hPa level, the final temperature of the air is found to be about  $-15^\circ\text{C}$ . The saturation mixing ratio at 620 hPa and  $-15^\circ\text{C}$  is  $\sim 1.9\text{ g kg}^{-1}$ . Therefore, about  $\sim 4.4 - 1.9 = 2.5\text{ g}$  of water must have condensed out of each kilogram of air during the rise from 820 to 620 hPa. ■

### 3.5.5 Equivalent Potential Temperature and Wet-Bulb Potential Temperature

We will now derive an equation that describes how temperature varies with pressure under conditions of

saturated adiabatic ascent or descent. Substituting (3.3) into (3.46) gives

$$\frac{dq}{T} = c_p \frac{dT}{T} - R \frac{dp}{p} \quad (3.66)$$

From (3.54) the potential temperature  $\theta$  is given by

$$\ln \theta = \ln T - \frac{R}{c_p} \ln p + \text{constant}$$

or, differentiating,

$$c_p \frac{d\theta}{\theta} = c_p \frac{dT}{T} - R \frac{dp}{p} \quad (3.67)$$

Combining (3.66) and (3.67) and substituting  $dq = -L_v dw_s$ , we obtain

$$-\frac{L_v}{c_p T} dw_s = \frac{d\theta}{\theta} \quad (3.68)$$

In Exercise 3.52 we show that

$$\frac{L_v}{c_p T} dw_s \approx d \left( \frac{L_v w_s}{c_p T} \right) \quad (3.69)$$

From (3.68) and (3.69)

$$-d \left( \frac{L_v w_s}{c_p T} \right) \approx \frac{d\theta}{\theta}$$

This last expression can be integrated to give

$$-\frac{L_v w_s}{c_p T} \approx \ln \theta + \text{constant} \quad (3.70)$$

We will define the constant of integration in (3.70) by requiring that at low temperatures, as  $w_s/T \rightarrow 0$ ,  $\theta \rightarrow \theta_e$ . Then

$$-\frac{L_v w_s}{c_p T} \approx \ln \left( \frac{\theta}{\theta_e} \right)$$

or

$$\theta_e \approx \theta \exp \left( \frac{L_v w_s}{c_p T} \right) \quad (3.71)$$

The quantity  $\theta_e$  given by (3.71) is called the *equivalent potential temperature*. It can be seen that  $\theta_e$  is the

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potential temperature  $\theta$  of a parcel of air when all the water vapor has condensed so that its saturation mixing ratio  $w_s$  is zero. Hence, recalling the definition of  $\theta$ , the equivalent potential temperature of an air parcel may be found as follows. The air is expanded (i.e., lifted) pseudoadiabatically until all the vapor has condensed, released its latent heat, and fallen out. The air is then compressed dry adiabatically to the standard pressure of 1000 hPa, at which point it will attain the temperature  $\theta_e$ . (If the air is initially unsaturated,  $w_s$  and  $T$  are the saturation mixing ratio and temperature at the point where the air first becomes saturated after being lifted dry adiabatically.) We have seen in Section 3.4.3 that potential temperature is a conserved quantity for adiabatic transformations. The equivalent potential temperature is conserved during both dry and saturated adiabatic processes.

If the line of constant equivalent potential temperature (i.e., the pseudoadiabat) that passes through the wet-bulb temperature of a parcel of air is traced back on a skew  $T - \ln p$  chart to the point where it intersects the 1000-hPa isobar, the temperature at this intersection is called the *wet-bulb potential temperature*  $\theta_w$  of the air parcel. Like the equivalent potential temperature, the wet-bulb potential temperature is conserved during both dry and saturated adiabatic processes. On skew  $T - \ln p$  charts, pseudoadiabats are labeled (along the 200-hPa isobar) with the wet-bulb potential temperature  $\theta_w$  (in °C) and the equivalent potential temperature  $\theta_e$  (in K) of air that rises or sinks along that pseudoadiabat. Both  $\theta_w$  and  $\theta_e$  provide equivalent information and are valuable as tracers of air parcels.

When height, rather than pressure, is used as the independent variable, the conserved quantity during adiabatic or pseudoadiabatic ascent or descent with water undergoing transitions between liquid and vapor phases is the *moist static energy* (MSE)<sup>30</sup>

$$MSE = c_p T + \Phi + L_v q \quad (3.72)$$

where  $T$  is the temperature of the air parcel,  $\Phi$  is the geopotential, and  $q_v$  is the specific humidity (nearly the same as  $w$ ). The first term on the right side of

(3.72) is the enthalpy per unit mass of air. The second term is the potential energy, and the third term is the latent heat content. The first two terms, which also appear in (3.51), are the *dry static energy*. When air is lifted dry adiabatically, enthalpy is converted into potential energy and the latent heat content remains unchanged. In saturated adiabatic ascent, energy is exchanged among all three terms on the right side of (3.72): potential energy increases, while the enthalpy and latent heat content both decrease. However, the sum of the three terms remains constant.

### 3.5.6 Normand's Rule

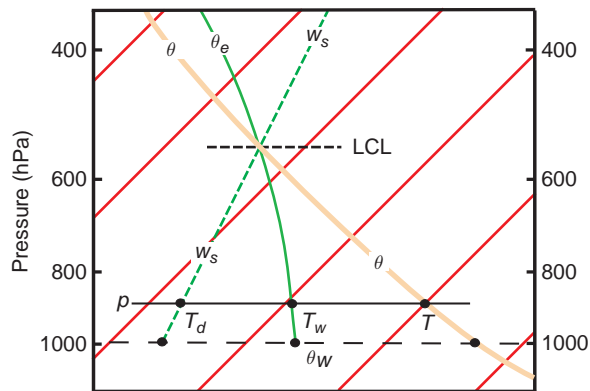
Many of the relationships discussed in this section are embodied in the following theorem, known as *Normand's*<sup>31</sup> *rule*, which is extremely helpful in many computations involving the skew  $T - \ln p$  chart. Normand's rule states that on a skew  $T - \ln p$  chart the lifting condensation level of an air parcel is located at the intersection of the potential temperature line that passes through the point located by the temperature and pressure of the air parcel, the equivalent potential temperature line (i.e., the pseudoadiabat) that passes through the point located by the wet-bulb temperature and pressure of the air parcel, and the saturation mixing ratio line that passes through the point determined by the dew point and pressure of the air. This rule is illustrated in Fig. 3.11 for the case of an air parcel with temperature  $T$ , pressure  $p$ , dew point  $T_d$ , and wet-bulb temperature  $T_w$ . It can be seen that if  $T$ ,  $p$ , and  $T_d$  are known,  $T_w$  may be readily determined using Normand's rule. Also, by extrapolating the  $\theta_e$  line that passes through  $T_w$  to the 1000-hPa level, the wet-bulb potential temperature  $\theta_w$  may be found (Fig. 3.11).

### 3.5.7 Net Effects of Ascent Followed by Descent

When a parcel of air is lifted above its LCL so that condensation occurs and if the products of the condensation fall out as precipitation, the latent heat gained by the air during this process will be retained by the air if the parcel returns to its original level.

<sup>30</sup> The word *static* derives from the fact that the kinetic energy associated with macroscale fluid motions is not included. The reader is invited to show that the kinetic energy per unit mass is much smaller than the other terms on the right side of (3.72), provided that the wind speed is small in comparison to the speed of sound.

<sup>31</sup> **Sir Charles William Blyth Normand** (1889–1982) British meteorologist. Director-General of Indian Meteorological Service, 1927–1944. A founding member of the National Science Academy of India. Improved methods for measuring atmospheric ozone.



**Fig. 3.11** Illustration of Normand's rule on the skew  $T - \ln p$  chart. The orange lines are isotherms. The method for determining the wet-bulb temperature ( $T_w$ ) and the wet-bulb potential temperature ( $\theta_w$ ) of an air parcel with temperature  $T$  and dew point  $T_d$  at pressure  $p$  is illustrated. LCL denotes the lifting condensation level of this air parcel.

The effects of the saturated ascent coupled with the adiabatic descent are:

- i. net increases in the temperature and potential temperature of the parcel;
- ii. a decrease in moisture content (as indicated by changes in the mixing ratio, relative humidity, dew point, or wet-bulb temperature); and,
- iii. no change in the equivalent potential temperature or wet-bulb potential temperature, which are conserved quantities for air parcels undergoing both dry and saturated processes.

The following exercise illustrates these points.

**Exercise 3.10** An air parcel at 950 hPa has a temperature of  $14^\circ\text{C}$  and a mixing ratio of  $8\text{ g kg}^{-1}$ . What is the wet-bulb potential temperature of the air? The air parcel is lifted to the 700-hPa level by passing over a mountain, and 70% of the water vapor that is condensed out by the ascent is removed by precipitation. Determine the temperature, potential temperature, mixing ratio, and wet-bulb potential temperature of the air parcel after it has descended to the 950-hPa level on the other side of the mountain.

**Solution:** On a skew  $T - \ln p$  chart (see the book Web site), locate the initial state of the air at 950 hPa and  $14^\circ\text{C}$ . The saturation mixing ratio for an air parcel with temperature and pressure is found from the chart to be  $10.6\text{ g kg}^{-1}$ . Therefore, because the air has a mixing ratio of only  $8\text{ g kg}^{-1}$ , it is unsaturated. The wet-bulb potential temperature ( $\theta_w$ ) can be determined using the method indicated schematically in Fig. 3.11, which is as follows. Trace the constant potential temperature line that passes through the initial state of the air parcel up to the point where it intersects the saturation mixing ratio line with value  $8\text{ g kg}^{-1}$ . This occurs at a pressure of about 890 hPa, which is the LCL of the air parcel. Now follow the equivalent potential temperature line that passes through this point back down to the 1000-hPa level and read off the temperature on the abscissa—it is  $14^\circ\text{C}$ . This is in the wet-bulb potential temperature of the air.

When the air is lifted over the mountain, its temperature and pressure up to the LCL at 890 hPa are given by points on the potential temperature line that passes through the point 950 hPa and  $14^\circ\text{C}$ . With further ascent of the air parcel to the 700-hPa level, the air follows the saturated adiabat that passes through the LCL. This saturated adiabat intersects the 700-hPa level at a point where the saturation mixing ratio is  $4.7\text{ g kg}^{-1}$ . Therefore,  $8 - 4.7 = 3.3\text{ g kg}^{-1}$  of water vapor has to condense out between the LCL and the 700-hPa level, and 70% of this, or  $2.3\text{ g kg}^{-1}$ , is precipitated out. Therefore, at the 700-hPa level  $1\text{ g kg}^{-1}$  of liquid water remains in the air. The air parcel descends on the other side of the mountain at the saturated adiabatic lapse rate until it evaporates all of its liquid water, at which point the saturation mixing ratio will have risen to  $4.7 + 1 = 5.7\text{ g kg}^{-1}$ . The air parcel is now at a pressure of 760 hPa and a temperature of  $1.8^\circ\text{C}$ . Thereafter, the air parcel descends along a dry adiabat to the 950-hPa level, where its temperature is  $20^\circ\text{C}$  and the mixing ratio is still  $5.7\text{ g kg}^{-1}$ . If the method indicated in Fig. 3.11 is applied again, the wet-bulb potential temperature of the air parcel will be found to be unchanged at  $14^\circ\text{C}$ . (The heating of air during its passage over a mountain,  $6^\circ\text{C}$  in this example, is responsible for the remarkable warmth of *Föhn* or *Chinook* winds, which often blow downward along the lee side of mountain ranges.<sup>32</sup>) ■

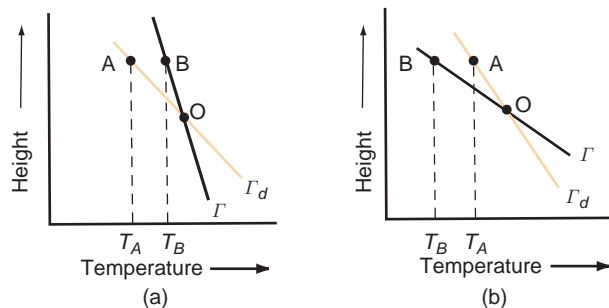
<sup>32</sup> The person who first explained the Föhn wind in this way appears to have been J. von Hann<sup>33</sup> in his classic book *Lehrbuch der Meteorologie*, Willibald Keller, Leipzig, 1901.

<sup>33</sup> **Julius F. von Hann** (1839–1921) Austrian meteorologist. Introduced thermodynamic principles into meteorology. Developed theories for mountain and valley winds. Published the first comprehensive treatise on climatology (1883).

## 3.6 Static Stability

### 3.6.1 Unsaturated Air

Consider a layer of the atmosphere in which the actual temperature lapse rate  $\Gamma$  (as measured, for example, by a radiosonde) is less than the dry adiabatic lapse rate  $\Gamma_d$  (Fig. 3.12a). If a parcel of unsaturated air originally located at level O is raised to the height defined by points A and B, its temperature will fall to  $T_A$ , which is lower than the ambient temperature  $T_B$  at this level. Because the parcel immediately adjusts to the pressure of the ambient air, it is clear from the ideal gas equation that the colder parcel of air must be denser than the warmer ambient air. Therefore, if left to itself, the parcel will tend to return to its original level. If the parcel is displaced downward from O it becomes warmer than the ambient air and, if left to itself, the parcel will tend to rise back to its original level. In both cases, the parcel of air encounters a restoring force after being displaced, which inhibits vertical mixing. Thus, the condition  $\Gamma < \Gamma_d$  corresponds to a *stable stratification* (or positive static stability) for unsaturated air parcels. In general, the larger the difference  $\Gamma_d - \Gamma$ , the greater the restoring force for a given displacement and the greater the static stability.<sup>34</sup>



**Fig. 3.12** Conditions for (a) positive static stability ( $\Gamma < \Gamma_d$ ) and (b) negative static instability ( $\Gamma > \Gamma_d$ ) for the displacement of unsaturated air parcels.

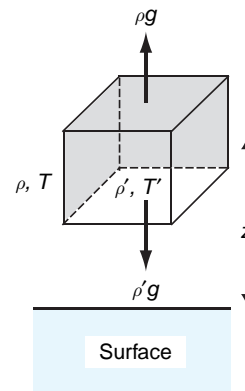
<sup>34</sup> A more general method for determining static stability is given in Section 9.3.4.

<sup>35</sup> **Archimedes** (287–212 B.C.) The greatest of Greek scientists. He invented engines of war and the water screw and he derived the principle of buoyancy named after him. When Syracuse was sacked by Rome, a soldier came upon the aged Archimedes absorbed in studying figures he had traced in the sand: “Do not disturb my circles” said Archimedes, but was killed instantly by the soldier. Unfortunately, right does not always conquer over might.

**Exercise 3.11** An unsaturated parcel of air has density  $\rho'$  and temperature  $T'$ , and the density and temperature of the ambient air are  $\rho$  and  $T$ . Derive an expression for the upward acceleration of the air parcel in terms of  $T$ ,  $T'$ , and  $g$ .

**Solution:** The situation is depicted in Fig. 3.13. If we consider a unit volume of the air parcel, its mass is  $\rho'$ . Therefore, the downward force acting on unit volume of the parcel is  $\rho'g$ . From the Archimedes<sup>35</sup> principle we know that the upward force acting on the parcel is equal in magnitude to the gravitational force that acts on the ambient air that is displaced by the air parcel. Because a unit volume of ambient air of density  $\rho$  is displaced by the air parcel, the magnitude of the upward force acting on the air parcel is  $\rho g$ . Therefore, the net upward force ( $F$ ) acting on a unit volume of the parcel is

$$F = (\rho - \rho')g$$



**Fig. 3.13** The box represents an air parcel of unit volume with its center of mass at height  $z$  above the Earth's surface. The density and temperature of the air parcel are  $\rho'$  and  $T'$ , respectively, and the density and temperature of the ambient air are  $\rho$  and  $T$ . The vertical forces acting on the air parcel are indicated by the thicker arrows.

Because the mass of a unit volume of the air parcel is  $\rho'$ , the upward acceleration of the parcel is

$$\frac{d^2z}{dt^2} = \frac{F}{\rho'} = \left( \frac{\rho - \rho'}{\rho'} \right) g$$

where  $z$  is the height of the air parcel. The pressure of the air parcel is the same as that of the ambient air, since they are at the same height in the atmosphere. Therefore, from the gas equation in the form of (3.2), the densities of the air parcel and the ambient air are inversely proportional to their temperatures. Hence,

$$\frac{d^2z}{dt^2} = \frac{\frac{1}{T} - \frac{1}{T'}}{\frac{1}{T'}} g$$

or

$$\frac{d^2z}{dt^2} = g \left( \frac{T' - T}{T} \right) \quad (3.73)$$

Strictly speaking, virtual temperature  $T_v$  should be used in place of  $T$  in all expressions relating to static stability. However, the virtual temperature correction is usually neglected except in certain calculations relating to the boundary layer.

**Exercise 3.12** The air parcel in Fig. 3.12a is displaced upward from its equilibrium level at  $z' = 0$  by a distance  $z'$  to a new level where the ambient temperature is  $T$ . The air parcel is then released. Derive an expression that describes the subsequent vertical displacement of the air parcel as a function of time in terms of  $T$ , the lapse rate of the ambient air ( $\Gamma$ ), and the dry adiabatic lapse rate ( $\Gamma_d$ ).

**Solution:** Let  $z = z_0$  be the equilibrium level of the air parcel and  $z' = z - z_0$  be the vertical displacement of the air parcel from its equilibrium level. Let  $T_0$  be the environmental air temperature at  $z = z_0$ . If the air parcel is lifted dry adiabatically through a distance  $z'$  from its equilibrium level, its temperature will be

$$T' = T_0 - (\Gamma_d) z'$$

Therefore

$$T' - T = -(\Gamma_d - \Gamma) z'$$

Substituting this last expression into (3.73), we obtain

$$\frac{d^2z'}{dt^2} = -\frac{g}{T} (\Gamma_d - \Gamma) z'$$

which may be written in the form

$$\frac{d^2z'}{dt^2} + N^2 z' = 0 \quad (3.74)$$

where

$$N = \left[ \frac{g}{T} (\Gamma_d - \Gamma) \right]^{1/2} \quad (3.75)$$

$N$  is referred to as the *Brunt<sup>36</sup>-Väisälä<sup>37</sup> frequency*. Equation (3.74) is a second order ordinary differential equation. If the layer in question is stably stratified (that is to say, if  $\Gamma_d > \Gamma$ ), then we can be assured that  $N$  is real,  $N^2$  is positive, and the solution of (3.74) is

$$z' = A \cos Nt + B \sin Nt$$

Making use of the conditions at the point of maximum displacement at time  $t = 0$ , namely that  $z' = z'(0)$  and  $dz'/dt = 0$  at  $t = 0$ , it follows that

$$z'(t) = z'(0) \cos Nt$$

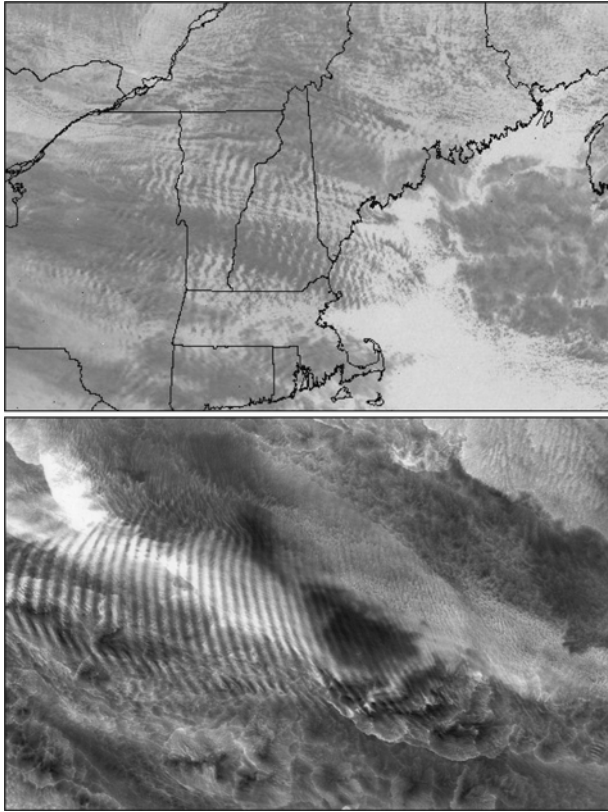
That is to say, the parcel executes a *buoyancy oscillation* about its equilibrium level  $z$  with amplitude equal to its initial displacement  $z'(0)$ , and frequency  $N$  (in units of radians per second). The Brunt-Väisälä frequency is thus a measure of the static stability: the higher the frequency, the greater the ambient stability. ■

Air parcels undergo buoyancy oscillations in association with *gravity waves*, a widespread phenomenon in planetary atmospheres, as illustrated in Fig. 3.14. Gravity waves may be excited by flow over

<sup>36</sup> **Sir David Brunt** (1886–1995) English meteorologist. First full-time professor of meteorology at Imperial College (1934–1952). His textbook *Physical and Dynamical Meteorology*, published in the 1930s, was one of the first modern unifying accounts of meteorology.

<sup>37</sup> **Vilho Väisälä** (1899–1969) Finnish meteorologist. Developed a number of meteorological instruments, including a version of the radiosonde in which readings of temperature, pressure, and moisture are telemetered in terms of radio frequencies. The modern counterpart of this instrument is one of Finland's successful exports.





**Fig. 3.14** Gravity waves, as revealed by cloud patterns. The upper photograph, based on NOAA GOES 8 visible satellite imagery, shows a wave pattern in west to east (right to left) airflow over the north-south-oriented mountain ranges of the Appalachians in the northeastern United States. The waves are transverse to the flow and their horizontal wavelength is  $\sim 20$  km. The atmospheric wave pattern is more regular and widespread than the undulations in the terrain. The bottom photograph, based on imagery from NASA's multiangle imaging spectro-radiometer (MISR), shows an even more regular wave pattern in a thin layer of clouds over the Indian Ocean.

mountainous terrain, as shown in the top photograph in Fig. 3.14 or by an intense local disturbance, as shown in the bottom photograph. The following exercise illustrates how buoyancy oscillations can be excited by flow over a mountain range.

**Exercise 3.13** A layer of unsaturated air flows over mountainous terrain in which the ridges are 10 km apart in the direction of the flow. The lapse rate is  $5^\circ\text{C km}^{-1}$  and the temperature is  $20^\circ\text{C}$ . For what value of the wind speed  $U$  will the period of the orographic (i.e., terrain-induced) forcing match the period of a buoyancy oscillation?

**Solution:** For the period  $\tau$  of the orographic forcing to match the period of the buoyancy oscillation, it is required that

$$\tau = \frac{L}{U} = \frac{2\pi}{N}$$

where  $L$  is the spacing between the ridges. Hence, from this last expression and (3.75),

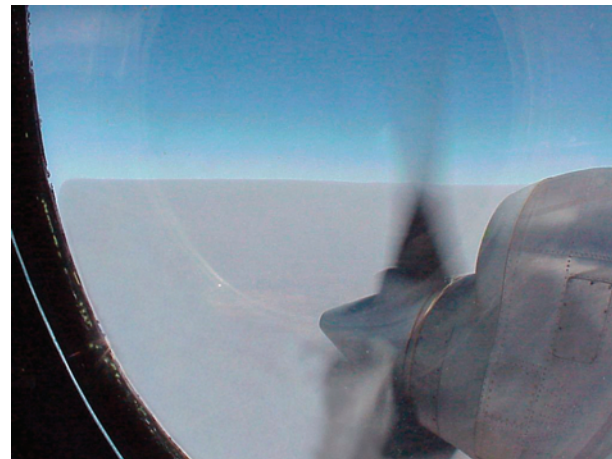
$$U = \frac{LN}{2\pi} = \frac{L}{2\pi} \left[ \frac{g}{T} (\Gamma_d - \Gamma) \right]^{1/2}$$

or, in SI units,

$$U = \frac{10^4}{2\pi} \left[ \frac{9.8}{293} \left( (9.8 - 5.0) \times 10^{-3} \right) \right]^{1/2} \\ \approx 20 \text{ m s}^{-1}$$

Layers of air with negative lapse rates (i.e., temperatures increasing with height) are called *inversions*. It is clear from the aforementioned discussion that these layers are marked by very strong static stability. A low-level inversion can act as a “lid” that traps pollution-laden air beneath it (Fig. 3.15). The layered structure of the stratosphere derives from the fact that it represents an inversion in the vertical temperature profile.

If  $\Gamma > \Gamma_d$  (Fig. 3.12b), a parcel of unsaturated air displaced upward from O will arrive at A with a temperature greater than that of its environment. Therefore, it will be less dense than the ambient air



**Fig. 3.15** Looking down onto widespread haze over southern Africa during the biomass-burning season. The haze is confined below a temperature inversion. Above the inversion, the air is remarkably clean and the visibility is excellent. (Photo: P. V. Hobbs.)

and, if left to itself, will continue to rise. Similarly, if the parcel is displaced downward it will be cooler than the ambient air, and it will continue to sink if left to itself. Such *unstable* situations generally do not persist in the free atmosphere, because the instability is eliminated by strong vertical mixing as fast as it forms. The only exception is in the layer just above the ground under conditions of very strong heating from below. ■

**Exercise 3.14** Show that if the potential temperature  $\theta$  increases with increasing altitude the atmosphere is stable with respect to the displacement of unsaturated air parcels.

**Solution:** Combining (3.1), (3.18), and (3.67), we obtain for a unit mass of air

$$c_p T \frac{d\theta}{\theta} = c_p dT + g dz$$

Letting  $d\theta = (\partial\theta/\partial z)dz$  and  $dT = (\partial T/\partial z)dz$  and dividing through by  $c_p T dz$  yields

$$\frac{1}{\theta} \frac{\partial\theta}{\partial z} = \frac{1}{T} \left( \frac{\partial T}{\partial z} + \frac{g}{c_p} \right) \quad (3.76)$$

Noting that  $-dT/dz$  is the actual lapse rate  $\Gamma$  of the air and the dry adiabatic lapse rate  $\Gamma_d$  is  $g/c_p$  (3.76) may be written as

$$\frac{1}{\theta} \frac{\partial\theta}{\partial z} = \frac{1}{T} (\Gamma_d - \Gamma) \quad (3.77)$$

However, it has been shown earlier that when  $\Gamma < \Gamma_d$  the air is characterized by positive static stability. It follows that under these same conditions  $\partial\theta/\partial z$  must be positive; that is, the potential temperature must increase with height. ■

### 3.6.2 Saturated Air

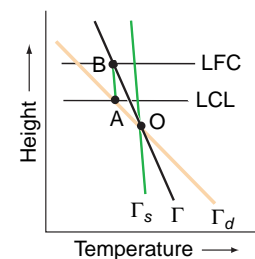
If a parcel of air is saturated, its temperature will decrease with height at the saturated adiabatic lapse rate  $\Gamma_s$ . It follows from arguments similar to those given in Section 3.6.1 that if  $\Gamma$  is the actual lapse rate of temperature in the atmosphere, saturated air parcels will be stable, neutral, or unstable with respect to vertical displacements, depending on whether  $\Gamma < \Gamma_s$ ,  $\Gamma = \Gamma_s$ , or  $\Gamma > \Gamma_s$ , respectively. When

an environmental temperature sounding is plotted on a skew  $T - \ln p$  chart the distinctions between  $\Gamma$ ,  $\Gamma_d$ , and  $\Gamma_s$  are clearly discernible (see Exercise 3.53).

### 3.6.3 Conditional and Convective Instability

If the actual lapse rate  $\Gamma$  of the atmosphere lies between the saturated adiabatic lapse rate  $\Gamma_s$  and the dry adiabatic lapse rate  $\Gamma_d$ , a parcel of air that is lifted sufficiently far above its equilibrium level will become warmer than the ambient air. This situation is illustrated in Fig. 3.16, where an air parcel lifted from its equilibrium level at O cools dry adiabatically until it reaches its lifting condensation level at A. At this level the air parcel is colder than the ambient air. Further lifting produces cooling at the moist adiabatic lapse rate so the temperature of the parcel of air follows the moist adiabat ABC. If the air parcel is sufficiently moist, the moist adiabat through A will cross the ambient temperature sounding; the point of intersection is shown as B in Fig. 3.16. Up to this point the parcel was colder and denser than the ambient air, and an expenditure of energy was required to lift it. If forced lifting had stopped prior to this point, the parcel would have returned to its equilibrium level at point O. However, once above point B, the parcel develops a positive buoyancy that carries it upward even in the absence of further forced lifting. For this reason, B is referred to as the *level of free convection* (LFC). The level of free convection depends on the amount of moisture in the rising parcel of air, as well as the magnitude of the lapse rate  $\Gamma$ .

From the aforementioned discussion it is clear that for a layer in which  $\Gamma_s < \Gamma < \Gamma_d$ , vigorous convective overturning will occur if forced vertical motions are



**Fig. 3.16** Conditions for conditional instability ( $\Gamma_s < \Gamma < \Gamma_d$ ).  $\Gamma_s$  and  $\Gamma_d$  are the saturated and dry adiabatic lapse rates, and  $\Gamma$  is the lapse rate of temperature of the ambient air. LCL and LFC denote the *lifting condensation level* and the *level of free convection*, respectively.

### 3.4 Analogs for Static Stability, Instability, Neutral Stability, and Conditional Instability

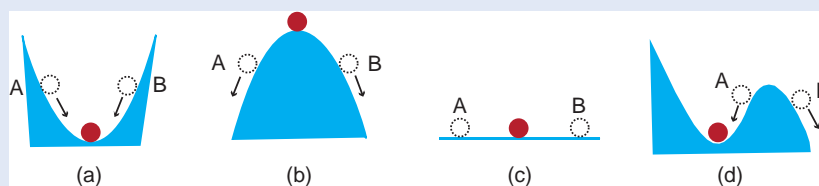
Sections 3.6.1 and 3.6.2 discussed the conditions for parcels of unsaturated air and saturated air to be stable, unstable, or neutral when displaced vertically in the atmosphere. Under stable conditions, if an air parcel is displaced either upward or downward and is then left to itself (i.e., the force causing the original displacement is removed), the parcel will return to its original position. An analogous situation is shown in Fig. 3.17a where a ball is originally located at the lowest point in a valley. If the ball is displaced in any direction and is then left to itself, it will return to its original location at the base of the valley.

Under unstable conditions in the atmosphere, an air parcel that is displaced either upward or downward, and then left to itself, will continue to move upward or downward, respectively. An analog is shown in Fig. 3.17b, where a ball is initially on top of a hill. If the ball is displaced in any direction, and is then left to itself, it will roll down the hill.

If an air parcel is displaced in a neutral atmosphere, and then left to itself, it will remain in the displaced location. An analog of this condition is a ball on a flat surface (Fig. 3.17c). If the ball is displaced, and then left to itself, it will not move.

If an air parcel is conditionally unstable, it can be lifted up to a certain height and, if left to itself, it will return to its original location. However, if the air parcel is lifted beyond a certain height (i.e., the level of free convection), and is then left to itself, it will continue rising (Section 3.6.3). An analog of this situation is shown in Fig. 3.17d, where a displacement of a ball to a point A, which lies to the left of the hillock, will result in the ball rolling back to its original position. However, if the displacement takes the ball to a point B on the other side of the hillock, the ball will not return to its original position but will roll down the right-hand side of the hillock.

It should be noted that in the analogs shown in Fig. 3.17 the only force acting on the ball after it is displaced is that due to gravity, which is always downward. In contrast, an air parcel is acted on by both a gravitational force and a buoyancy force. The gravitational force is always downward. The buoyancy force may be either upward or downward, depending on whether the air parcel is less dense or more dense than the ambient air.

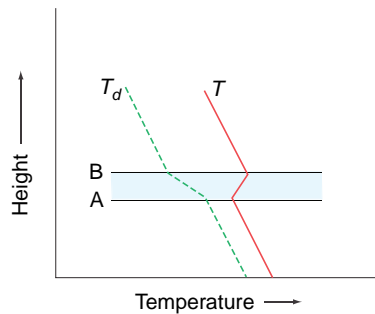


**Fig. 3.17** Analogs for (a) stable, (b) unstable, (c) neutral, and (d) conditional instability. The red circle is the original position of the ball, and the white circles are displaced positions. Arrows indicate the direction the ball will move from a displaced position if the force that produced the displacement is removed.

large enough to lift air parcels beyond their level of free convection. Such an atmosphere is said to be *conditionally unstable* with respect to convection. If vertical motions are weak, this type of stratification can be maintained indefinitely.

The potential for instability of air parcels is also related to the vertical stratification of water vapor. In the profiles shown in Fig. 3.18, the dew point decreases rapidly with height within the inversion layer AB that marks the top of a moist layer. Now,

suppose that this layer is lifted. An air parcel at A will reach its LCL quickly, and beyond that point it will cool moist adiabatically. In contrast, an air parcel starting at point B will cool dry adiabatically through a deep layer before it reaches its LCL. Therefore, as the inversion layer is lifted, the top part of it cools much more rapidly than the bottom part, and the lapse rate quickly becomes destabilized. Sufficient lifting may cause the layer to become conditionally unstable, even if the entire sounding is absolutely



**Fig. 3.18** Conditions for convective instability.  $T$  and  $T_d$  are the temperature and dew point of the air, respectively. The blue-shaded region is a dry inversion layer.

stable to begin with. It may be shown that the criterion for this so-called *convective (or potential) instability* is that  $\partial\theta_e/\partial z$  be negative (i.e.,  $\theta_e$  decrease with increasing height) within the layer.

Throughout large areas of the tropics,  $\theta_e$  decreases markedly with height from the mixed layer to the much drier air above. Yet deep convection breaks out only within a few percent of the area where there is sufficient lifting to release the instability.

## 3.7 The Second Law of Thermodynamics and Entropy

The first law of thermodynamics (Section 3.3) is a statement of the principle of conservation of energy. The *second law of thermodynamics*, which was deduced in various forms by Carnot,<sup>38</sup> Clausius,<sup>39</sup> and Lord Kelvin, is concerned with the maximum fraction of a quantity of heat that can be converted into work. The fact that for any given system there is a theoretical limit to this conversion was first clearly demonstrated by Carnot, who also introduced the important concepts of cyclic and reversible processes.

### 3.7.1 The Carnot Cycle

A *cyclic process* is a series of operations by which the state of a substance (called the *working substance*) changes but the substance is finally returned to its original state in all respects. If the volume of the working substance changes, the working sub-

stance may do external work, or work may be done on the working substance, during a cyclic process. Since the initial and final states of the working substance are the same in a cyclic process, and internal energy is a function of state, the internal energy of the working substance is unchanged in a cyclic process. Therefore, from (3.33), the *net* heat absorbed by the working substance is equal to the external work that it does in the cycle. A working substance is said to undergo a *reversible* transformation if each state of the system is in equilibrium so that a reversal in the direction of an infinitesimal change returns the working substance and the environment to their original states. A *heat engine* (or *engine* for short) is a device that does work through the agency of heat.

If during one cycle of an engine a quantity of heat  $Q_1$  is absorbed and heat  $Q_2$  is rejected, the amount of work done by the engine is  $Q_1 - Q_2$  and its *efficiency*  $\eta$  is defined as

$$\begin{aligned}\eta &= \frac{\text{Work done by the engine}}{\text{Heat absorbed by the working substance}} \\ &= \frac{Q_1 - Q_2}{Q_1}\end{aligned}\quad (3.78)$$

Carnot was concerned with the important practical problem of the efficiency with which heat engines can do useful mechanical work. He envisaged an ideal heat engine (Fig. 3.19) consisting of a working substance contained in a cylinder (Y) with insulating walls and a conducting base (B) that is fitted with an insulated, frictionless piston (P) to which a variable force can be applied, a nonconducting stand (S) on which the cylinder may be placed to insulate its base, an infinite warm reservoir of heat (H) at constant temperature  $T_1$ , and an infinite cold reservoir for heat (C) at constant temperature  $T_2$  (where  $T_1 > T_2$ ). Heat can be supplied from the warm reservoir to the working substance contained in the cylinder, and heat can be extracted from the working substance by the cold reservoir. As the working substance expands (or contracts), the piston moves outward (or inward) and external work is done by (or on) the working substance.

<sup>38</sup> **Nicholas Leonard Sadi Carnot** (1796–1832) Born in Luxenbourg. Admitted to the École Polytechnique, Paris, at age 16. Became a captain in the Corps of Engineers. Founded the science of thermodynamics.

<sup>39</sup> **Rudolf Clausius** (1822–1888) German physicist. Contributed to the sciences of thermodynamics, optics, and electricity.