## Serie de Fourier

$x(t)$ función periódica, con período $T$ (para cualquier $t_0$)

$$X[k] = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)e^{-jk\frac{2\pi}{T}t}dt \qquad x(t) = \sum_{k=-\infty}^{+\infty} X[k]e^{jk\frac{2\pi}{T}t}$$

$$\frac{1}{T}\int_{t_0}^{t_0+T} x(t)y^*(t)dt = \sum_{k=-\infty}^{\infty} X[k]Y^*[k]$$

$$\text{Im}[x(t)] = 0 \Longleftrightarrow X[-k] = X^*[k]$$

$$x(-t) = x^*(t) \Longleftrightarrow \text{Im}[X[k]] = 0$$

$$\text{Re}[x(t)] = 0 \Longleftrightarrow X[-k] = -X^*[k]$$

$$x(-t) = -x^*(t) \Longleftrightarrow \text{Re}[X[k]] = 0$$

| Función | Transformada |
|---|---|
| $ax(t) + by(t)$ | $aX[k] + bY[k]$ |
| $f(t-\tau)$ | $e^{-j2\pi k\tau/T}X[k]$ |
| $e^{j2\pi mt/T}x(t)$ | $X[k-m]$ |
| $\frac{1}{T}\int_0^T x(t-\tau)y(\tau)d\tau$ | $X[k]Y[k]$ |
| $x(t)y(t)$ | $X[k] * Y[k]$ |
| $\sum_n \Pi\left(\frac{t-nT}{\tau}\right)$ | ${}^{\tau}/_{T}\,\text{sinc}(k\tau/T)$ |

## Transformada de Fourier

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \qquad x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

$$\int_{-\infty}^{\infty} x(t)y^*(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)Y^*(j\omega)d\omega$$

$$\text{Im}[x(t)] = 0 \Longleftrightarrow X(j\omega) = X^*(-j\omega)$$

$$x(-t) = x^*(t) \Longleftrightarrow \text{Im}[X(j\omega)] = 0$$

$$\text{Re}[x(t)] = 0 \Longleftrightarrow X(-j\omega) = -X^*(j\omega)$$

$$x(-t) = -x^*(t) \Longleftrightarrow \text{Re}[X(j\omega)] = 0$$

| Función | Transformada |
|---|---|
| $ax(t) + by(t)$ | $aX(j\omega) + bY(j\omega)$ |
| $(x * y)(t)$ | $X(j\omega)Y(j\omega)$ |
| $x(t)y(t)$ | $\frac{1}{2\pi}(X * Y)(j\omega)$ |
| $x^*(t)$ | $X^*(-j\omega)$ |
| $X(t)$ | $2\pi\, x(-j\omega)$ |
| $x(t-t_d)$ | $X(j\omega)e^{-j\omega t_d}$ |
| $x(t)e^{j(\omega_c t+\phi)}$ | $X(j(\omega-\omega_c))e^{j\phi}$ |
| $x(t)\cos(\omega_c t)$ | $\frac{1}{2}\left(X(j(\omega-\omega_c)) + X(j(\omega+\omega_c))\right)$ |
| $x(\alpha t)$ | $\frac{1}{\lvert\alpha\rvert}X\left(\frac{j\omega}{\alpha}\right)$ |
| $\frac{d^n x(t)}{dt^n}$ | $(j\omega)^n X(j\omega)$ |
| $\int_{-\infty}^{t} x(\lambda)d\lambda$ | $\frac{1}{j\omega}X(f) + \pi X(0)\delta(j\omega)$ |
| $t^n x(t)$ | $(-j\omega)^{-n}\frac{d^n X(j\omega)}{d\omega^n}$ |
| $\delta(t-t_d)$ | $e^{-j\omega t_d}$ |
| $e^{j(\omega_c t+\phi)}$ | $2\pi\, e^{j\phi}\delta(\omega-\omega_c)$ |
| $e^{-\pi(at)^2}$ | $\frac{1}{a}e^{-\pi(\omega/a)^2}$ |
| $\sum_{k=-\infty}^{\infty}\delta(t-kT)$ | ${}^{2\pi}/_{T}\sum_{k=-\infty}^{\infty}\delta\left(\omega - {}^{2\pi k}/_{T}\right)$ |
| sign $t$ | ${}^{2}/_{j\omega}$ |
| $u(t)$ | ${}^{1}/_{j\omega} + \pi\delta(\omega)$ |
| $\Pi\left({}^{t}/_{\tau}\right)$ | $\tau\,\text{sinc}\left({}^{\omega\tau}/_{2\pi}\right)$ |
| $\Lambda\left({}^{t}/_{\tau}\right)$ | $\tau\,\text{sinc}^2\left({}^{\omega\tau}/_{2\pi}\right)$ |
| $\left({}^{W}/_{\pi}\right)\text{sinc}\left({}^{Wt}/_{\pi}\right)$ | $\Pi\left({}^{\omega}/_{2W}\right)$ |

## DTFT

$$X(e^{j\theta}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-jn\theta} \qquad x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})e^{j\theta n}d\theta$$

$$\sum_{n=-\infty}^{+\infty} x_1[n]x_2^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(e^{j\theta})X_2^*(e^{j\theta})d\theta$$

$$\text{Im}[x[n]] = 0 \Longleftrightarrow X(e^{-j\theta}) = X^*(e^{j\theta})$$

$$x[-n] = x^*[n] \Longleftrightarrow \text{Im}[X(e^{j\theta})] = 0$$

$$\text{Re}[x[n]] = 0 \Longleftrightarrow X(e^{-j\theta}) = -X^*(e^{j\theta})$$

$$x[-n] = -x^*[n] \Longleftrightarrow \text{Re}[X(e^{j\theta})] = 0$$

| Función | Transformada |
|---|---|
| $a_1x[n] + a_2y[n]$ | $a_1X(e^{j\theta}) + a_2Y(e^{j\theta})$ |
| $(x * y)[n]$ | $X(e^{j\theta})Y(e^{j\theta})$ |
| $x[n]y[n]$ | $\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\lambda})Y(e^{j(\theta-\lambda)})d\lambda$ |
| $x^*[n]$ | $X^*(e^{-j\theta})$ |
| $x[n-n_o]$ | $X(e^{j\theta})e^{-jn_o\theta}$ |
| $x[n]e^{jn\theta_o}$ | $X(e^{j(\theta-\theta_o)})$ |
| $n\,x[n]$ | $j\frac{d\,X(e^{j\theta})}{d\theta}$ |
| $x[-n]$ | $X(e^{-j\theta})$ |
| $\delta[n]$ | $1$ |
| $\delta[n-n_o]$ | $e^{-jn_o\theta}$ |
| $1$ | $\sum_{k=-\infty}^{+\infty} 2\pi\delta(\theta+2\pi k)$ |
| $a^n u[n] \quad (\lvert a\rvert < 1)$ | $\frac{1}{1-ae^{-j\theta}}$ |
| $u[n]$ | $\frac{1}{1-e^{-j\theta}} + \sum_{k=-\infty}^{\infty}\pi\delta(\theta+2\pi k)$ |
| $(n+1)a^n u[n] \quad (\lvert a\rvert < 1)$ | $\frac{1}{(1-ae^{-j\theta})^2}$ |
| $\frac{r^n\sin\theta_o(n+1)}{\sin\theta_o}u[n], (\lvert r\rvert < 1)$ | $\frac{1}{1-2r\cos\theta_o e^{-j\theta}+r^2e^{-j2\theta}}$ |
| $\frac{\sin\theta_o n}{\pi n}$ | $\sum_k \Pi\left(\frac{\theta+2\pi k}{2\theta_o}\right)$ |
| $x[n] = \begin{cases} 1 & 0 \le n \le M \\ 0 & \text{otro } n \end{cases}$ | $\frac{\sin(\theta(M+1)/2)}{\sin(\theta/2)}e^{-j\theta M/2}$ |
| $e^{j\theta_o n+\phi}$ | $\sum_{k=-\infty}^{\infty} 2\pi e^{j\phi}\delta(\theta-\theta_o+2\pi k)$ |

## DFT

$$X[k] = \sum_{n=0}^{N-1} x[n]W_N^{kn} \qquad x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]W_N^{-kn}$$

donde $W_N = e^{-j2\pi/N}$

$$\sum_{n=0}^{N-1} x[n]y^*[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]Y^*[k]$$

$$\text{Im}[x[n]] = 0 \Longleftrightarrow X[-n \bmod N] = X^*[k]$$

$$x[-n \bmod N] = x^*[n] \Longleftrightarrow \text{Im}[X[k]] = 0$$

$$\text{Re}[x[n]] = 0 \Longleftrightarrow X[-n \bmod N] = -X^*[k]$$

$$x[-n \bmod N] = -x^*[n] \Longleftrightarrow \text{Re}[X[k]] = 0$$

| Función | Transformada |
|---|---|
| $ax[n] + by[n]$ | $aX[k] + bY[k]$ |
| $X[n]$ | $Nx[-k \bmod N]$ |
| $x[(n-m) \bmod N]$ | $W_N^{km}X[k]$ |
| $W_N^{-ln}x[n]$ | $X[(k-l) \bmod N]$ |
| $\sum_{m=0}^{N-1} x[m]y[(n-m) \bmod N]$ | $X[k]Y[k]$ |
| $x[n]y[n]$ | $\frac{1}{N}\sum_{l=0}^{N-1} X[l]Y[(k-l) \bmod N]$ |
| $x^*[n]$ | $X^*[-k \bmod N]$ |
| $x^*[-n \bmod N]$ | $X^*[k]$ |

## Transformada de Laplace

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt \qquad x(t) = \sum_{Polos} \text{Res}[X(s)e^{st}]$$

donde la sumatoria se realiza en todos los polos a la izquierda de la ROC.

| Función | Transformada | ROC |
|---|---|---|
| $ax(t) + by(t)$ | $aX(s) + bY(s)$ | contiene $R_x \cap R_y$ |
| $(x * y)(t)$ | $X(s)Y(s)$ | contiene $R_x \cap R_y$ |
| $x(t)y(t)$ | $(X * Y)(s)$ | contiene $R_x \cap R_y$ |
| $x^*(t)$ | $X^*(s^*)$ | $R_x$ |
| $x(t - t_d)$ | $X(s)e^{-st_d}$ | $R_x$ |
| $x(t)e^{s_0 t}$ | $X(s - s_0)$ | $R_x + \text{Re}\{s_0\}$ |
| $x(\alpha t)$ | $\frac{1}{\lvert\alpha\rvert} X\left(\frac{s}{\alpha}\right)$ | $\alpha\, R_x$ |
| $\frac{d^n x(t)}{dt^n}$ | $s^n\, X(s)$ | $R_x$ |
| $\int_{-\infty}^{t} x(\lambda)d\lambda$ | $s^{-1}X(s)$ | $R_x \cap \{\text{Re}[s] > 0\}$ |
| $t^n x(t)$ | $(-s)^{-n} \frac{d^n X(s)}{ds}$ | $R_x$ |
| $\delta(t - t_d)$ | $e^{-st_d}$ | $\forall s$ |
| $u(t)$ | $1/s$ | $\text{Re}[s] > 0$ |
| $-u(-t)$ | $1/s$ | $\text{Re}[s] < 0$ |
| $\frac{t^{n-1}}{(n-1)!}u(t)$ | $1/s^n$ | $\text{Re}[s] > 0$ |
| $-\frac{t^{n-1}}{(n-1)!}u(-t)$ | $1/s^n$ | $\text{Re}[s] < 0$ |
| $e^{-at}u(t)$ | $1/s+a$ | $\text{Re}[s] > -\text{Re}[a]$ |
| $-e^{-at}u(-t)$ | $1/s+a$ | $\text{Re}[s] < -\text{Re}[a]$ |
| $\frac{t^{n-1}}{(n-1)!}e^{-at}u(t)$ | $1/(s+a)^n$ | $\text{Re}[s] > -\text{Re}[a]$ |
| $e^{-at}\cos(\omega_0 t)u(t)$ | $(s+a)/((s+a)^2 + \omega_0^2)$ | $\text{Re}[s] > -\text{Re}[a]$ |
| $e^{-at}\sin(\omega_0 t)u(t)$ | $\omega_0/((s+a)^2 + \omega_0^2)$ | $\text{Re}[s] > -\text{Re}[a]$ |
| $u_n(t) = \frac{d^n \delta(t)}{dt^n}$ | $s^n$ | $\forall s$ |

## Transformada de Laplace Unilateral

$$X(s) = \int_{0^-}^{\infty} x(t)e^{-st}dt$$

Con $x(t) = y(t) = 0, \forall t < 0$
$(x * y)(t) \longleftrightarrow X(s)Y(s), ROC \supset R_x \cap R_y$

$$\frac{d^n x(t)}{dt^n} \longleftrightarrow s^n X(s) - \sum_{k=0}^{n-1} s^k \frac{d^{n-1-k}x(t)}{dt^{n-1-k}}, ROC : R_x$$

Teorema de Valor Inicial (TVI): $x(0^+) = \lim_{s\to\infty} sX(s)$
Teorema de Valor Final (TVF): $\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$

El TVI y TVF son válidos si $x(t)$ no contiene impulso o singularidades de mayor orden en $t = 0$. Además, para el TVF es necesario que $x(t)$ tenga límite finito con $t \to +\infty$ (sin polos en el semiplano derecho).

## Algunas funciones e identidades útiles

$$\text{sinc}\ t = \frac{\sin \pi t}{\pi t} \qquad \text{sign}\ t = \begin{cases} 1, & t > 0 \\ -1, & t < 0 \end{cases}$$

$$u(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases} \qquad u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

$$\Pi\left(\frac{t}{\tau}\right) = \begin{cases} 1 & \lvert t\rvert < \frac{\tau}{2} \\ 0 & \lvert t\rvert > \frac{\tau}{2} \end{cases} \qquad \Lambda\left(\frac{t}{\tau}\right) = \begin{cases} 1 - \frac{\lvert t\rvert}{\tau} & \lvert t\rvert < \tau \\ 0 & \lvert t\rvert > \tau \end{cases}$$

$$\sum_{k=-\infty}^{+\infty} e^{-jk\alpha t} = \frac{2\pi}{\alpha} \sum_{k=-\infty}^{+\infty} \delta\left(t - k\frac{2\pi}{\alpha}\right), \text{Fórmula de Poisson}$$

$$\sum_{k=N_1}^{N_2} \alpha^k = \frac{\alpha^{N_1} - \alpha^{N_2+1}}{1 - \alpha}, N_2 > N_1, \alpha \neq 1, \text{Serie Geométrica}$$

## Transformada Z

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n]z^{-n} \qquad x(n) = \frac{1}{2\pi j} \oint_C X(z)z^{n-1}dz$$

donde C es una curva antihoraria en la región de convergencia y que envuelve al origen.

| Secuencia | Transformada Z | ROC |
|---|---|---|
| $ax[n] + by[n]$ | $aX(z) + bY(z)$ | contiene $R_x \cap R_y$ |
| $x[n - n_o]$ | $z^{-n_o}X(z)$ | $R_x$, quizás $\pm\, 0$ ó $\infty$ |
| $z_o^n x[n]$ | $X(z/z_o)$ | $\lvert z_o\rvert R_x$ |
| $n^k x[n]$ | $\left(-z\frac{d}{dz}\right)^k X(z)$ | $R_x$, quizás $\pm\, 0$ ó $\infty$ |
| $x^*[n]$ | $X^*(z^*)$ | $R_x$ |
| $x[-n]$ | $X(1/z)$ | $1/R_x$ |
| $(x * y)[n]$ | $X(z)Y(z)$ | contiene $R_x \cap R_y$ |
| $\delta[n]$ | $1$ | $\forall z$ |
| $a^n u[n]$ | $\frac{1}{1-az^{-1}}$ | $\lvert z\rvert > \lvert a\rvert$ |
| $-a^n u[-n-1]$ | $\frac{1}{1-az^{-1}}$ | $\lvert z\rvert < \lvert a\rvert$ |
| $\delta[n - n_o]$ | $z^{-n_o}$ | $\forall z$ excepto 0 ó $\infty$ |
| $\cos(\omega_o n)u[n]$ | $\frac{1-z^{-1}\cos\omega_o}{1-2z^{-1}\cos\omega_o + z^{-2}}$ | $\lvert z\rvert > 1$ |
| $\sin(\omega_o n)u[n]$ | $\frac{z^{-1}\sin\omega_o}{1-2z^{-1}\cos\omega_o + z^{-2}}$ | $\lvert z\rvert > 1$ |

## Transformada Z unilateral

$$X(z) = \sum_{n=0}^{+\infty} x[n]z^{-n}$$

$$x[n] \longleftrightarrow X_u(z)$$

$$x[n + n_o] \longleftrightarrow z^{n_o}X_u(z) - x[0]z^{n_o} - \cdots - x[n_o - 1]z$$

$$x[n - n_o] \longleftrightarrow z^{-n_o}X_u(z) + x[-1]z^{-n_o+1} + \cdots + x[-n_o]$$

Teorema de valor inicial: $x[0] = \lim_{z\to\infty} X(z)$

## Identidades trigonométricas

$e^{j\theta} = \cos\theta + j\sin\theta$
$\cos\theta = 1/2\,(e^{j\theta} + e^{-j\theta})$
$\sin\theta = 1/2j\,(e^{j\theta} - e^{-j\theta})$
$\cos\theta = \sin(\theta + 90°)$
$\sin\theta = \cos(\theta - 90°)$
$\sin^2\theta + \cos^2\theta = 1$
$\cos^2\theta = 1/2\,(1 + \cos 2\theta)$
$\cos^3\theta = 1/4\,(3\cos\theta + \cos 3\theta)$
$\sin^2\theta = 1/2\,(1 - \cos 2\theta)$
$\sin^3\theta = 1/4\,(3\sin\theta - \sin 3\theta)$
$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$
$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$
$\tan(\alpha \pm \beta) = (\tan\alpha \pm \tan\beta)\,/\,(1 \mp \tan\alpha\tan\beta)$
$\sin\alpha\sin\beta = 1/2\,\cos(\alpha - \beta) - 1/2\,\cos(\alpha + \beta)$
$\cos\alpha\cos\beta = 1/2\,\cos(\alpha - \beta) + 1/2\,\cos(\alpha + \beta)$
$\sin\alpha\cos\beta = 1/2\,\sin(\alpha - \beta) + 1/2\,\sin(\alpha + \beta)$

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