Solución Examen Electromagnetismo (1128) 23/12/2021.-

(1) a)

$$V(x,y,z) = \frac{1}{4\pi\varepsilon_0}\left\{\frac{Q_0}{\sqrt{(x-a)^2+(y-b)^2+z^2}} + \frac{Q_1^{=-Q_0}}{\sqrt{(x+a)^2+(y-b)^2+z^2}} + \frac{Q_2}{\sqrt{(x+a)^2+(y+b)^2+z^2}} + \frac{Q_3^{=-Q_0}}{\sqrt{(x-a)^2+(y+b)^2+z^2}}\right\}$$

C.B.
$$\begin{cases} V(x,0,z)=0 & (1)\\ V(0,y,z)=0 & (2)\\ V(r)\xrightarrow[r\to\infty]{} 0 & (3)\end{cases}$$

Consideremos (1):

$$V(x,0,z) = \frac{1}{4\pi\varepsilon_0}\left\{\cancel{\frac{Q_0}{\sqrt{(x-a)^2+b^2+z^2}}} - \frac{Q_0}{\sqrt{(x+a)^2+b^2+z^2}} + \frac{Q_2}{\sqrt{(x+a)^2+b^2+z^2}} - \cancel{\frac{Q_0}{\sqrt{(x-a)^2+b^2+z^2}}}\right\} = 0$$

$$\Rightarrow \boxed{Q_2 = +Q_0}$$

Análogamente se puede verificar que $Q_2 = Q_0$ satisface (2) y (3).

b) semiplano correspondiente a $y=0$:

$$\sigma(x,z) = \varepsilon_0 \underbrace{\vec{E}(x,y,z)}_{=-\nabla V(x,y,z)}\cdot\hat{n}^{=\hat{\jmath}}\Big|_{y=0} = -\varepsilon_0 \frac{\partial V}{\partial y}\Big|_{y=0}$$

$$= -\frac{\cancel{\varepsilon_0} Q_0}{4\pi\cancel{\varepsilon_0}}\left\{\frac{(-\frac{1}{\cancel{2}}\cancel{2})(y-b)}{[(x-a)^2+(y-b)^2+z^2]^{3/2}} - \frac{(-\frac{1}{\cancel{2}}\cancel{2})(y-b)}{[(x+a)^2+(y-b)^2+z^2]^{3/2}} - \frac{(-\frac{1}{\cancel{2}}\cancel{2})(y-b)}{[(x+a)^2+(y+b)^2+z^2]^{3/2}} + \frac{(-\frac{1}{\cancel{2}}\cancel{2})(y+b)}{[(x-a)^2+(y-b)^2+z^2]^{3/2}}\right\}\Bigg|_{y=0}$$

$$= -\frac{Q_0}{4\pi}\left\{\frac{b}{[(x-a)^2+b^2+z^2]^{3/2}} - \frac{b}{[(x+a)^2+b^2+z^2]^{3/2}} - \frac{b}{[(x+a)^2+b^2+z^2]^{3/2}} + \frac{b}{[(x-a)^2+b^2+z^2]^{3/2}}\right\}$$

$$\sigma(x,z) = -\frac{Q_0}{2\cancel{4}\pi}\left\{\frac{\cancel{z}b}{\left[(x-a)^2+b^2+z^2\right]^{3/2}} - \frac{zb}{\left[(x+a)^2+b^2+z^2\right]^{3/2}}\right\}$$

$$\boxed{\sigma(x,z) = -\frac{Q_0 b}{2\pi}\left\{\frac{1}{\left[(x-a)^2+b^2+z^2\right]^{3/2}} - \frac{1}{\left[(x+a)^2+b^2+z^2\right]^{3/2}}\right\}}$$

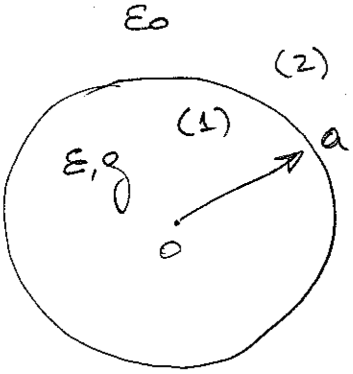
Analogamente, para el semiplano correspondiente a $x=0$:

$$\sigma(y,z) = \varepsilon_0 \vec{E}\cdot\hat{\imath}\Big|_{x=0} = -\varepsilon_0 \frac{\partial V}{\partial x}\Big|_{x=0}$$

$$\boxed{\sigma(y,z) = -\frac{Q_0 a}{2\pi}\left\{\frac{1}{\left[a^2+(y-b)^2+z^2\right]^{3/2}} - \frac{1}{\left[a^2+(y+b)^2+z^2\right]^{3/2}}\right\}}$$

—x—

(2)



a) Dentro de la esfera (región 1)

$$\nabla\cdot\vec J_1 + \frac{\partial\rho}{\partial t} = 0 \quad (i) \qquad \Big| \begin{array}{l}\vec D_1 = \varepsilon \vec E_1 \\ \vec J_1 = g\vec E_1\end{array}$$

$$\Rightarrow \frac{g}{\varepsilon}\underbrace{\nabla\cdot\vec D_1}_{=\rho} + \frac{\partial\rho}{\partial t} = 0$$

$$\Rightarrow \boxed{\rho(r,t) = \rho(r,0)\, e^{-\frac{g}{\varepsilon}t} = \left(\frac{\rho_0 a}{r}\right) e^{-\frac{g}{\varepsilon}t}}$$

De (i): $$\underbrace{\int_V \nabla\cdot\vec J_1\, dv}_{=} = -\int_V \frac{\partial\rho}{\partial t}\, dv = \cancel{-}\left(\cancel{-}\frac{g}{\varepsilon}\rho_0 a\int_0^r \frac{1}{r}\underbrace{(4\pi r^2 dr)}_{dv}\right) e^{-\frac{g}{\varepsilon}t}$$

← teo. divergencia

$$\oint_S \vec J_1\cdot\hat n\, da = J_1(r)(\cancel{4\pi} r^{\cancel{2}}) = \frac{g\rho_0 a}{\varepsilon}\cancel{4\pi}\underbrace{\left(\frac{r}{2}\Big|_0^r\right)}_{=\frac{\cancel{r^2}}{2}} e^{-\frac{g}{\varepsilon}t}$$

$$\Rightarrow \boxed{\vec J_1(r,t) = \left(\frac{g\rho_0 a}{2\varepsilon}\right) e^{-\frac{g}{\varepsilon}t}\,\hat e_r} \quad \text{(adentro)}$$

$$\boxed{\vec E_1(r,t) = \frac{\vec J_1}{g} = \left(\frac{\rho_0 a}{2\varepsilon}\right) e^{-\frac{g}{\varepsilon}t}\,\hat e_r}$$

Afuera (región 2): $\boxed{\vec J_2 = 0}$, $\vec D_2 = \varepsilon_0 \vec E_2$

$$\oint_S \vec D_2\cdot\hat n\, da = q\big|_{encerrada}$$

← la carga encerrada (carga neta en la esfera) se conserva, entonces podemos calcularla por ej. en $t=0$ en que la densidad superficial de carga es cero (*)

S ← superf. gaussiana (superficie esfera radio r con $r>a$)

$$\varepsilon_0 E_2(r,t)(4\pi r^2) = \int_V \rho(r,t=0)\, dv = \int_V \frac{\rho_0 a}{r}\underbrace{(4\pi r^2)\, dr}_{dv} = \rho_0 a 4\pi \underbrace{\int_0^a r\, dr}_{=\frac{a^2}{2}}$$

$$\Rightarrow \boxed{\vec E_2(r,t) = \frac{\rho_0 a^3}{2\varepsilon_0 r^2}\,\hat e_r}$$

$$\boxed{\sigma_a(t)} = D_{2n}\big|_a - D_{1n}\big|_a$$

$$= \varepsilon_0 E_2\big|_{r=a} - \varepsilon E_1\big|_{r=a} = \frac{\rho_0 a^3}{2r^2}\bigg|_{r=a} - \frac{\rho_0 a}{2} e^{-\frac{g}{\varepsilon}t} = \boxed{\frac{\rho_0 a}{2}\left(1 - e^{-\frac{g}{\varepsilon}t}\right)}$$

(se verifica $\sigma_a(t=0)=0$, lo que usamos en (*) al calcular la carga encerrada integrando ρ en el instante inicial)

b) $$P(t) = \int_V \underbrace{\vec{J}_1 \cdot \vec{E}_1}_{\frac{J_1^2}{g}}\, dv = \frac{J_1^2}{g}\left(\frac{4}{3}\pi a^3\right) = \frac{g^{\cancel{2}}\rho_0^2 a^2 e^{-\frac{2g}{\varepsilon}t}}{\cancel{4}\varepsilon^2}\,\frac{\cancel{4}\pi a^3}{3\,\cancel{g}}$$

como J_1^2 cte en V

$$= \frac{g\rho_0^2 a^5 \pi}{3\varepsilon^2}\, e^{-\frac{2g}{\varepsilon}t}$$

$$E_{dis} = \int_0^\infty P(t)\,dt = \frac{g\rho_0^2 a^5\pi}{3\varepsilon^2}\int_0^\infty e^{-\frac{2g}{\varepsilon}t} = \frac{\cancel{g}\rho_0^2 a^5 \pi}{3\varepsilon^{\cancel{2}}}\,\frac{\cancel{\varepsilon}}{2\cancel{g}}$$

$$\int_0^\infty e^{-\frac{2g}{\varepsilon}t} = -\frac{\varepsilon}{2g}\left(e^{-\frac{2g}{\varepsilon}t}\Big|_0^t\right) \quad (0-1)$$

$$\boxed{E_{dis} = \frac{\pi\rho_0^2 a^5}{6\varepsilon}}$$

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(3)

$$N_1 I_1 + N_2 I_2 = \phi_\mu \left(\frac{4l}{\mu S}\right) + \phi_e \left(\frac{2e}{\mu_0 S}\right)$$

$$= \phi \left[\frac{4l}{\mu S} + \frac{2e}{\mu_0 S}\right]$$

Cont. de B_n
$\Rightarrow B_e = B_\mu$
$\Rightarrow \phi_e = B_e S = B_\mu S = \phi_\mu$
$\phi_e = \phi_\mu (= \phi)$

$$\Rightarrow \phi = \frac{N_1 I_1 + N_2 I_2}{\left(\frac{4l\mu_0 + 2e\mu}{\mu_0 \mu S}\right)}$$

$$\boxed{L_1 = N_1 \frac{d\phi}{dI_1} = N_1^2 \left(\frac{\mu_0 \mu S}{4l\mu_0 + 2e\mu}\right)}, \quad \boxed{L_2 = N_2 \frac{d\phi}{dI_2} = N_2^2 \left(\frac{\mu_0 \mu S}{4l\mu_0 + 2e\mu}\right)}$$

$$\boxed{M = N_1 \frac{d\phi}{dI_2} = N_1 N_2 \left(\frac{\mu_0 \mu S}{4l\mu_0 + 2\mu e}\right)}$$ (idem a partir de $N_2 \frac{d\phi}{dI_1}$)

$$\boxed{H_\mu = \frac{B_\mu}{\mu} = \frac{\phi}{\mu S} = \frac{N_1 I_1 + N_2 I_2}{\mu S \left(\frac{4l\mu_0 + 2\mu e}{\mu_0 \mu S}\right)} = \frac{\mu_0 (N_1 I_1 + N_2 I_2)}{4l\mu_0 + 2\mu e}}$$

$$\boxed{H_e = \frac{B_e}{\mu_0} = \frac{\phi}{\mu_0 S} = \frac{\mu (N_1 I_1 + N_2 I_2)}{4l\mu_0 + 2\mu e}}$$

b) $$U = \frac{1}{2} \int_V \vec{B} \cdot \vec{H} \, dv = \frac{1}{2} \left[\underset{\mu H_\mu}{\underset{\shortparallel}{B_\mu}} H_\mu S(4l) + \underset{\mu_0 H_e}{\underset{\shortparallel}{B_e}} H_e S(2e) \right]$$

$$= \frac{1}{2} \left[\mu H_\mu^2 S(4l) + \mu_0 H_e^2 S(2e) \right]$$

$$= \frac{1}{2} \left[\mu \mu_0^2 \frac{(N_1 I_1 + N_2 I_2)^2}{(4l\mu_0 + 2\mu e)^2} S(4l) + \mu_0 \mu \frac{(N_1 I_1 + N_2 I_2)^2}{(4l\mu_0 + 2\mu e)^2} S(2e) \right]$$

$$= \frac{1}{2} \mu \mu_0 \frac{(N_1 I_1 + N_2 I_2)^2}{(4l\mu_0 + 2\mu e)^{\cancel{2}}} \cancel{(4l\mu_0 + 2e\mu)} S$$

$$\Rightarrow \boxed{U(e) = \frac{1}{2} \mu \mu_0 S \frac{(N_1 I_1 + N_2 I_2)^2}{(4l\mu_0 + 2\mu e)}}$$

La fuerza para "lograr desprender" la barra lateral debe ser hecha por un agente externo para contrarrestar la fuerza atractiva del electroimán cuando $e=0$:

$$\vec{F}_{ext} = -\vec{F}\Big|_{e=0}$$, siendo $\vec{F}$ la fuerza magnética sobre la barra lateral

$$\vec{F} = \frac{\partial U}{\partial e}\Big|_{\substack{corr.\\cte.}} \hat{e} = \frac{1}{2}\mu\mu_0 S\,(N_1I_1+N_2I_2)^2\,\frac{\partial\,(4l\mu_0+2\mu e)^{-1}}{\partial e}\,\hat{e}$$

← versor hacia la derecha

$$= \frac{1}{\cancel{2}}\mu\mu_0\,\frac{S(N_1I_1+N_2I_2)^2}{(4l\mu_0+2\mu e)^2}\,(-\cancel{2}\mu)\,\hat{e} = -\mu^2\mu_0\,\frac{S(N_1I_1+N_2I_2)^2}{(4l\mu_0+2\mu e)^2}\,\hat{e}$$

$$\Rightarrow \vec{F}_{ext} = -\vec{F}\Big|_{e=0} = -\left\{-\mu^2\mu_0\,\frac{S\,(N_1I_1+N_2I_2)^2}{(4l\mu_0)^2}\,\hat{e}\right\}$$

$$\boxed{\vec{F}_{ext} = \frac{\mu^2 S\,(N_1I_1+N_2I_2)^2}{16\,l^2\,\mu_0}\,\hat{e}}$$

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