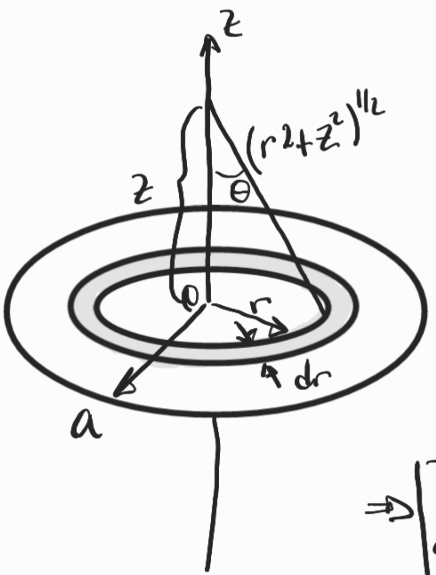
①

a)



$\sigma_0 da = \sigma_0 2\pi r\, dr$

anillo

$$d\vec{E}(z) = \frac{dq}{4\pi\varepsilon_0 \left[(r^2+z^2)^{1/2}\right]^2} \cos\theta \; \hat{z}, \quad \cos\theta = \frac{z}{(r^2+z^2)^{1/2}}$$

$$= \frac{\sigma_0 (\cancel{2}\cancel{\pi} r\, dr)}{\cancel{4}\cancel{\pi}_{2}\varepsilon_0} \frac{z}{(r^2+z^2)^{3/2}} \hat{z}$$

$$\Rightarrow \boxed{d\vec{E}(z) = \frac{\sigma_0 r\, dr\, z}{2\varepsilon_0 (r^2+z^2)^{3/2}} \hat{z}}$$

b)

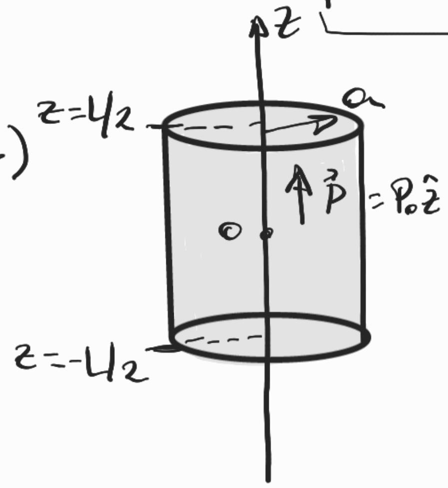
$$\vec{E}(z) = \hat{z} \frac{\sigma_0 z}{2\varepsilon_0} \int_0^a \frac{r}{(r^2+z^2)^{3/2}} dr = \hat{z} \frac{\sigma_0 z}{2\varepsilon_0} \left[-\frac{1}{(r^2+z^2)^{1/2}}\right]\Bigg|_{r=0}^{r=a}$$

$$= \hat{z} \frac{\sigma_0 z}{2\varepsilon_0} \left[-\frac{1}{(a^2+z^2)^{1/2}} + \frac{1}{(z^2)^{1/2}}\right] = \hat{z} \frac{\sigma_0 z}{2\varepsilon_0} \left[\frac{1}{|z|} - \frac{1}{(a^2+z^2)^{1/2}}\right]$$

$$\Rightarrow \boxed{\vec{E}(z) = \left[\pm \frac{\sigma_0}{2\varepsilon_0} - \frac{\sigma_0}{2\varepsilon_0} \frac{z}{(a^2+z^2)^{1/2}}\right] \hat{z} \begin{cases} z>0 \\ z<0 \end{cases}}$$

Para $a \to \infty$: $\boxed{\vec{E}(z) \xrightarrow[a\to\infty]{} \pm \frac{\sigma_0}{2\varepsilon_0} \hat{z} \begin{cases} z>0 \\ z<0 \end{cases}}$

c)



$$\boxed{\rho_P = -\nabla\cdot\vec{P} \overset{\text{cil}}{=} 0}$$

$$\boxed{\sigma_P\big|_{z=L/2} = \vec{P}\cdot\underbrace{\hat{n}}_{\hat{z}} = P_0}$$

$$\boxed{\sigma_P\big|_{z=-L/2} = \vec{P}\cdot\underbrace{\hat{n}}_{-\hat{z}} = -P_0}$$

d) Los campos serán los debidos a dos discos trasladados en z con respecto al origen, con densidades superficiales opuestas: $\underbrace{\sigma_P(L/2)}_{=\sigma_1} = P_0$ y $\underbrace{\sigma_P(-L/2)}_{=\sigma_2} = -P_0$

Usando b):

$$\vec{E}_1(z) = \left[\frac{\pm \overset{=P_0}{\sigma_1}}{2\varepsilon_0} - \frac{\overset{=P_0}{\sigma_1}(z-L/2)}{2\varepsilon_0\left[a^2+(z-L/2)^2\right]^{1/2}}\right]\hat{z} \begin{cases} z>L/2 \\ z<L/2 \end{cases}$$

(disco en $L/2$)

$$\vec{E}_2(z) = \left[\frac{\pm \overset{=-P_0}{\sigma_2}}{2\varepsilon_0} - \frac{\overset{=-P_0}{\sigma_2}(z+L/2)}{2\varepsilon_0\left[a^2+(z+L/2)^2\right]^{1/2}}\right]\hat{z} \begin{cases} z>-L/2 \\ z<-L/2 \end{cases}$$

(disco en $-L/2$)

$$\Rightarrow \vec{E}(z) = \vec{E}_1(z) + \vec{E}_2(z) \quad \text{(superposición)}$$

$z>L/2$

$$E(z) = \pm\frac{P_0}{2\varepsilon_0} \mp \frac{P_0}{2\varepsilon_0} - \frac{P_0(z-L/2)}{2\varepsilon_0\left[a^2+(z-L/2)^2\right]^{1/2}} + \frac{P_0(z+L/2)}{2\varepsilon_0\left[a^2+(z+L/2)^2\right]^{1/2}}$$

$$\boxed{\vec{E}(z) = \frac{P_0}{2\varepsilon_0}\left[\frac{(z+L/2)}{\left[a^2+(z+L/2)^2\right]^{1/2}} - \frac{(z-L/2)}{\left[a^2+(z-L/2)^2\right]^{1/2}}\right]\hat{z}}$$

$$\boxed{\vec{D} = \varepsilon_0\vec{E}}$$

$-L/2<z<L/2$

$$E(z) = -\frac{P_0}{2\varepsilon_0} - \frac{P_0}{2\varepsilon_0} - \frac{P_0(z-L/2)}{2\varepsilon_0\left[a^2+(z-L/2)^2\right]^{1/2}} + \frac{P_0(z+L/2)}{2\varepsilon_0\left[a^2+(z+L/2)^2\right]^{1/2}}$$

$$\boxed{\vec{E}(z) = \left[-\frac{P_0}{\varepsilon_0} - \frac{P_0}{2\varepsilon_0}\left[\frac{(z-L/2)}{\left[a^2+(z-L/2)^2\right]^{1/2}} + \frac{(z+L/2)}{\left[a^2+(z+L/2)^2\right]^{1/2}}\right]\right]\hat{z}}$$

$$\boxed{\vec{D} = \varepsilon_0\vec{E} + \underbrace{\vec{P}}_{=P_0\hat{z}}}$$

$z<-L/2$

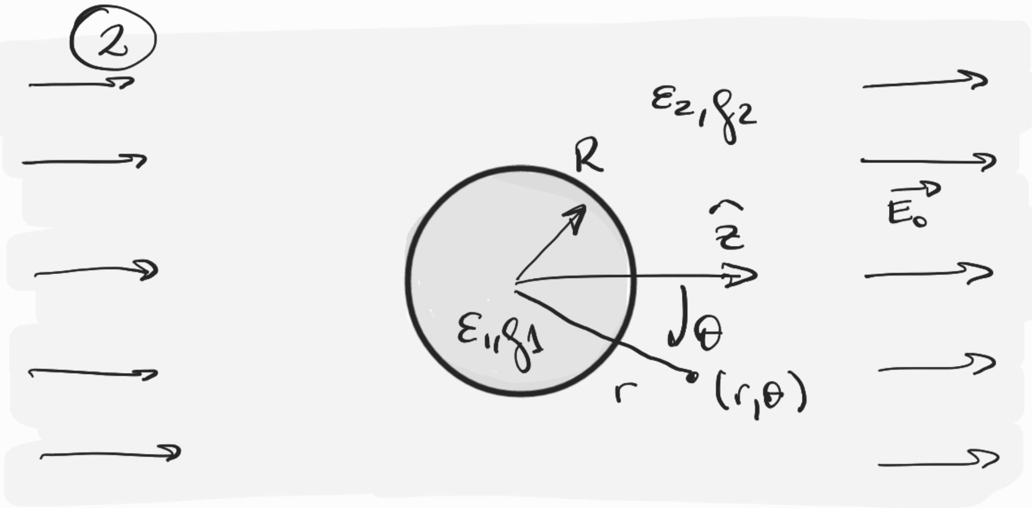
$$E = \cancel{-\frac{P_0}{2\varepsilon_0}} - \frac{P_0(z-L/2)}{2\varepsilon_0\left[a^2+(z-L/2)^2\right]^{1/2}} + \cancel{\frac{P_0}{2\varepsilon_0}} + \frac{P_0(z+L/2)}{2\varepsilon_0\left[a^2+(z+L/2)^2\right]^{1/2}}$$

$$\boxed{\vec{E}(z) = \frac{P_0}{2\varepsilon_0}\left[\frac{z+L/2}{\left[a^2+(z+L/2)^2\right]^{1/2}} - \frac{(z-L/2)}{\left[a^2+(z-L/2)^2\right]^{1/2}}\right]\hat{z}}$$

$$\boxed{\vec{D} = \varepsilon_0\vec{E}}$$

— x —

2



a) $\nabla \times \vec{E}_i = 0 \rightarrow \exists \phi_i$ (estacionario indep. del tiempo)

tal que $\vec{E}_i = -\nabla \phi_i$

$\nabla \cdot \vec{D}_i = 0 \; (\rho_L = 0)$

$\vec{D}_i = \varepsilon_i \vec{E}_i \rightarrow \nabla \cdot \vec{E}_i = 0$

"de"

$\Rightarrow \boxed{\nabla^2 \phi_i = 0}$

(con $i=1, r<R$ e $i=2, r>R$)

condiciones de frontera para los potenciales:

$\boxed{\phi_1 \underset{r\to 0}{} \text{ finito}}$ (I)

$\boxed{\phi_1(r=R) = \phi_2(r=R)}$ (II)

$\boxed{\phi_2 \xrightarrow[r\to\infty]{} -E_0 r\cos\theta}$ (III)

$\left.(\vec{J}_2 - \vec{J}_1)\cdot \underset{\hat{e}_r}{\hat{n}}\right|_{r=R} = 0$ (estacionario) $\rightarrow \boxed{\left. g_2 \frac{\partial \phi_2}{\partial r}\right|_{r=R} = \left. g_1 \frac{\partial \phi_1}{\partial r}\right|_{r=R}}$ (IV)

b) $\phi_1(r,\theta) = \left(Ar + \frac{B}{r^2}\right)\cos\theta \;;\; r<R$

$\phi_2(r,\theta) = \left(Cr + \frac{D}{r^2}\right)\cos\theta \;;\; r>R$

De (I): $\phi_1 \underset{r\to 0}{}$ no diverge $\Rightarrow \boxed{B = 0}$

De (III): $Cr\cos\theta = -E_0 r\cos\theta \rightarrow \boxed{C = -E_0}$

De (II): $AR\cos\theta = \left(-E_0 R + \frac{D}{R^2}\right)\cos\theta \Rightarrow A = -E_0 + \frac{D}{R^3}$

De (IV): $g_2\left(-E_0 - \frac{2D}{R^3}\right) = g_1 A = g_1\left(-E_0 + \frac{D}{R^3}\right)$

$-E_0(g_2 - g_1) = \frac{D}{R^3}(g_1 + 2g_2) \Rightarrow \boxed{D = R^3 E_0 \frac{(g_1 - g_2)}{(g_1 + 2g_2)}}$

$\Rightarrow \boxed{A = -E_0 + E_0\frac{(g_1 - g_2)}{(g_1 + 2g_2)} = \frac{-3g_2 E_0}{(g_1 + 2g_2)}}$

$$\Rightarrow \boxed{\underset{r<R}{\phi_1(r,\theta)} = \left(\frac{-3\varrho_2 E_0}{\varrho_1+2\varrho_2}\right) r\cos\theta}$$

$$\boxed{\underset{r>R}{\phi_2(r,\theta)} = -E_0\left[r - \frac{(\varrho_1-\varrho_2)R^3}{(\varrho_1+2\varrho_2)r^2}\right]\cos\theta}$$

Obs: La validez de ϕ_1 y ϕ_2 sin usar todos los órdenes se justifica por el teorema de unicidad de la solución de la ec. de Laplace, conocidas las condiciones de frontera del problema dado.

$$\boxed{\vec{E}_1(r,\theta)} = -\nabla\phi_1 \overset{esf}{=} -\left[\frac{\partial\phi_1}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial\phi_1}{\partial\theta}\hat{e}_\theta\right] = \boxed{\frac{3\varrho_2 E_0}{\varrho_1+2\varrho_2}(\cos\theta\,\hat{e}_r - \sin\theta\,\hat{e}_\theta)}\;;\; r<R$$

$$\boxed{\vec{E}_2(r,\theta)} = -\nabla\phi_2 = -\left[\frac{\partial\phi_2}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial\phi_2}{\partial\theta}\hat{e}_\theta\right] =$$

$$= \boxed{E_0\left[1 + \frac{2(\varrho_1-\varrho_2)R^3}{(\varrho_1+2\varrho_2)r^3}\right]\cos\theta\,\hat{e}_r - E_0\left[1 - \frac{(\varrho_1-\varrho_2)R^3}{(\varrho_1+2\varrho_2)r^3}\right]\sin\theta\,\hat{e}_\theta}\;;\; r>R$$

c) $$\phi^{(dipolo)} = \frac{1}{4\pi\varepsilon_2}\frac{p\cos\theta}{r^2} \quad \leftarrow \text{potencial debido a un dipolo } \vec{p} = p\hat{z}$$

Comparando con la expresión hallada en (b) para ϕ_2:

$$\Rightarrow \frac{p}{4\pi\varepsilon_2} = \frac{E_0(\varrho_1-\varrho_2)R^3}{(\varrho_1+2\varrho_2)} \Rightarrow \boxed{\vec{p} = 4\pi\varepsilon_2 E_0 R^3\frac{(\varrho_1-\varrho_2)}{(\varrho_1+2\varrho_2)}\hat{z}}$$

momento dipolar efectivo de la esfera

d)

$$\sigma_L = (\vec{D}_2 - \vec{D}_1)\cdot\hat{e}_r\Big|_{r=R} \Rightarrow \sigma_L = \varepsilon_2\vec{E}_2\cdot\hat{e}_r\Big|_R - \varepsilon_1\vec{E}_1\cdot\hat{e}_r\Big|_R$$

$$\Rightarrow \sigma_L(r=R,\theta) = \varepsilon_2 E_0\left[1 + \frac{2(\varrho_1-\varrho_2)\cancel{R^3}}{(\varrho_1+2\varrho_2)\cancel{R^3}}\right]\cos\theta - \varepsilon_1\frac{3\varrho_2 E_0\cos\theta}{\varrho_1+2\varrho_2}$$

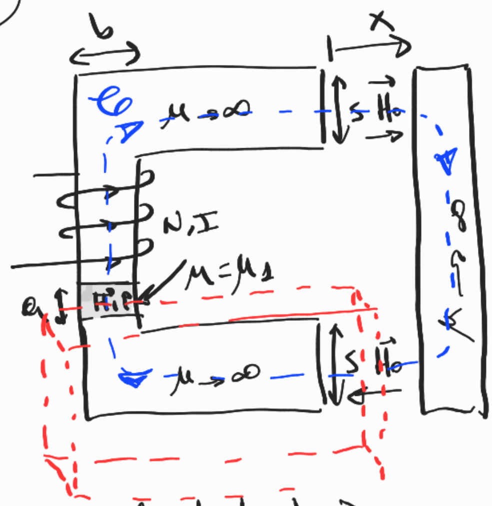
$$= E_0\cos\theta\,\frac{\varepsilon_2(\varrho_1+\cancel{2\varrho_2}) + 2\varepsilon_2(\varrho_1-\cancel{\varrho_2}) - 3\varepsilon_1\varrho_2}{\varrho_1+2\varrho_2}$$

$$\boxed{\sigma_L(r=R,\theta) = 3E_0\cos\theta\left(\frac{\varepsilon_2\varrho_1 - \varepsilon_1\varrho_2}{\varrho_1+2\varrho_2}\right)}$$

$$\sigma_L = 0\ \forall\theta \iff \boxed{\varepsilon_2\varrho_1 = \varepsilon_1\varrho_2} \iff \boxed{\frac{\varepsilon_2}{\varrho_2} = \frac{\varepsilon_1}{\varrho_1}}$$

— x —

③



profundidad D

$\oint \vec{H} \cdot d\vec{l} = NI$ (Ampère forma integral)
$\mathcal{C}$

$H_1 a + 2H_0 x = NI \quad (I)$

$\oint_S \vec{B} \cdot \hat{n}\, da = 0$ (cons. del flujo)

$B_1 = \mu_1 H_1$ y $B_0 = \mu_0 H_0$

$\Rightarrow (\mu_1 H_1)(bD) - (\mu_0 H_0)(sD) = 0$

$H_1 = \dfrac{\mu_0 H_0 s}{\mu_1 b} \quad (II)$

Sustituyendo (II) en (I):

$H_1 a + 2H_0 x = NI \Rightarrow \dfrac{\mu_0 H_0 s a}{\mu_1 b} + 2H_0 x = NI$

$H_0 \left(2x + \dfrac{\mu_0 s a}{\mu_1 b} \right) = NI \rightarrow \boxed{H_0 = \dfrac{\mu_1 b N I}{2\mu_1 b x + \mu_0 s a}}$ (en el sentido indicado en la figura)

$\boxed{H_1 = \dfrac{\mu_0 s}{\cancel{\mu_1 b}} \left(\dfrac{\cancel{\mu_1 b} NI}{2\mu_1 b x + \mu_0 s a} \right) = \dfrac{\mu_0 s N I}{2\mu_1 b x + \mu_0 s a}}$ (en el sentido indicado en la figura)

b) Por conservación del flujo: $\boxed{\Phi = \mu_0 H_0 s D = \mu_1 H_1 b D = \dfrac{\mu_0 s D \mu_1 b N I}{2\mu_1 b x + \mu_0 s a}}$

$\boxed{L = N \dfrac{d\Phi}{dI} = \dfrac{N^2 \mu_0 \mu_1 s b D}{2\mu_1 b x + \mu_0 s a}}$

c) $\boxed{U = \dfrac{1}{2} L I^2 = \dfrac{1}{2} \dfrac{\mu_0 \mu_1 s b D \, (NI)^2}{(2\mu_1 b x + \mu_0 s a)}}$

$\boxed{U = \dfrac{1}{2} \int \vec{B} \cdot \vec{H}\, dv} = 2\left[\dfrac{1}{2} (\mu_0 H_0) H_0 (x s D) \right] + \dfrac{1}{2} (\mu_1 H_1) H_1 (a b D)$

$= \mu_0 \left[\dfrac{\mu_1 b N I}{2\mu_1 b x + \mu_0 s a} \right]^2 (x s D) + \dfrac{1}{2} \mu_1 \left[\dfrac{\mu_0 s N I}{2\mu_1 b x + \mu_0 s a} \right]^2 (a b D)$

$$= \frac{1}{2}\frac{\mu_0\mu_1 sbD(NI)^2}{(2\mu_1 bx+\mu_0 sa)}\left[\frac{\cancel{2\mu_1 bx+\mu_0 sa}}{\cancel{2\mu_1 bx+\mu_0 sa}}\right] = \boxed{\frac{1}{2}\frac{\mu_0\mu_1 sbD(NI)^2}{(2\mu_1 bx+\mu_0 sa)}}$$

d) $$\vec{F} = \frac{\partial U}{\partial x}\Big|_I \hat{x} = \frac{1}{2} I^2 \frac{dL}{dx} = \frac{1}{\cancel{2}}\frac{(NI)^2(\mu_0\mu_1 sbD)(-\cancel{2}\mu_1 b)}{(2\mu_1 bx+\mu_0 sa)^2}$$

$$= \boxed{-\frac{(NI)^2\mu_0\mu_1^2 sb^2 D}{(2\mu_1 bx+\mu_0 sa)^2}\hat{x}}$$

— x —