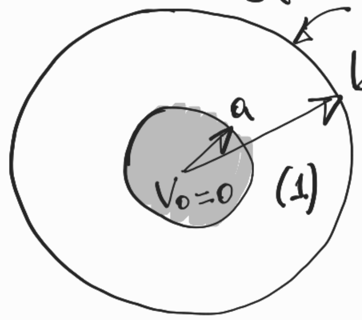
ELECTROMAGNETISMO 1120 – PRIMER PARCIAL 2022

(1) $\sigma_b(r=b,\theta) = \sigma_0 \cos\theta$



$\phi_1(r,\theta) \leftarrow$ potencial en la región (1) $(a<r<b)$

$\phi_2(r,\theta) \leftarrow$ potencial en la región (2) $(r>b)$

a) Condiciones sobre el potencial electrostático:

$\boxed{\phi_1(r=a,\theta) = V_0 = 0}$ (i)

$\boxed{\phi_2(r=b,\theta) = \phi_1(r=b,\theta)}$ (ii)

$\boxed{\phi_2(r,\theta) \xrightarrow[r\to\infty]{} 0}$ (iii)

$(\vec{D}_2 - \vec{D}_1)\cdot\hat{e}_r = \sigma_0\cos\theta \longrightarrow \boxed{-\left.\frac{\partial\phi_2}{\partial r}\right|_{r=b} + \left.\frac{\partial\phi_1}{\partial r}\right|_{r=b} = \frac{\sigma_0}{\varepsilon_0}\cos\theta}$ (iv)

$\vec{D}_2 = \varepsilon_0\vec{E}_2 = -\nabla\phi_2$, $\vec{D}_1 = \varepsilon_0\vec{E}_1 = -\nabla\phi_1$

b) $\phi_1(r,\theta) = \left(A_1 r + \frac{B_1}{r^2}\right)\cos\theta \leftarrow$ propongo solución para la región (1)

$\phi_2(r,\theta) = \left(A_2 r + \frac{B_2}{r^2}\right)\cos\theta \leftarrow$ propongo solución para la región (2)

De (i): $V_0 = \left(A_1 a + \frac{B_1}{a^2}\right)\cos\theta \quad \forall\theta \rightarrow \boxed{B_1 = -a^3 A_1}$, ($V_0 = 0$)

$\Rightarrow \phi_1(r,\theta) = A_1\left(r - \frac{a^3}{r^2}\right)\cos\theta$

De (iii): $\boxed{A_2 = 0} \Rightarrow \phi_2(r,\theta) = \frac{B_2}{r^2}\cos\theta$

De (ii): $\frac{B_2}{\cancel{b^2}} = A_1\left(\frac{b^3 - a^3}{\cancel{b^2}}\right) \Rightarrow B_2 = (b^3 - a^3)A_1$

$\Rightarrow \phi_2(r,\theta) = A_1\frac{(b^3 - a^3)}{r^2}\cos\theta$

Sustituyendo en (iv):

$$2A_1\frac{(b^3 - a^3)}{b^3}\cos\theta + A_1\left(1 + \frac{2a^3}{b^3}\right)\cos\theta = \frac{\sigma_0}{\varepsilon_0}\cos\theta$$

$$A_1\left[2 - \cancel{\frac{2a^3}{b^3}} + 1 + \cancel{\frac{2a^3}{b^3}}\right] = \frac{\sigma_0}{\varepsilon_0} \Rightarrow \boxed{A_1 = \frac{\sigma_0}{3\varepsilon_0}}$$

$$\Rightarrow \boxed{\phi_1(r,\theta) = \frac{(r^3 - a^3)}{3\varepsilon_0 r^2}\sigma_0\cos\theta}$$

$$\boxed{\phi_2(r,\theta) = \frac{b^3 - a^3}{3\varepsilon_0 r^2}\sigma_0\cos\theta}$$

Como el resultado final verifica la ecuación de Laplace y las condiciones de frontera, entonces es la solución correcta por el teorema de unicidad.

c) $\underbrace{\varepsilon(\vec{D}_1}_{-\varepsilon_0 \frac{\partial \phi_1}{\partial r}\big|_{r=a}} - \underbrace{\vec{D}_0}_{0}).\hat{e}_r = \sigma_a(r=a,\theta) \Rightarrow \sigma_a(\theta) = -\varepsilon_0 \frac{\partial \phi_1}{\partial r}\Big|_{r=a}$

$\vec{E}=0$ (conductor en equilibrio electrostático)

$$\Rightarrow \sigma_a(\theta) = -\varepsilon_0 \frac{\partial \phi_1}{\partial r}\Big|_{r=a} = -\frac{\cancel{\varepsilon_0}}{3\cancel{\varepsilon_0}} \sigma_0 \cos\theta \underbrace{\frac{d}{dr}\left(r - \frac{a^3}{r^2}\right)\Big|_{r=a}}_{1+2\frac{a^3}{r^3}\big|_{r=a} = 3} = -\frac{\cancel{3}}{\cancel{3}}\sigma_0\cos\theta$$

$$\Rightarrow \boxed{\sigma_a(\theta) = -\sigma_0 \cos\theta}$$

d) $\phi_e(r,\theta) = \frac{(b^3-a^3)\sigma_0 \cos\theta}{3\varepsilon_0 r^2}$

Desarrollo multipolar de una distribución de carga hasta el término dipolar es de la forma:

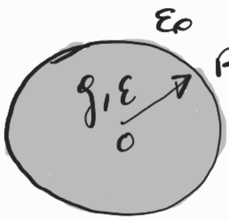
$$\phi(r,\theta) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r} + \frac{1}{4\pi\varepsilon_0} P\frac{\cos\theta}{r^2} \qquad \text{con } \vec{P} = p\hat{z}$$

Comparando término a término, vemos que la carga neta es cero ($Q=0$), no hay término en $\frac{1}{r}$ en ϕ_e.

Y $\frac{1}{4\pi\cancel{\varepsilon_0}} P \frac{\cos\theta}{\cancel{r^2}} = \frac{(b^3-a^3)\sigma_0}{3\cancel{\varepsilon_0}}\frac{\cos\theta}{\cancel{r^2}} \Rightarrow \boxed{P = \frac{4\pi}{3}(b^3-a^3)\sigma_0}$

—x—

(2)



$$\rho(\vec{r}, t=0) = \rho_0 \frac{r}{R}$$

a) Ec. continuidad:

$$\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0 \quad \text{dentro del material}$$

$$\vec{J} = g\vec{E} = \frac{g}{\varepsilon}\vec{D}$$

$$\frac{g}{\varepsilon} \underbrace{\nabla \cdot \vec{D}}_{\rho} + \frac{\partial \rho}{\partial t} = 0$$

$$\Rightarrow \frac{\partial \rho}{\partial t} + \frac{g}{\varepsilon}\rho = 0 \rightarrow \rho(r,t) = \rho(r,0)\, e^{-\frac{g}{\varepsilon}t} \Rightarrow \boxed{\rho(r,t) = \rho_0 \frac{r}{R} e^{-\frac{g}{\varepsilon}t}}$$

b) En el material

$$\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t} = \rho_0 \frac{r}{R} \frac{g}{\varepsilon} e^{-\frac{g}{\varepsilon}t} \Rightarrow \frac{\partial (r^2 J)}{\partial r} = \rho_0 \frac{r^3}{R} \frac{g}{\varepsilon} e^{-\frac{g}{\varepsilon}t}$$

$$\frac{1}{r^2}\frac{\partial (r^2 J)}{\partial r} \text{ (esf.)}$$

$$r^2 \vec{J} = \frac{\rho_0}{4} \frac{r^4}{R} \frac{g}{\varepsilon} e^{-\frac{g}{\varepsilon}t}$$

$$\Rightarrow \boxed{\vec{J}(r,t) = \underbrace{\frac{\rho_0}{4} \frac{r^2}{R} \frac{g}{\varepsilon}}_{=\vec{J}(r,0)} e^{-\frac{g}{\varepsilon}t}\, \hat{e_r}} \quad (r<R)$$

$$\Rightarrow \boxed{\vec{E}(r,t) = \frac{\vec{J}}{g} = \frac{\rho_0}{4\varepsilon} \frac{r^2}{R} e^{-\frac{g}{\varepsilon}t}\, \hat{e_r}} \quad (r<R)$$

Afuera $(r>R)$ $\boxed{\vec{J} = 0}$ ($g = 0$, vacío)

Como la carga encerrada se conserva, la puedo calcular en $t=0$

$$\underset{\text{Gauss}}{\varepsilon_0 E(r,t)(4\pi r^2)} = \int_V \left(\rho_0 \frac{r}{R}\right) dv = \frac{\rho_0}{R} 4\pi \int_0^R r^3\, dr$$

$$E(r,t)(\cancel{4\pi} r^2) = \frac{\rho_0}{\cancel{R}} \frac{\cancel{4\pi}}{\varepsilon_0} \frac{R^{\cancel{4}3}}{4} \Rightarrow \boxed{\vec{E}(r,t) = \frac{\rho_0 R^3}{4\varepsilon_0 r^2}\, \hat{e_r}} \quad (r>R)$$

(indep. de t)

c) $$\left.(\vec{D_2} - \vec{D_1}) \cdot \hat{n}\right|_{r=R} = \sigma_R(R,t)$$

$$\vec{D_2} = \varepsilon_0 \vec{E_2}, \quad \vec{D_1} = \varepsilon \vec{E_1}, \quad \hat{n} = \hat{e_r}$$

$\vec{E_1}$: $\vec{E}$ para $r<R$

$\vec{E_2}$: $\vec{E}$ para $r>R$

$$\begin{bmatrix} (1)\ \text{región } r<R \\ (2)\ \text{región } r>R \\ \hat{n} = \hat{e_r} \end{bmatrix}$$

$$\Rightarrow \sigma_R(R,t) = \cancel{\varepsilon_0} \frac{\rho_0 R^{\cancel{3}}}{4\cancel{\varepsilon_0} \cancel{R^2}} - \cancel{\varepsilon} \frac{\rho_0}{4\cancel{\varepsilon}} \frac{R^{\cancel{2}}}{\cancel{R}} e^{-\frac{g}{\varepsilon}t} \Rightarrow \boxed{\sigma_R(R,t) = \frac{\rho_0 R}{4}\left(1 - e^{-\frac{g}{\varepsilon}t}\right)}$$

d) $$Q(t) = \int_V \rho(r,t)\, dv + \underbrace{\int_S \sigma_R(t)\, da}_{\sigma_R(t)(4\pi R^2)} = \left[\frac{\rho_0}{\cancel{R}} 4\pi \underbrace{\int_0^R r^3 dr}_{\frac{R^{\cancel{4}3}}{\cancel{4}}}\right] e^{-\frac{g}{\varepsilon}t} + \frac{\rho_0 R}{4}\left(1 - e^{-\frac{g}{\varepsilon}t}\right)$$

$$= \rho_0 R^3 \pi\, \cancel{e^{-\frac{g}{\varepsilon}t}} + \rho_0 R^3 \pi \left(1 - \cancel{e^{-\frac{g}{\varepsilon}t}}\right) \Rightarrow \boxed{Q(t) = \rho_0 \pi R^3} = \text{cte}$$