Electromagnetismo (1128) Examen 22/7/23

1

a) Gauss $\oint_S \vec{D}\cdot\hat{n}\,dS \underset{\substack{\uparrow \\ \text{Por la simetría } \vec{D}=\vec{D}(r)}}{=} D(r)(4\pi r^2) = q \rightarrow \boxed{\vec{D}(r) = \frac{q}{4\pi r^2}\hat{e}_r}, \forall r$

$$\boxed{\vec{E}(r)} = \begin{cases} \frac{\vec{D}}{\varepsilon_0} = \frac{q}{4\pi\varepsilon_0 r^2}\hat{e}_r ; & 0<r<b \text{ y } r>a \\ \frac{\vec{D}}{\varepsilon} = \boxed{\frac{q}{4\pi\varepsilon r^2}\hat{e}_r} ; & b<r<a \end{cases}$$

$$\vec{D} = \varepsilon_0\vec{E}+\vec{P} \rightarrow \boxed{\vec{P}} = \vec{D}-\varepsilon_0\vec{E} = \begin{cases} \boxed{0} ; & 0<r<b \text{ y } r>a \\ \boxed{\left(1-\frac{\varepsilon_0}{\varepsilon}\right)\frac{q}{4\pi r^2}\hat{e}_r} ; & b<r<a \end{cases}$$

b) $\rho_P = -\nabla\cdot\vec{P}$

$$\begin{cases} \boxed{\rho_P = 0} \ (\vec{P}=0) ; & 0<r<b \text{ y } r>a \\ \boxed{\rho_P = -\nabla\cdot\vec{P} \overset{\text{esf}}{=} \frac{1}{r^2}\frac{\partial(\overbrace{r^2 P_r}^{\text{de indep } r})}{\partial r} = 0} , & b<r<a \end{cases} \quad \Bigg| \quad \underline{\underline{\rho_P = 0 \ \forall r}}$$

$$\boxed{\sigma_P\big|_{r=b} = \vec{P}\cdot\underset{-\hat{e}_r}{\hat{n}}\Big|_{r=b} = -\left(1-\frac{\varepsilon_0}{\varepsilon}\right)\frac{q}{4\pi b^2}}$$

$$\boxed{\sigma_P\big|_{r=a} = \vec{P}\cdot\underset{\hat{e}_r}{\hat{n}}\Big|_{r=a} = \left(1-\frac{\varepsilon_0}{\varepsilon}\right)\frac{q}{4\pi a^2}}$$

c) $$\boxed{Q_P\big|_{r=b}} = \int_{S_b}\sigma_P\big|_{r=b}\,dS = \underbrace{\left(\sigma_P\big|_{r=b}\right)}_{\text{cte en } r=b}(4\pi b^2) = -\left(1-\frac{\varepsilon_0}{\varepsilon}\right)\frac{q}{\cancel{4\pi b^2}}\cancel{(4\pi b^2)} = \boxed{-q+\frac{\varepsilon_0}{\varepsilon}q}$$

$$\boxed{Q_{tot} = \underbrace{\int_{S_b} \sigma_P|_{r=b}\, dS}_{\overset{\shortparallel}{-q+\frac{\varepsilon_0}{\varepsilon}q}} + \underbrace{\int_{S_a} \sigma_P|_{r=a}\, dS}_{\overset{\shortparallel}{\left(1-\frac{\varepsilon_0}{\varepsilon}\right)\frac{q}{\cancel{4\pi a^2}}\cancel{4\pi a^2}}} + \underbrace{\int_{V_{ab}} \overset{\overset{0}{\shortparallel}}{\rho_P}\, dV}_{\overset{\shortparallel}{0}} = 0}$$

Obs:
Si bien las moléculas del material dieléctrico se polarizan en respuesta al campo debido a la carga q, dando lugar a cargas superficiales de polarización, el material en su conjunto permanece eléctricamente neutro ($Q_{tot}=0$)

d) límites $a \to \infty$
$b \to 0$

$$\boxed{\vec{E}(r) = \frac{q}{4\pi\varepsilon r^2}\,\hat{e}_r} \quad ; \; 0 < r < \infty$$

$$\boxed{\vec{E}_0(r) = \frac{q}{4\pi\varepsilon_0 r^2}\,\hat{e}_r} \quad ; \; 0 < r < \infty \leftarrow$$ campo debido a carga en el vacío

$$\Rightarrow \vec{E}(r) = \underset{\overset{\wedge}{1}}{\left(\frac{\varepsilon_0}{\varepsilon}\right)} \vec{E}_0(r) \leftarrow$$ Si la carga está rodeada por dieléctrico, su campo se ve apantallado por ese dieléctrico y se reduce en un factor $\frac{\varepsilon_0}{\varepsilon}$

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(2)

a) Consideramos soluciones de la forma: $\phi(\cancel{\rho},\varphi,\cancel{z}) = \phi(\varphi)$

$\phi_1(\varphi) = A_1\varphi + B_1 \quad ; 0<\varphi<\pi/2 \quad$ (región 1)

$\phi_2(\varphi) = A_2\varphi + B_2 \quad ; \pi/2<\varphi<\pi \quad$ (región 2)

Ec. de Laplace: $\nabla^2\phi \overset{\text{cil}}{\underset{\phi=\phi(\varphi)}{=}} 0 \rightarrow \dfrac{d^2\phi}{d\varphi^2} = 0$

$\dfrac{d\phi_1}{d\varphi} = A_1 \rightarrow \dfrac{d^2\phi_1}{d\varphi^2} = 0$ ✓ $\quad \Big\| \quad \dfrac{d\phi_2}{d\varphi} = A_2 \rightarrow \dfrac{d^2\phi_2}{d\varphi^2} = 0$ ✓

Condiciones de frontera:

* $\phi_1(\varphi=0) = V_0 \rightarrow \boxed{B_1 = V_0}$ (i)
* $\phi_2(\varphi=\pi) = 0 \rightarrow A_2\pi + B_2 = 0 \rightarrow B_2 = -A_2\pi$ (ii)
* $\phi_1(\varphi=\pi/2) = \phi_2(\varphi=\pi/2) \rightarrow A_1\frac{\pi}{2} + B_1 = A_2\frac{\pi}{2} + B_2 \rightarrow$

  $A_1\frac{\pi}{2} + V_0 \overset{(ii)}{=} A_2\left(\frac{\pi}{2} - \pi\right) = -\frac{\pi}{2}A_2$ (iii)
* $\underbrace{\vec{J}_1}_{g_1\vec{E}_1 = -g_1\nabla\phi}\cdot\hat{e}_\varphi\Big|_{\varphi=\frac{\pi}{2}} \overset{(\text{estac.})}{=} \underbrace{\vec{J}_2}_{g_2\vec{E}_2 = -g_2\nabla\phi_2}\cdot\hat{e}_\varphi\Big|_{\varphi=\frac{\pi}{2}} \overset{(\text{cil})}{\rightarrow} g_1\dfrac{d\phi_1}{d\varphi}\Big|_{\frac{\pi}{2}} = g_2\dfrac{d\phi_2}{d\varphi}\Big|_{\frac{\pi}{2}} \rightarrow g_1A_1 = g_2A_2$ (iv)

Sust (iv) en (iii): $A_1\frac{\pi}{2} + V_0 = -\dfrac{g_1A_1}{g_2}\dfrac{\pi}{2} \rightarrow \frac{\pi}{2}A_1\left(1 + \dfrac{g_1}{g_2}\right) = -V_0$

$\rightarrow \boxed{A_1 = \dfrac{-2g_2}{(g_1+g_2)}\dfrac{V_0}{\pi}} \xrightarrow{(iv)} \boxed{A_2 = \dfrac{g_1}{g_2}A_1 = \dfrac{-2g_1}{(g_1+g_2)}\dfrac{V_0}{\pi}} \xrightarrow{(ii)} \boxed{B_2 = \dfrac{2g_1V_0}{g_1+g_2}}$

b) $\boxed{\vec{E}_1} = -\nabla\phi_1 \overset{\text{cil}}{=} -\dfrac{1}{\rho}\dfrac{d\phi_1}{d\varphi}\hat{e}_\varphi = -\dfrac{1}{\rho}A_1\hat{e}_\varphi = \boxed{\dfrac{2g_2V_0}{(g_1+g_2)\pi}\left(\dfrac{1}{\rho}\right)\hat{e}_\varphi}$

$\phi=\phi(\varphi)$

$\boxed{\vec{J}_1 = g_1\vec{E}_1 = \dfrac{2g_1g_2V_0}{(g_1+g_2)\pi}\left(\dfrac{1}{\rho}\right)\hat{e}_\varphi}$

$\boxed{\vec{E}_2} = -\nabla\phi_2 \overset{\text{cil}}{=} -\dfrac{1}{\rho}\dfrac{d\phi_2}{d\varphi}\hat{e}_\varphi = -\dfrac{1}{\rho}A_2\hat{e}_\varphi = \boxed{\dfrac{2g_1V_0}{(g_1+g_2)\pi}\left(\dfrac{1}{\rho}\right)\hat{e}_\varphi}$

$\boxed{\vec{J}_2 = g_2\vec{E}_2 = \dfrac{2g_1g_2V_0}{(g_1+g_2)\pi}\left(\dfrac{1}{\rho}\right)\hat{e}_\varphi}$

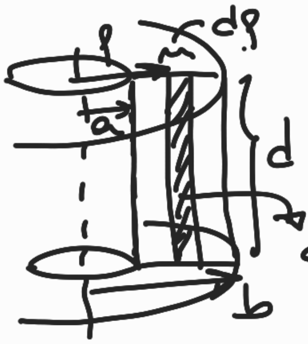
c) $$\boxed{\sigma_L(\rho,\varphi=0)} = \underbrace{\vec{D}_1}_{\varepsilon_1\vec{E}_1}\cdot\underbrace{\hat{n}}_{\hat{e}_\varphi}\Big|_{\varphi=0} = \boxed{\frac{2\varepsilon_1 g_2 V_0}{(g_1+g_2)\pi}\left(\frac{1}{\rho}\right)}$$

$$\boxed{\sigma_L\left(\rho,\varphi=\frac{\pi}{2}\right)} = (\underbrace{\vec{D}_2}_{\varepsilon_2\vec{E}_2}-\underbrace{\vec{D}_1}_{\varepsilon_1\vec{E}_1})\cdot\underbrace{\hat{n}}_{\hat{e}_\varphi}\Big|_{\varphi=\frac{\pi}{2}} = \boxed{\frac{2V_0}{(g_1+g_2)\pi}\left(\frac{1}{\rho}\right)\left[\varepsilon_2 g_1-\varepsilon_1 g_2\right]}$$

$$\boxed{\sigma_L(\rho,\varphi=\pi)} = \underbrace{\vec{D}_2}_{\varepsilon_2\vec{E}_2}\cdot\underbrace{\hat{n}}_{-\hat{e}_\varphi}\Big|_{\varphi=\pi} = \boxed{-\frac{2\varepsilon_2 g_1 V_0}{(g_1+g_2)\pi}\left(\frac{1}{\rho}\right)}$$

d) $$I=\int_S \vec{J}\cdot\underbrace{\hat{n}}_{\hat{e}_\varphi}\,dS$$

$(\vec{J}_1=\vec{J}_2)$



$\Rightarrow$

$$\boxed{I=\int_{r=b}^{a}\frac{2g_1g_2V_0}{(g_1+g_2)\pi}\left(\frac{1}{\rho}\right)(d\rho)d = \frac{2g_1g_2V_0(d)}{(g_1+g_2)\pi}\underbrace{\int_b^a\frac{d\rho}{\rho}}_{\ln\left(\frac{a}{b}\right)} = \frac{2g_1g_2V_0(d)}{(g_1+g_2)\pi}\ln\left(\frac{a}{b}\right)}$$

$$\boxed{R=\frac{V_0}{I}=\frac{(g_1+g_2)\pi}{2g_1g_2(d)\ln(a/b)}}$$

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(3)

Ampère: $\oint_{\mathcal{C}} \vec{H}\cdot d\vec{l} = NI \rightarrow H_n l_n + H_m l_m + H_0 l_g = NI \quad (*)$

cons. flujo: $B_m \cancel{A} = B_n \cancel{A} = B_0 \cancel{A} \quad (2) \rightarrow \underline{\underline{B_0 = B_n = B_m}}$

$H_n = \frac{B_n}{\mu} = \frac{B_m}{\mu} \Rightarrow \boxed{H_n = 0}$ $\quad \mu \to \infty$

(núcleo)

$\boxed{H_0 = \frac{B_0}{\mu_0} = \frac{B_m}{\mu_0}}$

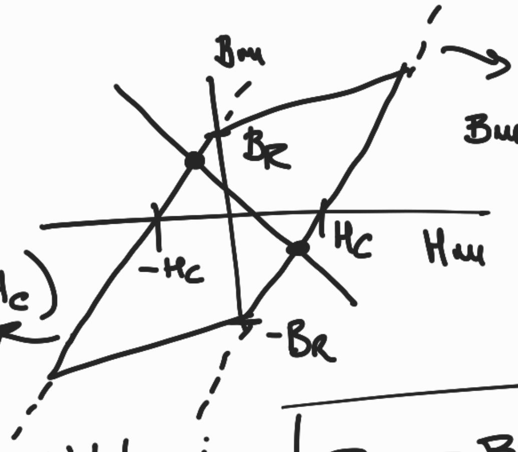
(entrehierro)

sust. H_n y H_0 en función de B_m en (*):

$$H_m l_m + \frac{B_m}{\mu_0} l_g = NI = \underbrace{2H_c l_g}_{\text{(letra)}} \Rightarrow \frac{B_m}{\mu_0} l_g + H_m l_m = 2H_c l_g$$

$$\boxed{B_m = -\left(\mu_0 \frac{l_m}{l_g}\right) H_m + 2\mu_0 H_c} \quad (1)$$

← los posibles valores (B_m, H_m) en el imán deben estar en esta recta y a la vez pertenecer a la curva de histéresis.



$B_m = \frac{B_R}{H_c}(H_m - H_c)$

$B_m = \frac{B_R}{H_c}(H_m + H_c)$

De la curva de histéresis: $\boxed{B_m = \frac{B_R}{H_c}(H_m \mp H_c)} \quad (2)$

Igualando (2) y (1):

$$\underbrace{\left(\frac{B_R}{\mu_0 H_c}\right)}_{\simeq 3} (H_m \mp H_c) = -\left(\frac{l_m}{l_g}\right) H_m + 2H_c$$

← letra

$$\left(3+\frac{l_m}{l_g}\right)H_m = H_c\,(2\pm 3)\ \begin{cases}(+)\\(-)\end{cases}$$

$$\boxed{H_m = \frac{5}{\left(3+\frac{l_m}{l_g}\right)}H_c} \rightarrow B_m = \underbrace{\frac{B_R}{H_c}}_{=3\mu_0}(H_m - H_c) = 3\mu_0\left[\frac{5H_c}{\left(3+\frac{l_m}{l_g}\right)} - H_c\right] = \boxed{3\mu_0\frac{\left(2-\frac{l_m}{l_g}\right)H_c}{\left(3+\frac{l_m}{l_g}\right)}}$$

$$\boxed{H_m = \frac{-1}{\left(3+\frac{l_m}{l_g}\right)}H_c} \rightarrow B_m = \underbrace{\frac{B_R}{H_c}}_{=3\mu_0}(H_m + H_c) = 3\mu_0\left[\frac{-1}{\left(3+\frac{l_m}{l_g}\right)}H_c + H_c\right] = \boxed{3\mu_0\frac{\left(2+\frac{l_m}{l_g}\right)H_c}{\left(3+\frac{l_m}{l_g}\right)}}$$

b) B_m sólo puede anularse en el primer caso (+), para $\boxed{\frac{l_m}{l_g} = 2}$

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