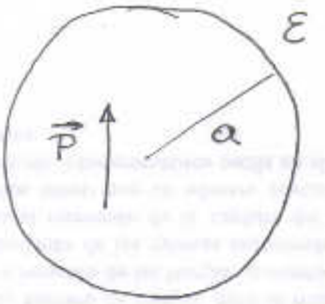
Primer Parcial Electromagnetismo (1128)
17/10/2020

Ejercicio Nº 1



parte a:

$$\nabla \wedge \vec{E} = 0 \Rightarrow \vec{E} = -\nabla \varphi$$

Medio lineal: $\nabla . \vec{D} = 0$

$$\vec{D} = \varepsilon \vec{E} \Rightarrow \nabla . \vec{E} = 0 \Rightarrow \nabla^2 \varphi = 0$$

Esfera: $\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$

$$\nabla . \vec{D} = \varepsilon_0 \nabla . \vec{E} + \underbrace{\nabla . \vec{P}}_{= 0} = \varepsilon_0 \nabla . \vec{E} = 0 \Rightarrow \nabla^2 \varphi = 0$$

parte b: $r < a$: $\varphi(\vec{r}) = \varphi_1(\vec{r}) = \sum_{n=0}^{\infty} \left(A_n r^n + \frac{B_n}{r^{n+1}} \right) P_n(\cos\theta)$

$r > a$: $\varphi(\vec{r}) = \varphi_2(\vec{r}) = \sum_{n=0}^{\infty} \left(C_n r^n + \frac{D_n}{r^{n+1}} \right) P_n(\cos\theta)$

$\varphi(0)$ acotado $\Rightarrow B_n = 0 \ \forall n$

$\varphi(r \to \infty) = 0 \Rightarrow C_n = 0 \ \forall n$

Condiciones de borde $(r = a)$: $\vec{D}_2 . \vec{e}_r = \vec{D}_1 . \vec{e}_r$ porque no hay carga libre en la interfaz

$$\vec{E}_2 . \vec{e}_\theta = \vec{E}_1 . \vec{e}_\theta$$

$\vec{e}_r, \vec{e}_\theta$ de coordenadas esféricas

$\vec{D}_2 = \varepsilon \vec{E}_2$

$$\vec{E}_2 = -\nabla \varphi_2 = -\frac{\partial \varphi_2}{\partial r} \vec{e}_r - \frac{1}{r} \frac{\partial \varphi_2}{\partial \theta} \vec{e}_\theta =$$

$$= + \sum_{n=0}^{\infty} \frac{(n+1) D_n}{r^{n+2}} P_n(\cos\theta) \vec{e}_r - \sum_{n=0}^{\infty} \frac{D_n}{r^{n+2}} \frac{d}{d\theta} P_n(\cos\theta) \vec{e}_\theta$$

$$\vec{D}_1 = \varepsilon_0 \vec{E}_1 + \vec{P} = \varepsilon_0 \vec{E}_1 + P \vec{k}$$

$$\vec{E}_1 = -\nabla \varphi_1 = -\sum_{n=0}^{\infty} n A_n r^{n-1} P_n(\cos\theta) \vec{e}_r - \sum_{n=0} A_n r^{n-1} \frac{d}{d\theta} P_n(\cos\theta) \vec{e}_\theta$$

$$\Rightarrow \varepsilon \sum_{n=0}^{\infty} \frac{(n+1) D_n}{a^{n+2}} P_n(\cos\theta) = -\varepsilon_0 \sum_{n=0}^{\infty} n A_n a^{n-1} P_n(\cos\theta) + P \cos\theta$$

$(\vec{k} . \vec{e}_r = \cos\theta)$

$\Rightarrow D_0 = 0$

$$\frac{\varepsilon 2 D_1}{a^3} = -\varepsilon_0 A_1 + P$$

$$\frac{\varepsilon(n+1)D_n}{a^{n+2}} = -\varepsilon_0 n A_n a^{n-1} \quad n\geq 2$$

$$-\sum_{n=0}^{\infty} \frac{D_n}{a^{n+2}} \frac{d}{d\theta}P_n(\cos\theta) = -\sum_{n=0}^{\infty} A_n a^{n-1}\frac{d}{d\theta}P_n(\cos\theta)$$

$$P_0(\cos\theta)=1 \Rightarrow \frac{dP_0(\cos\theta)}{d\theta}=0$$

$$-\frac{D_n}{a^{n+2}} = -A_n a^{n-1} \quad n\geq 1$$

$$\frac{D_n}{a^{n+2}} = +A_n a^{n-1} \Rightarrow +\varepsilon(n+1)A_n \cancel{a^{n-1}} = -\varepsilon_0 n A_n \cancel{a^{n-1}} \quad n\geq 2$$

$$A_n\underbrace{\left[\varepsilon(n+1)+\varepsilon_0 n\right]}_{\text{En principio } \neq 0} = 0 \Rightarrow A_n=0 \text{ y } D_n=0$$

para $n\geq 2$

$$\frac{D_1}{a^3} = +A_1$$

$$\frac{\varepsilon\, 2D_1}{a^3} = -\varepsilon_0\frac{D_1}{a^3} + P \Rightarrow (2\varepsilon+\varepsilon_0)\frac{D_1}{a^3} = P$$

$$D_1 = \frac{Pa^3}{2\varepsilon+\varepsilon_0} \qquad A_1=\frac{P}{2\varepsilon+\varepsilon_0}$$

$$\vec E_1 = \frac{P}{2\varepsilon+\varepsilon_0}\left(-\cos\theta\,\vec e_r + \operatorname{sen}\theta\,\vec e_\theta\right)$$

$$\vec k = \cos\theta\,\vec e_r - \operatorname{sen}\theta\,\vec e_\theta \Rightarrow \boxed{\vec E_1 = -\frac{P\vec k}{2\varepsilon+\varepsilon_0}}$$

$$\boxed{\vec E_2 = \frac{Pa^3}{2\varepsilon+\varepsilon_0}\frac{\left(2\cos\theta\,\vec e_r + \operatorname{sen}\theta\,\vec e_\theta\right)}{r^3}}$$

partec: $\rho_p = -\nabla\cdot\vec P = 0 \ \forall \vec r$

$\sigma_p = \vec P\cdot\vec n$

Para esfera: $\vec n=\vec e_r \Rightarrow \sigma_p = \vec P\cdot\vec e_r = \boxed{P\cos\theta = \sigma_p}$

" dieléctrico exterior: $\vec n = -\vec e_r$

$$\vec{P} = \vec{D} - \varepsilon_0 \vec{E} = (\varepsilon - \varepsilon_0)\vec{E}$$

$$\sigma_p = -(\varepsilon - \varepsilon_0)\vec{E}_2 \cdot \vec{e}_r = -\frac{(\varepsilon-\varepsilon_0) P a^3}{2\varepsilon + \varepsilon_0}\,\frac{2\cos\theta}{a^3}$$

$$\boxed{\sigma_p = -\frac{2(\varepsilon-\varepsilon_0) P\cos\theta}{2\varepsilon+\varepsilon_0}}$$

$$\sigma_{total} = P\cos\theta - \frac{2(\varepsilon-\varepsilon_0)P\cos\theta}{2\varepsilon+\varepsilon_0} = \frac{(\cancel{2\varepsilon}+\varepsilon_0-\cancel{2\varepsilon}+2\varepsilon_0)P\cos\theta}{2\varepsilon+\varepsilon_0}$$

$$\boxed{\sigma_{total} = \frac{3\varepsilon_0 P\cos\theta}{2\varepsilon+\varepsilon_0}}$$

parte d: $\varphi(0) = \varphi_1(0) = A_0$

$$\varphi_1(a) = \varphi_2(a) \Rightarrow A_0 + \cancel{A_1 a\cos\theta = \frac{D_1}{a^2}\cos\theta} \Rightarrow A_0 = 0$$

$$\boxed{\varphi(0) = 0}$$

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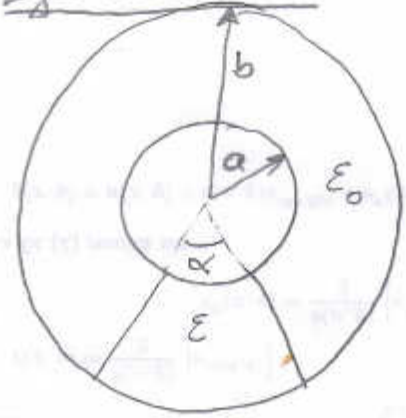
Ejercicio Nº2



parte a: $\sigma_\varepsilon \rightarrow$ densidad superficial de carga en conductor interior sobre el dieléctrico

$\sigma_0 \rightarrow$ Idem en el vacío

$$\nabla \cdot \vec{D} = 0$$

$$\vec{D} = D\vec{e}_r \Rightarrow \frac{1}{\rho}\frac{\partial(\rho D)}{\partial \rho} = 0$$

$$\rho D = A \Rightarrow D = \frac{A}{\rho}$$

$$D(a) = \sigma = \frac{A}{a} \Rightarrow A = \sigma a \qquad \vec{D}_\varepsilon = \frac{\sigma_\varepsilon a \vec{e}_\rho}{\rho}; \; \vec{D}_0 = \frac{\sigma_0 a \vec{e}_\rho}{\rho}$$

$$\vec{E}_\varepsilon = \frac{\vec{D}_\varepsilon}{\varepsilon} = \frac{\sigma_\varepsilon a \vec{e}_\rho}{\varepsilon \rho} \qquad \vec{E}_0 = \frac{\vec{D}_0}{\varepsilon_0} = \frac{\sigma_0 a}{\varepsilon_0 \rho}\vec{e}_\rho$$

En la frontera del dieléctrico: $\vec{E}_\varepsilon \cdot \vec{e}_r = \vec{E}_0 \cdot \vec{e}_r$

$$\frac{\sigma_\varepsilon a}{\varepsilon \rho} = \frac{\sigma_0 a}{\varepsilon_0 \rho} \Rightarrow \sigma_\varepsilon = \frac{\varepsilon \sigma_0}{\varepsilon_0}$$

$$\varphi(a) - \varphi(b) = V_0 = -\int_b^a \vec{E}\cdot d\vec{r} = \int_a^b \vec{E}\cdot d\vec{r} = \frac{\sigma_0 a}{\varepsilon_0}\int_a^b \frac{d\rho}{\rho}$$

$$V_0 = \frac{\sigma_0 a}{\varepsilon_0} \ln\frac{b}{a}$$

$$\sigma_0 = \frac{\varepsilon_0 V_0}{a \ln\frac{b}{a}} \qquad \sigma_\varepsilon = \frac{\varepsilon V_0}{a \ln\frac{b}{a}}$$

$$Q = \sigma_\varepsilon \alpha a L + \sigma_0(2\pi - \alpha) a L =$$

$$= \frac{\varepsilon V_0 \alpha L}{\ln\frac{b}{a}} + \frac{\varepsilon_0 V_0 (2\pi - \alpha) L}{\ln\frac{b}{a}}$$

$$C = \frac{\varepsilon \alpha L + \varepsilon_0(2\pi - \alpha)L}{\ln\frac{b}{a}} \qquad \boxed{\frac{C}{L} = \frac{2\pi\varepsilon_0 + \alpha(\varepsilon - \varepsilon_0)}{\ln\frac{b}{a}}}$$

parte b:

$$\vec{E}_\varepsilon = \frac{V_0}{\rho \ln\frac{b}{a}}\,\vec{e}_\rho \Rightarrow \vec{J} = g\vec{E}_\varepsilon = \frac{g V_0}{\rho \ln\frac{b}{a}}\,\vec{e}_\rho$$

$$I = \int_S \vec{J}\cdot\vec{n}\,da = \int_0^{\alpha}\int_0^{L} \frac{g V_0}{\rho \ln\frac{b}{a}}\,\rho\,d\phi\,dz = \frac{g V_0 \alpha L}{\ln\frac{b}{a}}$$

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$$V_0 = \frac{I \ln b/a}{g\alpha L} \Rightarrow \boxed{R = \frac{\ln b/a}{g\alpha L}}$$

La capacidad C se puede ver como el paralelo de dos capacidades: $C = C_\varepsilon + C_0$ c/ $C_\varepsilon = \frac{\varepsilon \alpha L}{\ln b/a}$

$$R C_\varepsilon = \frac{\varepsilon}{g} \checkmark$$

parte c:

$$\oint \vec{J}\cdot\vec{n} = -\frac{dQ}{dt}$$

$$\text{"} \quad \frac{g V(t)}{a \ln\frac{b}{a}}\,\alpha a L \qquad Q(t) = \sigma_\varepsilon \alpha a L + \sigma_0 (2\pi-\alpha) a L$$

$$\sigma_\varepsilon = \frac{\varepsilon V(t)}{a\ln\frac{b}{a}} \qquad \sigma_0 = \frac{\varepsilon_0 V(t)}{a\ln\frac{b}{a}}$$

$$g\frac{V(t)\alpha \cancel{L}}{\cancel{\ln\frac{b}{a}}} = -\frac{\varepsilon\alpha\cancel{L}}{\cancel{\ln\frac{b}{a}}}\frac{dV}{dt} - \frac{\varepsilon_0(2\pi-\alpha)\cancel{L}}{\cancel{\ln\frac{b}{a}}}\frac{dV}{dt}$$

$$\frac{dV}{dt} + \frac{\alpha g}{\varepsilon\alpha + \varepsilon_0(2\pi-\alpha)}V(t) = 0$$

$$V(t) = B e^{-\frac{t}{\tau}} \Rightarrow -\frac{1}{\tau} + \frac{\alpha g}{\varepsilon\alpha + \varepsilon_0(2\pi-\alpha)} = 0 \Rightarrow \boxed{\tau = \frac{2\pi\varepsilon_0 + \alpha(\varepsilon-\varepsilon_0)}{\alpha g}}$$

$$\tau = RC$$

$$\boxed{V(t) = V(0)e^{-\frac{t}{\tau}}} \qquad \boxed{\sigma_\varepsilon = \frac{\varepsilon V(0) e^{-\frac{t}{\tau}}}{a\ln\frac{b}{a}}} \qquad \boxed{\sigma_0 = \frac{\varepsilon_0 V(0) e^{-\frac{t}{\tau}}}{a\ln\frac{b}{a}}}$$

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$$P = \int_V \vec{J}\cdot\vec{E}\,dV = \int_0^{\alpha}\int_a^{b}\int_0^{L} g\underbrace{\vec{E}^2}_{\left(\frac{\sigma_\varepsilon}{\varepsilon}\frac{a}{\rho}\right)^2 = \frac{V^2}{\left(\ln\frac{b}{a}\right)^2\rho^2}}\rho\,d\phi\,d\rho\,dz = \frac{gV^2}{\left(\ln\frac{b}{a}\right)^2}\alpha L\underbrace{\int_a^b \frac{d\rho}{\rho}}_{\ln\frac{b}{a}}$$

$$P = \frac{gV^2\alpha L}{\ln\frac{b}{a}} = \frac{g\alpha L}{\ln\frac{b}{a}}V(0)^2 e^{-\frac{2t}{\tau}}$$

$$E = \int_0^\infty dt\,P(t) = \frac{g\alpha L}{\ln\frac{b}{a}}V(0)^2\underbrace{\int_0^\infty dt\,e^{-\frac{2t}{\tau}}}_{-\frac{\tau}{2}e^{-\frac{2t}{\tau}}\Big|_0^\infty = \frac{\tau}{2}} = \frac{\cancel{g}\cancel{\alpha}LV(0)^2}{\ln\frac{b}{a}}\frac{1}{2}\frac{2\pi\varepsilon_0+\alpha(\varepsilon-\varepsilon_0)}{\cancel{\alpha}\cancel{g}}$$

$$\boxed{E = \frac{1}{2}\underbrace{\frac{2\pi\varepsilon_0+\alpha(\varepsilon-\varepsilon_0)L}{\ln\frac{b}{a}}}_{C}V(0)^2 = \frac{CV(0)^2}{2}}$$

$$U(0) = \frac{1}{2}\int\vec{D}\cdot\vec{E}\,dV = \frac{1}{2}\int_0^{\alpha}\int_a^{b}\int_0^{L}\varepsilon\vec{E}^2\rho\,d\phi\,d\rho\,dz + \frac{1}{2}\int_\alpha^{2\pi}\int_a^{b}\int_0^{L}\varepsilon_0\vec{E}^2\rho\,d\phi\,d\rho\,dz$$

$$= \frac{V(0)^2}{\left(\ln\frac{b}{a}\right)^2}\frac{1}{2}\left\{\varepsilon\,\alpha L\int_a^b\frac{d\rho}{\rho} + \varepsilon_0(2\pi-\alpha)L\int_a^b\frac{d\rho}{\rho}\right\} =$$

$$= \frac{V(0)^2}{2}\,\frac{\varepsilon_0 2\pi+\alpha(\varepsilon-\varepsilon_0)}{\ln\frac{b}{a}}\cdot L = \frac{CV(0)^2}{2}$$