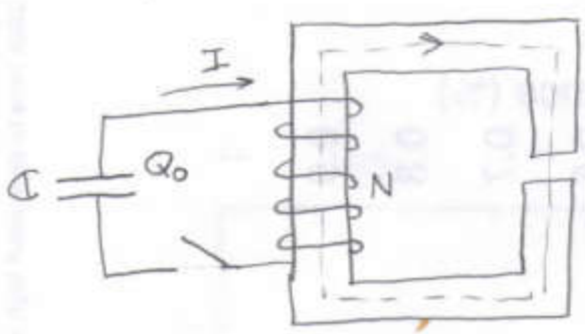
Examen Electromagnetismo (1128) 29/1/2021

Ejercicio Nº 1



μ, ℓ, S

parte a:

$$\Phi = N\phi = N B_{hierro} S$$

$$\ell H_{hierro} + e H_{aire} = NI$$

$$H_{hierro} = \frac{B_{hierro}}{\mu} \qquad H_{aire} = \frac{B_{aire}}{\mu_0}$$

$$B_{hierro} \cancel{S} = B_{aire} \cancel{S}$$

$$\Rightarrow \left(\frac{\ell}{\mu} + \frac{e}{\mu_0}\right) B_{hierro} = NI \qquad \Phi = \frac{N^2 I S}{\frac{\ell}{\mu} + \frac{e}{\mu_0}} \Rightarrow \boxed{L = \frac{N^2 S}{\frac{\ell}{\mu} + \frac{e}{\mu_0}}}$$

parte b: $L\frac{dI}{dt} - \frac{Q}{C} = 0 \qquad I = -\frac{dQ}{dt}$

$$-L\frac{d^2Q}{dt^2} - \frac{Q}{C} = 0 \qquad \frac{d^2Q}{dt^2} + \frac{1}{LC}Q = 0$$

$$\Rightarrow Q(t) = A\cos\omega t + B\,\mathrm{sen}\,\omega t \qquad \omega = \frac{1}{\sqrt{LC}}$$

$$Q(0) = Q_0 = A$$

$$I(0) = -\frac{dQ}{dt}(0) = 0 = +A\omega\,\mathrm{sen}\,\omega t - B\omega\cos\omega t\Big|_{t=0} = -B\omega$$

$$\Rightarrow B = 0 \Rightarrow Q(t) = Q_0 \cos\omega t \Rightarrow \boxed{I(t) = \frac{Q_0}{\sqrt{LC}}\,\mathrm{sen}\,\omega t}$$

parte c:

$$B_{aire}^{max} = B_{hierro}^{max} = \frac{N I^{max}}{\frac{\ell}{\mu} + \frac{e}{\mu_0}} \qquad I^{max} = \frac{Q_0}{\sqrt{LC}}$$

$$\Rightarrow \boxed{B_{aire}^{max} = \frac{N Q_0}{\left(\frac{\ell}{\mu} + \frac{e}{\mu_0}\right)\sqrt{LC}}}$$

$I(t) = I^{max}$ cuando $\mathrm{sen}\,\omega t = 1$

$$\Rightarrow \boxed{t = \frac{\pi}{2\omega} = \frac{\pi\sqrt{LC}}{2}}$$

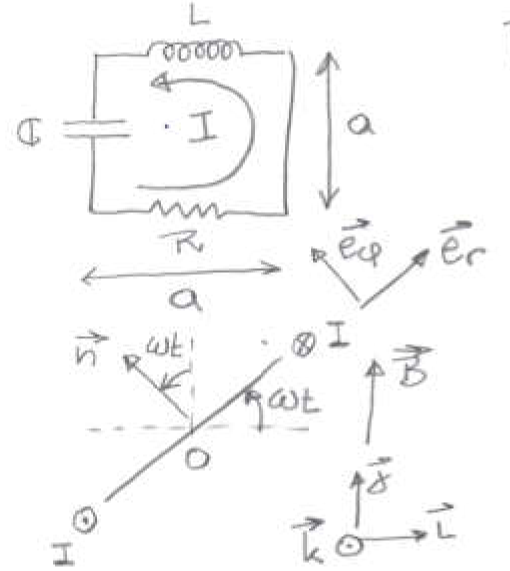
parte d:

$$W = \int \frac{\vec{B}\cdot\vec{H}}{2}\,dV = \frac{B_{hierro}^2}{2\mu}\ell S + \frac{B_{aire}^2}{2\mu_0} e S =$$

$$= \frac{N^2 Q_0^2}{\left(\frac{\ell}{\mu} + \frac{e}{\mu_0}\right)^{\cancel{2}} LC} S \cdot \cancel{\left(\frac{\ell}{\mu} + \frac{e}{\mu_0}\right)} \cdot \frac{1}{2} = \boxed{\frac{Q_0^2}{2C} = W}$$

Es la energía inicial almacenada en C.

Ejercicio Nº2



parte a:

$$\mathcal{E} = L\frac{dI}{dt} + \frac{q}{C} + RI$$

$$\mathcal{E} = -\frac{d\Phi}{dt}$$

$$\Phi = \int \vec{B}\cdot\vec{n}\,dS = \int B\cos\omega t\,dS = $$

$$= Ba^2\cos\omega t$$

$$\frac{d\Phi}{dt} = -Ba^2\omega\,\mathrm{sen}\,\omega t$$

$$\mathcal{E} = Ba^2\omega\,\mathrm{sen}\,\omega t = L\frac{dI}{dt} + \frac{q}{C} + RI$$

Para tiempos largos I será sinusoidal: $I = I_0 e^{j\omega t}$

$$\mathcal{E} = -jBa^2\omega\, e^{j\omega t} = \left(j\omega L + \frac{1}{j\omega C} + R\right) I_0 e^{j\omega t}$$

$$I_0 = -j\frac{Ba^2\omega}{j\left(\omega L - \frac{1}{\omega C}\right) + R} = -j\frac{Ba^2\omega\left(R - j\left(\omega L - \frac{1}{\omega C}\right)\right)}{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

$$\boxed{I(t) = \frac{Ba^2\omega}{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}\left[R\,\mathrm{sen}\,\omega t - \left(\omega L - \frac{1}{\omega C}\right)\cos\omega t\right]}$$

$$|I_0| = \frac{Ba^2\omega}{\sqrt{\left(\omega L - \frac{1}{\omega C}\right)^2 + R^2}}$$

$|I_0|$ máxima si $\boxed{\omega = \frac{1}{\sqrt{LC - (RC)^2/2}}}$

Si $R \ll \frac{L}{C} \Rightarrow \boxed{\omega \cong \frac{1}{\sqrt{LC}}}$

parte b: $I\dot{\vec{\omega}} = \vec{M}_0^{(ext)} + \vec{M}_0^{(mag)} \Rightarrow \vec{M}_0^{(ext)} = -\vec{M}_0^{(mag)}$

$\dot{\vec{\omega}} = 0$

$$\vec{M}_0^{(mag)} = \int_C \vec{r}\wedge d\vec{F} = \int_C \vec{r}\wedge I(d\vec{l}\wedge\vec{B}) = IB\int_C \vec{r}\wedge(d\vec{l}\wedge\vec{j})$$

$$\int_C \vec{r}\wedge(d\vec{l}\wedge\vec{k}) = \int_{-a/2}^{a/2} r\vec{e}_r\wedge(dr\,\vec{e}_r\wedge\vec{j}) + \int_0^a \frac{a}{2}\vec{e}_r\wedge(-dz\,\vec{k}\wedge\vec{j}) +$$

$$+\int_{-a/2}^{a/2} r\vec{e}_r\wedge(-dr\,\vec{e}_r\wedge\vec{j}) + \int_0^a -\frac{a}{2}\vec{e}_r\wedge(dz\,\vec{k}\wedge\vec{j})$$

$$\int_{-\frac{a}{2}}^{a/2} r\,dr = \frac{r^2}{2}\Big|_{-\frac{a}{2}}^{a/2} = 0$$

$$\vec{e}_r \wedge (\vec{k} \wedge \vec{j}) = \vec{e}_r \wedge (-\vec{i}) = (\cos\omega t\,\vec{i} + \operatorname{sen}\omega t\,\vec{j}) \wedge (-\vec{i}) = +\operatorname{sen}\omega t\,\vec{k}$$

$$\vec{M}_o^{(mag)} = IB\left(-\frac{a^2}{2}\operatorname{sen}\omega t\,\vec{k} - \frac{a^2}{2}\operatorname{sen}\omega t\,\vec{k}\right) = -IBa^2\operatorname{sen}\omega t\,\vec{k}$$

$$\boxed{\vec{M}_o^{(ext)} = IBa^2\operatorname{sen}\omega t\,\vec{k}}$$

parte c: $P_R = R\,I^2 \quad \langle P_R\rangle = R\langle I^2\rangle$

$$I = |I_o|\cos(\omega t - \varphi) \Rightarrow \langle P_R\rangle = R|I_o|^2\underbrace{\langle\cos^2(\omega t - \varphi)\rangle}_{1/2}$$

$$\boxed{P_R = \frac{RB^2a^4\omega^2}{2\left[R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2\right]}}$$

$$P^{(ext)} = \vec{M}_o^{(ext)}\cdot\underbrace{\vec{\omega}}_{\omega\vec{k}} = IBa^2\omega\operatorname{sen}\omega t = \frac{B^2a^4\omega^2}{R^2+\left(\omega L - \frac{1}{\omega C}\right)^2}\left(R\operatorname{sen}^2\omega t - \left(\omega L - \frac{1}{\omega C}\right)\operatorname{sen}\omega t\cos\omega t\right)$$

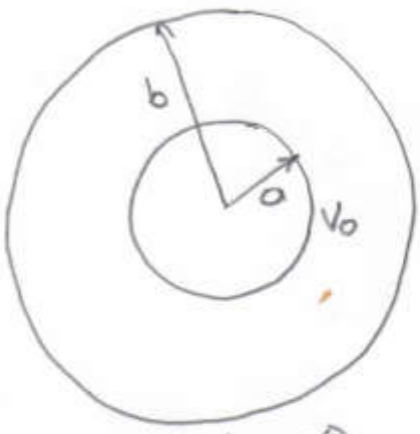
$$\langle\operatorname{sen}^2\omega t\rangle = \frac{1}{2}$$

$$\langle\operatorname{sen}\omega t\cos\omega t\rangle = \frac{1}{T}\int_0^T dt\,\operatorname{sen}\omega t\cos\omega t = \frac{1}{T\omega}\operatorname{sen}^2\omega t\Big|_0^T = 0$$

$$\boxed{\langle P^{ext}\rangle = \frac{B^2a^4\omega^2 R}{2\left[R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2\right]}}$$

$P_R = P_{ext}$

Ejercicio Nº 3



$\sigma(\theta) = k\cos\theta$

parte a:

$r > b$

$$\varphi_1(\vec r) = A_0 + \frac{B_0}{r} + \left(A_1 r + \frac{B_1}{r^2}\right)\cos\theta$$

$a < r < b$

$$\varphi_2(\vec r) = A_0' + \frac{B_0'}{r} + \left(A_1' r + \frac{B_1'}{r^2}\right)\cos\theta$$

$$\varphi_1(\vec r) \xrightarrow[r\to\infty]{} 0 = A_0 + A_1 r\cos\theta \Rightarrow A_0, A_1 = 0$$

$$\varphi_2(\vec r)\big|_{r=a} = V_0 = A_0' + \frac{B_0'}{a} + \left(A_1' a + \frac{B_1'}{a^2}\right)\cos\theta$$

$$\Rightarrow V_0 = A_0' + \frac{B_0'}{a} \qquad A_1' a + \frac{B_1'}{a^2} = 0$$

$$\vec D_1 \cdot \vec e_r\big|_{r=b} - \vec D_2 \cdot \vec e_r\big|_{r=b} = \sigma(\theta) \Rightarrow \vec E_1 \cdot \vec e_r\big|_{r=b} - \vec E_2 \cdot \vec e_r\big|_{r=b} = \frac{k\cos\theta}{\varepsilon_0}$$

$$\vec E = -\frac{\partial\varphi}{\partial r}\vec e_r - \frac{1}{r}\frac{\partial\varphi}{\partial\theta}\vec e_\theta \Rightarrow \vec E_1 = \left(+\frac{B_0}{r^2} + \frac{2B_1\cos\theta}{r^3}\right)\vec e_r + \frac{B_1}{r^3}\operatorname{sen}\theta\,\vec e_\theta$$

$$\vec E_2 = \left[+\frac{B_0'}{r^2} - \left(A_1' - \frac{2B_1'}{r^3}\right)\cos\theta\right]\vec e_r + \left(A_1' + \frac{B_1'}{r^3}\right)\operatorname{sen}\theta\,\vec e_\theta$$

$$\Rightarrow \frac{B_0}{b^2} + \frac{2B_1}{b^3}\cos\theta - \frac{B_0'}{b^2} + \left(A_1' - \frac{2B_1'}{b^3}\right)\cos\theta = \frac{k\cos\theta}{\varepsilon_0}$$

$$\Rightarrow B_0 = B_0' \qquad \frac{2B_1}{b^3} + A_1' - \frac{2B_1'}{b^3} = \frac{k}{\varepsilon_0}$$

$$\vec E_1 \cdot \vec e_\theta\big|_{r=b} = \vec E_2 \cdot \vec e_\theta\big|_{r=b} \Rightarrow \frac{B_1}{b^3}\cancel{\operatorname{sen}\theta} = \left(A_1' + \frac{B_1'}{b^3}\right)\cancel{\operatorname{sen}\theta}$$

$$A_1' = -\frac{B_1'}{a^3} \Rightarrow \frac{B_1}{b^3} = B_1'\left(\frac{1}{b^3} - \frac{1}{a^3}\right) \Rightarrow \cancel{\frac{2B_1'}{b^3}} - \frac{2B_1'}{a^3} - \frac{B_1'}{a^3} - \cancel{\frac{2B_1'}{b^3}} = \frac{k}{\varepsilon_0}$$

$$B_1' = -\frac{ka^3}{3\varepsilon_0},\ A_1' = \frac{k}{3\varepsilon_0},\ B_1 = -\frac{ka^3}{3\varepsilon_0}\left(1 - \frac{b^3}{a^3}\right) \Rightarrow B_1 = \frac{b^3 - a^3}{3\varepsilon_0}k$$

$$\varphi_1(b) = \varphi_2(b) \Rightarrow \cancel{\frac{B_0}{b}} = A_0' + \cancel{\frac{B_0'}{b}} \Rightarrow A_0' = 0,\ B_0 = B_0' = aV_0$$

$$\Rightarrow \boxed{\begin{aligned} \varphi_1(\vec r) &= \frac{aV_0}{r} + \frac{b^3 - a^3}{3\varepsilon_0}\frac{k}{r^2}\cos\theta \\ \varphi_2(\vec r) &= \frac{aV_0}{r} + \frac{k}{3\varepsilon_0}\left(r - \frac{a^3}{r^2}\right)\cos\theta \end{aligned}}$$

parte b:

Porque la solución de la ecuación de Laplace para determinadas condiciones de borde es única. Ese potencial verifica la ecuación de Laplace y verifica las condiciones de borde, entonces es la solución única del problema.

parte c:

$$\sigma_a = \vec{D}_2 \cdot \vec{e}_r \Big|_{r=a} = \varepsilon_0 \left(\frac{V_0}{a} - \frac{k}{3\varepsilon_0}\left(1+2\right)\cos\theta \right)$$

$$\boxed{\sigma_a = \frac{\varepsilon_0 V_0}{a} - k\cos\theta}$$

parte d:

$$Q_{tot} = \int_0^{\pi}\int_0^{2\pi} a^2 \operatorname{sen}\theta \, d\theta \, d\varphi \left(\frac{\varepsilon_0 V_0}{a} - k\cos\theta \right) + \int_0^{\pi}\int_0^{2\pi} b^2 \operatorname{sen}\theta \, d\theta \, d\varphi \, k\cos\theta$$

$$\int_0^{\pi} d\theta \operatorname{sen}\theta\cos\theta = \frac{\operatorname{sen}^2\theta}{2}\Big|_0^{\pi} = 0$$

$$Q_{tot} = a^{\cancel{2}} 2\pi \frac{\varepsilon_0 V_0}{\cancel{a}} \underbrace{\int_0^{\pi} \operatorname{sen}\theta \, d\theta}_{-\cos\theta\big|_0^{\pi} = +1+1=2} \Rightarrow \boxed{Q_{tot} = 4\pi\varepsilon_0 a V_0}$$

$$Q_{tot} = \int_{r\to\infty} \vec{D}_1 \cdot \vec{n} \, da = \varepsilon_0 \int_0^{\pi}\int_0^{2\pi} \left(\frac{B_0}{\cancel{r^2}} + \underbrace{\frac{2B_1\cos\theta}{r^3}}_{\to 0} \right) \cancel{r^2} \operatorname{sen}\theta \, d\theta \, d\varphi =$$

$$= \varepsilon_0 B_0 4\pi = \varepsilon_0 a V_0 4\pi \checkmark$$