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always exhibit their greatest intensity near the ground and diminish with increasing height above the ground.

- ii. Some upper level lows do not extend downward to the ground, as indicated in Fig. 3.3b. It follows from the hypsometric equation that these lows must be *cold core* below the level at which they achieve their greatest intensity and *warm core* above that level, as shown in Fig. 3.3b.

3.2.4 Reduction of Pressure to Sea Level

In mountainous regions the difference in surface pressure from one observing station to another is largely due to differences in elevation. To isolate that part of the pressure field that is due to the passage of weather systems, it is necessary to reduce the pressures to a common reference level. For this purpose, sea level is normally used.

Let the subscripts g and 0 refer to conditions at the ground and at sea level ($Z = 0$), respectively. Then, for the layer between the Earth's surface and sea level, the hypsometric equation (3.29) assumes the form

$$Z_g = \bar{H} \ln \frac{p_0}{p_g} \quad (3.30)$$

which can be solved to obtain the sea-level pressure

$$p_0 = p_g \exp\left(\frac{Z_g}{\bar{H}}\right) = p_g \exp\left(\frac{g_0 Z_g}{R_d \bar{T}_v}\right) \quad (3.31)$$

If Z_g is small, the scale height \bar{H} can be evaluated from the ground temperature. Also, if $Z_g / \bar{H} \ll 1$, the exponential in (3.31) can be approximated by $1 + Z_g / \bar{H}$, in which case (3.31) becomes

$$p_0 - p_g \approx p_g \frac{Z_g}{\bar{H}} = p_g \left(\frac{g_0 Z_g}{R_d \bar{T}_v}\right) \quad (3.32)$$

Because $p_g \approx 1000$ hPa and $\bar{H} \approx 8000$ m, the pressure correction (in hPa) is roughly equal to Z_g (in

meters) divided by 8. In other words, for altitudes up to a few hundred meters above (or below) sea level, the pressure decreases by about 1 hPa for every 8 m of vertical ascent.

3.3 The First Law of Thermodynamics¹⁵

In addition to the macroscopic kinetic and potential energy that a system as a whole may possess, it also contains *internal energy* due to the kinetic and potential energy of its molecules or atoms. Increases in internal kinetic energy in the form of molecular motions are manifested as increases in the temperature of the system, whereas changes in the potential energy of the molecules are caused by changes in their relative positions by virtue of any forces that act between the molecules.

Let us suppose that a closed system¹⁶ of unit mass takes in a certain quantity of thermal energy q (measured in joules), which it can receive by thermal conduction and/or radiation. As a result the system may do a certain amount of *external work* w (also measured in joules). The excess of the energy supplied to the body over and above the external work done by the body is $q - w$. Therefore, if there is no change in the macroscopic kinetic and potential energy of the body, it follows from the principle of conservation of energy that the internal energy of the system must increase by $q - w$. That is,

$$q - w = u_2 - u_1 \quad (3.33)$$

where u_1 and u_2 are the internal energies of the system before and after the change. In differential form (3.33) becomes

$$dq - dw = du \quad (3.34)$$

where dq is the differential increment of heat added to the system, dw is the differential element

¹⁵ The first law of thermodynamics is a statement of the conservation of energy, taking into account the conversions between the various forms that it can assume and the exchanges of energy between a system and its environment that can take place through the transfer of heat and the performance of mechanical work. A general formulation of the first law of thermodynamics is beyond the scope of this text because it requires consideration of conservation laws, not only for energy, but also for momentum and mass. This section presents a simplified formulation that ignores the macroscopic kinetic and potential energy (i.e., the energy that air molecules possess by virtue of their height above sea level and their organized fluid motions). As it turns out, the expression for the first law of thermodynamics that emerges in this simplified treatment is identical to the one recovered from a more complete treatment of the conservation laws, as is done in J. R. Holton, *Introduction to Dynamic Meteorology*, 4th Edition, Academic Press, New York, 2004, pp. 146–149.

¹⁶ A closed system is one in which the total amount of matter, which may be in the form of gas, liquid, solid or a mixture of these phases, is kept constant.

of work done by the system, and du is the differential increase in internal energy of the system. Equations (3.33) and (3.34) are statements of the *first law of thermodynamics*. In fact (3.34) provides a definition of du . The change in internal energy du depends only on the initial and final states of the system and is therefore independent of the manner by which the system is transferred between these two states. Such parameters are referred to as *functions of state*.¹⁷

To visualize the work term dw in (3.34) in a simple case, consider a substance, often called the *working substance*, contained in a cylinder of fixed cross-sectional area that is fitted with a movable, frictionless piston (Fig. 3.4). The volume of the substance is proportional to the distance from the base of the cylinder to the face of the piston and can be represented on the horizontal axis of the graph shown in Fig. 3.4. The pressure of the substance in the cylinder can be represented on the vertical axis of this graph. Therefore, every state of the substance, corresponding to a given position of the piston, is represented by a point on this pressure–volume (p – V) diagram. When the substance is in equilibrium at a state represented by point P on the graph, its pressure is p and its volume is V (Fig. 3.4). If the piston moves outward through an incremental distance dx while its pressure remains essentially constant at p , the work dW done by the substance in pushing the external force F through a distance dx is

$$dW = Fdx$$

or, because $F = pA$ where A is the cross-sectional area of the face of the piston,

$$dW = pA dx = pdV \quad (3.35)$$

In other words, the work done by the substance when its volume increases by a small increment dV is equal to the pressure of the substance multiplied by its increase in volume, which is equal to the blue-shaded area in the graph shown in Fig. 3.4; that is, it is equal to the area under the curve PQ .

When the substance passes from state A with volume V_1 to state B with volume V_2 (Fig. 3.4), during which its pressure p changes, the work W done by the material is equal to the area under the curve AB . That is,

$$W = \int_{V_1}^{V_2} pdV \quad (3.36)$$

Equations (3.35) and (3.36) are quite general and represent work done by any substance (or system) due to a change in its volume. If $V_2 > V_1$, W is positive, indicating that the *substance does work on its environment*. If $V_2 < V_1$, W is negative, which indicates that the *environment does work on the substance*.

The p – V diagram shown in Fig. 3.4 is an example of a *thermodynamic diagram* in which the physical state of a substance is represented by two thermodynamic variables. Such diagrams are very useful in meteorology; we will discuss other examples later in this chapter.

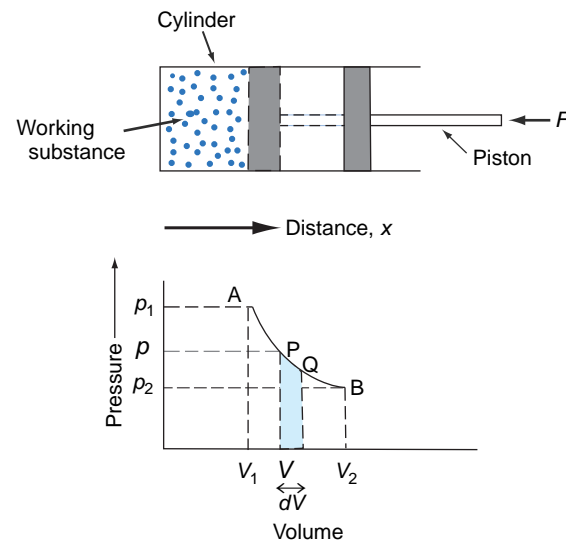


Fig. 3.4 Representation of the state of a working substance in a cylinder on a p – V diagram. The work done by the working substance in passing from P to Q is $p dV$, which is equal to the blue-shaded area. [Reprinted from *Atmospheric Science: An Introductory Survey*, 1st Edition, J. M. Wallace and P. V. Hobbs, p. 62, Copyright 1977, with permission from Elsevier.]

¹⁷ Neither the heat q nor the work w are functions of state, since their values depend on *how* a system is transformed from one state to another. For example, a system may or may not receive heat and it may or may not do external work as it undergoes transitions between different states.

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If we are dealing with a unit mass of a substance, the volume V is replaced by the specific volume α . Therefore, the work dw that is done when the specific volume increases by $d\alpha$ is

$$dw = p d\alpha \quad (3.37)$$

Combination of (3.34) and (3.37) yields

$$dq = du + p d\alpha \quad (3.38)$$

which is an alternative statement of the first law of thermodynamics.¹⁸

3.3.1 Joule's Law

Following a series of laboratory experiments on air, Joule¹⁹ concluded in 1848 that when a gas expands without doing external work, by expanding into a chamber that has been evacuated, and without taking in or giving out heat, the temperature of the gas does

not change. This statement, which is known as *Joule's law*, is strictly true only for an ideal gas, but air (and many other gases) behaves very similarly to an ideal gas over a wide range of conditions.

Joule's law leads to an important conclusion concerning the internal energy of an ideal gas. If a gas neither does external work nor takes in or gives out heat, $dw = 0$ and $dq = 0$ in (3.38), so that $du = 0$. Also, according to Joule's law, under these conditions the temperature of the gas does not change, which implies that the kinetic energy of the molecules remains constant. Therefore, because the total internal energy of the gas is constant, that part of the internal energy due to the potential energy must also remain unchanged, even though the volume of the gas changes. In other words, the internal energy of an ideal gas is independent of its volume if the temperature is kept constant. This can be the case only if the molecules of an ideal gas do not exert forces on each other. In this case, the internal energy of an ideal gas will depend only on its temperature.²⁰

3.2 More Handball?

Box 3.1. showed that the gas laws can be illustrated by picturing the molecules of a gas as elastic balls bouncing around randomly in a handball court. Suppose now that the walls of the court are permitted to move outward when subjected to a force. The force on the walls is supplied by the impact of the balls, and the work required to move the walls outward comes from a decrease in the kinetic energy of the balls that rebound from the walls with lower velocities

than they struck them. This decrease in kinetic energy is in accordance with the first law of thermodynamics under adiabatic conditions. The work done by the system by pushing the walls outward is equal to the decrease in the internal energy of the system [see (3.38)]. Of course, if the outside of the walls of the court are bombarded by balls in a similar manner to the inside walls, there will be no net force on the walls and no work will be done.

¹⁸ We have assumed here that the only work done by or on a system is due to a change in the volume of the system. However, there are other ways in which a system may do work, e.g., by the creation of new surface area between two phases (such as between liquid and air when a soap film is formed). Unless stated otherwise, we will assume that the work done by or on a system is due entirely to changes in the volume of the system.

¹⁹ **James Prescott Joule** (1818–1889) Son of a wealthy English brewer; one of the great experimentalists of the 19th century. He started his scientific work (carried out in laboratories in his home and at his own expense) at age 19. He measured the mechanical equivalent of heat, recognized the dynamical nature of heat, and developed the principle of conservation of energy.

²⁰ Subsequent experiments carried out by Lord Kelvin²¹ revealed the existence of small forces between the molecules of a gas.

²¹ **Lord Kelvin 1st Baron (William Thomson)** (1824–1907) Scottish mathematician and physicist. Entered Glasgow University at age 11. At 22 became Professor of Natural Philosophy at the same university. Carried out incomparable work in thermodynamics, electricity, and hydrodynamics.

3.3.2 Specific Heats

Suppose a small quantity of heat dq is given to a unit mass of a material and, as a consequence, the temperature of the material increases from T to $T + dT$ without any changes in phase occurring within the material. The ratio dq/dT is called the *specific heat* of the material. The specific heat defined in this way could have any number of values, depending on how the material changes as it receives the heat. If the volume of the material is kept constant, a *specific heat at constant volume* c_v is defined

$$c_v = \left(\frac{dq}{dT} \right)_{v \text{ const}} \quad (3.39)$$

However, if the volume of the material is constant (3.38) becomes $dq = du$. Therefore

$$c_v = \left(\frac{du}{dT} \right)_{v \text{ const}}$$

For an ideal gas, Joule's law applies and therefore u depends only on temperature. Therefore, regardless of whether the volume of a gas changes, we may write

$$c_v = \left(\frac{du}{dT} \right) \quad (3.40)$$

From (3.38) and (3.40), the first law of thermodynamics for an ideal gas can be written in the form²²

$$dq = c_v dT + p d\alpha \quad (3.41)$$

Because u is a function of state, no matter how the material changes from state 1 to state 2, the change in its internal energy is, from (3.40),

$$u_2 - u_1 = \int_{T_1}^{T_2} c_v dT$$

We can also define a specific heat at constant pressure c_p

$$c_p = \left(\frac{dq}{dT} \right)_{p \text{ const}} \quad (3.42)$$

where the material is allowed to expand as heat is added to it and its temperature rises, but its pressure remains constant. In this case, a certain amount of the heat added to the material will have to be expended to do work as the system expands against the constant pressure of its environment. Therefore, a larger quantity of heat must be added to the material to raise its temperature by a given amount than if the volume of the material were kept constant. For the case of an ideal gas, this inequality can be seen mathematically as follows. Equation (3.41) can be rewritten in the form

$$dq = c_v dT + d(p\alpha) - \alpha dp \quad (3.43)$$

From the ideal gas equation (3.3), $d(p\alpha) = R dT$. Therefore (3.43) becomes

$$dq = (c_v + R) dT - \alpha dp \quad (3.44)$$

At constant pressure, the last term in (3.44) vanishes; therefore, from (3.42) and (3.44),

$$c_p = c_v + R \quad (3.45)$$

The specific heats at constant volume and at constant pressure for dry air are 717 and 1004 J K⁻¹ kg⁻¹, respectively, and the difference between them is 287 J K⁻¹ kg⁻¹, which is the gas constant for dry air. It can be shown that for ideal monatomic gases $c_p:c_v:R = 5:3:2$, and for ideal diatomic gases $c_p:c_v:R = 7:5:2$.

By combining (3.44) and (3.45) we obtain an alternate form of the first law of thermodynamics:

$$dq = c_p dT - \alpha dp \quad (3.46)$$

3.3.3 Enthalpy

If heat is added to a material at constant pressure so that the specific volume of the material increases from α_1 to α_2 , the work done by a unit mass of the material is $p(\alpha_2 - \alpha_1)$. Therefore, from (3.38), the finite quantity of heat Δq added to

²² The term dq is sometimes called the *diabatic* (or nonadiabatic) *heating* or *cooling*, where "diabatic" means involving the transfer of heat. The term "diabatic" would be redundant if "heating" and "cooling" were always taken to mean "the addition or removal of heat." However, "heating" and "cooling" are often used in the sense of "to raise or lower the temperature of," in which case it is meaningful to distinguish between that part of the temperature change dT due to diabatic effects (dq) and that part due to adiabatic effects ($p d\alpha$).

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a unit mass of the material at constant pressure is given by

$$\begin{aligned}\Delta q &= (u_2 - u_1) + p(\alpha_2 - \alpha_1) \\ &= (u_2 + p\alpha_2) - (u_1 + p\alpha_1)\end{aligned}$$

where u_1 and u_2 are, respectively, the initial and final internal energies for a unit mass of the material. Therefore, at constant pressure,

$$\Delta q = h_2 - h_1$$

where h is the *enthalpy* of a unit mass of the material, which is defined by

$$h \equiv u + p\alpha \quad (3.47)$$

Because u , p , and α are functions of state, h is a function of state. Differentiating (3.47), we obtain

$$dh = du + d(p\alpha)$$

Substituting for du from (3.40) and combining with (3.43), we obtain

$$dq = dh - \alpha dp \quad (3.48)$$

which is yet another form of the first law of thermodynamics.

By comparing (3.46) and (3.48) we see that

$$dh = c_p dT \quad (3.49)$$

or, in integrated form,

$$h = c_p T \quad (3.50)$$

where h is taken as zero when $T = 0$. In view of (3.50), h corresponds to the heat required to raise the temperature of a material from 0 to T K at constant pressure.

When a layer of air that is at rest and in hydrostatic balance is heated, for example, by radiative transfer, the weight of the overlying air pressing down on it

remains constant. Hence, the heating is at constant pressure. The energy added to the air is realized in the form of an increase in enthalpy (or *sensible heat*, as atmospheric scientists commonly refer to it) and

$$dq = dh = c_p dT$$

The air within the layer expands as it warms, doing work on the overlying air by lifting it against the Earth's gravitational attraction. Of the energy per unit mass imparted to the air by the heating, we see from (3.40) and (3.41) that $du = c_v dT$ is reflected in an increase in internal energy and $p d\alpha = R dT$ is expended doing work on the overlying air. Because the Earth's atmosphere is made up mainly of the diatomic gases N_2 and O_2 , the energy added by the heating dq is partitioned between the increase in internal energy du and the expansion work $p d\alpha$ in the ratio 5:2.

We can write a more general expression that is applicable to a moving air parcel, the pressure of which changes as it rises or sinks relative to the surrounding air. By combining (3.20), (3.48), and (3.50) we obtain

$$dq = d(h + \Phi) = d(c_p T + \Phi) \quad (3.51)$$

Hence, if the material is a parcel of air with a fixed mass that is moving about in an hydrostatic atmosphere, the quantity $(h + \Phi)$, which is called the *dry static energy*, is constant provided the parcel neither gains nor loses heat (i.e., $dq = 0$).²³

3.4 Adiabatic Processes

If a material undergoes a change in its physical state (e.g., its pressure, volume, or temperature) without any heat being added to it or withdrawn from it, the change is said to be *adiabatic*.

Suppose that the initial state of a material is represented by the point A on the p - V diagram in Fig. 3.5 and that when the material undergoes an isothermal transformation it moves along the line AB. If the same material underwent a similar change in volume but under adiabatic conditions, the transformation would

²³ Strictly speaking, Eq. (3.51) holds only for an atmosphere in which there are no fluid motions. However, it is correct to within a few percent for the Earth's atmosphere where the kinetic energy of fluid motions represents only a very small fraction of the total energy. An exact relationship can be obtained by using Newton's second law of motion and the continuity equation in place of Eq. (3.20) in the derivation. See J. R. Holton, *An Introduction to Dynamic Meteorology*, 4th ed., Academic Press, pp. 46–49 (2004).

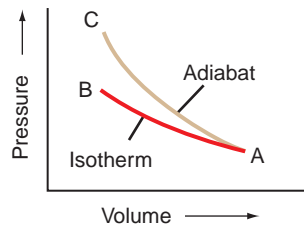


Fig. 3.5 An isotherm and an adiabat on a p - V diagram.

be represented by a curve such as AC, which is called an *adiabat*. The reason why the adiabat AC is steeper than the isotherm AB on a p - V diagram can be seen as follows. During adiabatic compression, the internal energy increases [because $dq = 0$ and $p d\alpha$ is negative in (3.38)] and therefore the temperature of the system rises. However, for isothermal compression, the temperature remains constant. Hence, $T_C > T_B$ and therefore $p_C > p_B$.

3.4.1 Concept of an Air Parcel

In many fluid mechanics problems, mixing is viewed as a result of the random motions of individual molecules. In the atmosphere, molecular mixing is important only within a centimeter of the Earth's surface and at levels above the turbopause (~ 105 km). At intermediate levels, virtually all mixing in the vertical is accomplished by the exchange of macroscale "air parcels" with horizontal dimensions ranging from millimeters to the scale of the Earth itself.

To gain some insights into the nature of vertical mixing in the atmosphere, it is useful to consider the behavior of an air parcel of infinitesimal dimensions that is assumed to be

- i. thermally insulated from its environment so that its temperature changes adiabatically as it rises or sinks, always remaining at exactly the same pressure as the environmental air at the same level,²⁴ which is assumed to be in hydrostatic equilibrium; and
- ii. moving slowly enough that the macroscopic kinetic energy of the air parcel is a negligible fraction of its total energy.

Although in the case of real air parcels one or more of these assumptions is nearly always violated

to some extent, this simple, idealized model is helpful in understanding some of the physical processes that influence the distribution of vertical motions and vertical mixing in the atmosphere.

3.4.2 The Dry Adiabatic Lapse Rate

We will now derive an expression for the rate of change of temperature with height of a parcel of dry air that moves about in the Earth's atmosphere while always satisfying the conditions listed at the end of Section 3.4.1. Because the air parcel undergoes only adiabatic transformations ($dq = 0$) and the atmosphere is in hydrostatic equilibrium, for a unit mass of air in the parcel we have, from (3.51),

$$d(c_p T + \Phi) = 0 \quad (3.52)$$

Dividing through by dz and making use of (3.20) we obtain

$$-\left(\frac{dT}{dz}\right)_{\text{dry parcel}} = \frac{g}{c_p} \equiv \Gamma_d \quad (3.53)$$

where Γ_d is called the *dry adiabatic lapse rate*. Because an air parcel expands as it rises in the atmosphere, its temperature will decrease with height so that Γ_d defined by (3.53) is a positive quantity. Substituting $g = 9.81 \text{ m s}^{-2}$ and $c_p = 1004 \text{ J K}^{-1} \text{ kg}^{-1}$ into (3.53) gives $\Gamma_d = 0.0098 \text{ K m}^{-1}$ or 9.8 K km^{-1} , which is the numerical value of the dry adiabatic lapse rate.

It should be emphasized again that Γ_d is the rate of change of temperature following a parcel of dry air that is being raised or lowered adiabatically in the atmosphere. The actual lapse rate of temperature in a column of air, which we will indicate by $\Gamma = \partial T / \partial z$, as measured, for example, by a radiosonde, averages $6\text{--}7 \text{ K km}^{-1}$ in the troposphere, but it takes on a wide range of values at individual locations.

3.4.3 Potential Temperature

The *potential temperature* θ of an air parcel is defined as the temperature that the parcel of air would have if it were expanded or compressed adiabatically from its existing pressure and temperature to a standard pressure p_0 (generally taken as 1000 hPa).

²⁴ Any pressure differences between the parcel and its environment give rise to sound waves that produce an almost instantaneous adjustment. Temperature differences, however, are eliminated by much slower processes.

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We can derive an expression for the potential temperature of an air parcel in terms of its pressure p , temperature T , and the standard pressure p_0 as follows. For an adiabatic transformation ($dq = 0$) (3.46) becomes

$$c_p dT - \alpha dp = 0$$

Substituting α from (3.3) into this expression yields

$$\frac{c_p}{R} \frac{dT}{T} - \frac{dp}{p} = 0$$

Integrating upward from p_0 (where, by definition, $T = \theta$) to p , we obtain

$$\frac{c_p}{R} \int_{\theta}^T \frac{dT}{T} = \int_{p_0}^p \frac{dp}{p}$$

or

$$\frac{c_p}{R} \ln \frac{T}{\theta} = \ln \frac{p}{p_0}$$

Taking the antilog of both sides

$$\left(\frac{T}{\theta}\right)^{c_p/R} = \frac{p}{p_0}$$

or

$$\theta = T \left(\frac{p_0}{p}\right)^{R/c_p} \quad (3.54)$$

Equation (3.54) is called *Poisson's²⁵ equation*. It is usually assumed that $R \approx R_d = 287 \text{ J K}^{-1} \text{ kg}^{-1}$ and $c_p \approx c_{pd} = 1004 \text{ J K}^{-1} \text{ kg}^{-1}$; therefore, $R/c_p \approx 0.286$.

Parameters that remain constant during certain transformations are said to be *conserved*. Potential temperature is a conserved quantity for an air parcel that moves around in the atmosphere under adiabatic conditions (see Exercise 3.36). Potential temperature is an extremely useful parameter in atmospheric thermodynamics, since atmospheric processes are often close to adiabatic, and therefore θ remains essentially constant, like density in an incompressible fluid.

3.4.4 Thermodynamic Diagrams

Poisson's equation may be conveniently solved in graphical form. If pressure is plotted on the ordinate on a distorted scale, in which the distance from the origin is proportional to p^{R_d/c_p} , or $p^{0.286}$ is used, regardless of whether air is dry or moist, and temperature (in K) is plotted on the abscissa, then (3.54) becomes

$$p^{0.286} = \left(\frac{p_0^{0.286}}{\theta}\right) T \quad (3.55)$$

For a constant value of θ , Eq. (3.55) is of the form $y \propto x$ where $y = p^{0.286}$, $x = T$, and the constant of proportionality is $p_0^{0.286}/\theta$. Each constant value of θ represents a dry adiabat, which is defined by a straight line with a particular slope that passes through the point $p = 0$, $T = 0$. If the pressure scale is inverted so that p increases downward, the relation takes the form shown in Fig. 3.6, which is the basis for the *pseudoadiabatic chart* that used to be widely used for meteorological computations. The region of the chart of greatest interest in the atmosphere is the portion shown within the dotted lines in Fig. 3.6, and this is generally the only portion of the chart that is printed.

In the pseudoadiabatic chart, isotherms are vertical and dry adiabats (constant θ) are oriented at an acute angle relative to isotherms (Fig. 3.6). Because changes in temperature with height in the atmosphere generally lie between isothermal and dry adiabatic, most temperature soundings lie within a narrow range of angles when plotted on a pseudoadiabatic chart. This restriction is overcome in the so-called *skew $T - \ln p$ chart*, in which the ordinate (y) is $-\ln p$ (the minus sign ensures that lower pressure levels are located above higher pressure levels on the chart) and the abscissa (x) is

$$x = T + (\text{constant})y = T - (\text{constant}) \ln p \quad (3.56)$$

Since, from (3.56),

$$y = \frac{x - T}{(\text{constant})}$$

and for an isotherm T is constant, the relationship between y and x for an isotherm is of the form

²⁵ **Simeon Denis Poisson** (1781–1840) French mathematician. Studied medicine but turned to applied mathematics and became the first professor of mechanics at the Sorbonne in Paris.