where ρ'_d is the density that the same mass of dry air would have if it alone occupied all of the volume V and ρ'_v is the density that the same mass of water vapor would have if it alone occupied all of the volume V. We may call these *partial densities*. Because $\rho = \rho'_d + \rho'_v$, it might appear that the density of moist air is greater than that of dry air. However, this is not the case because the partial density ρ'_v is less than the true density of dry air.[12] Applying the ideal gas equation in the form of (3.2) to the water vapor and dry air in turn, we have

$$e = \rho'_v R_v T$$

and

$$p'_d = \rho'_d R_d T$$

where e and p'_d are the partial pressures exerted by the water vapor and the dry air, respectively. Also, from Dalton's law of partial pressures,

$$p = p'_d + e$$

Combining the last four equations

$$\rho = \frac{p - e}{R_d T} + \frac{e}{R_v T}$$

or

$$\rho = \frac{p}{R_d T} \left[1 - \frac{e}{p} (1 - \varepsilon) \right]$$

where ε is defined by (3.14). The last equation may be written as

$$p = \rho R_d T_v \qquad (3.15)$$

where

$$T_v \equiv \frac{T}{1 - \dfrac{e}{p}(1 - \varepsilon)} \qquad (3.16)$$

T_v is called the *virtual temperature*. If this fictitious temperature, rather than the actual temperature, is used for moist air, the total pressure p and density ρ of the moist air are related by a form of the ideal gas equation [namely, (3.15)], but with the gas constant the same as that for a unit mass of *dry air* (R_d) and the actual temperature T replaced by the virtual temperature T_v. It follows that the virtual temperature is the temperature that dry air would need to attain in order to have the same density as the moist air at the same pressure. Because moist air is less dense than dry air at the same temperature and pressure, the virtual temperature is always greater than the actual temperature. However, even for very warm and moist air, the virtual temperature exceeds the actual temperature by only a few degrees (e.g., see Exercise 3.7 in Section 3.5).

3.2 The Hydrostatic Equation

Air pressure at any height in the atmosphere is due to the force per unit area exerted by the weight of all of the air lying above that height. Consequently, atmospheric pressure decreases with increasing height above the ground (in the same way that the pressure at any level in a stack of foam mattresses depends on how many mattresses lie above that level). The net upward force acting on a thin horizontal slab of air, due to the decrease in atmospheric pressure with height, is generally very closely in balance with the downward force due to gravitational attraction that acts on the slab. If the net upward force on the slab is equal to the downward force on the slab, the atmosphere is said to be in *hydrostatic balance*. We will now derive an important equation for the atmosphere in hydrostatic balance.

Consider a vertical column of air with unit horizontal cross-sectional area (Fig. 3.1). The mass of air between heights z and $z + \delta z$ in the column is $\rho \delta z$, where ρ is the density of the air at height z. The downward force acting on this slab of air due to the weight of the air is $g\rho\delta z$, where g is the acceleration due to gravity at height z. Now let us consider the net

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[12] The fact that moist air is less dense than dry air was first clearly stated by Sir Isaac Newton[13] in his "*Opticks*" (1717). However, the basis for this relationship was not generally understood until the latter half of the 18th century.

[13] **Sir Isaac Newton** (1642–1727) Renowned English mathematician, physicist, and astronomer. A posthumous, premature ("I could have been fitted into a quart mug at birth"), and only child. Discovered the laws of motion, the universal law of gravitation, calculus, the colored spectrum of white light, and constructed the first reflecting telescope. He said of himself: "I do not know what I may appear to the world, but to myself I seem to have been only like a boy playing on the seashore, and diverting myself in now and then finding a smoother pebble or a prettier shell than ordinary, while the great ocean of truth lay all undiscovered before me."
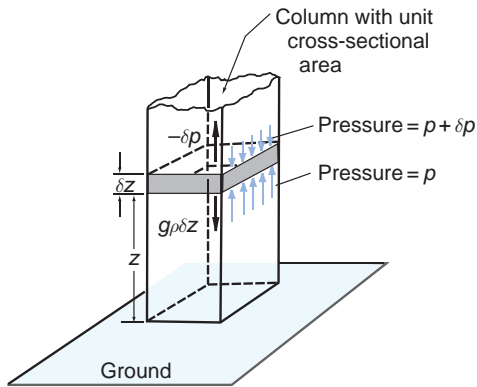
Fig. 3.1 Balance of vertical forces in an atmosphere in which there are no vertical accelerations (i.e., an atmosphere in hydrostatic balance). Small blue arrows indicate the downward force exerted on the air in the shaded slab due to the pressure of the air above the slab; longer blue arrows indicate the upward force exerted on the shaded slab due to the pressure of the air below the slab. Because the slab has a unit cross-sectional area, these two pressures have the same numerical values as forces. The net upward force due to these pressures $(-\delta p)$ is indicated by the upward-pointing thick black arrow. Because the incremental pressure change δp is a negative quantity, $-\delta p$ is positive. The downward-pointing thick black arrow is the force acting on the shaded slab due to the mass of the air in this slab.

vertical force that acts on the slab of air between z and $z + \delta z$ due to the pressure of the surrounding air. Let the change in pressure in going from height z to height $z + \delta z$ be δp, as indicated in Fig. 3.1. Because we know that pressure decreases with height, δp must be a negative quantity, and the upward pressure on the lower face of the shaded block must be slightly greater than the downward pressure on the upper face of the block. Therefore, the net vertical force on the block due to the vertical gradient of pressure is upward and given by the positive quantity $-\delta p$, as indicated in Fig. 3.1. For an atmosphere in hydrostatic balance, the balance of forces in the vertical requires that

$$-\delta p = g\rho\delta z$$

or, in the limit as $\delta z \to 0$,

$$\frac{\partial p}{\partial z} = -g\rho \tag{3.17}$$

Equation (3.17) is the *hydrostatic equation*.[14] It should be noted that the negative sign in (3.17) ensures that the pressure decreases with increasing height. Because $\rho = 1/\alpha$ (3.17) can be rearranged to give

$$gdz = -\alpha dp \tag{3.18}$$

If the pressure at height z is $p(z)$, we have, from (3.17), above a fixed point on the Earth

$$-\int_{p(z)}^{p(\infty)} dp = \int_z^\infty g\rho dz$$

or, because $p(\infty) = 0$,

$$p(z) = \int_z^\infty g\rho dz \tag{3.19}$$

That is, the pressure at height z is equal to the weight of the air in the vertical column of unit cross-sectional area lying above that level. If the mass of the Earth's atmosphere were distributed uniformly over the globe, retaining the Earth's topography in its present form, the pressure at sea level would be 1.013×10^5 Pa, or 1013 hPa, which is referred to as *1 atmosphere* (or *1 atm*).

3.2.1 Geopotential

The *geopotential* Φ at any point in the Earth's atmosphere is defined as the work that must be done against the Earth's gravitational field to raise a mass of 1 kg from sea level to that point. In other words, Φ is the gravitational potential per unit mass. The units of geopotential are J kg^{-1} or m^2 s^{-2}. The force (in newtons) acting on 1 kg at height z above sea level is numerically equal to g. The work (in joules) in raising 1 kg from z to $z + dz$ is gdz; therefore

$$d\Phi \equiv gdz$$

or, using (3.18),

$$d\Phi \equiv gdz = -\alpha dp \tag{3.20}$$

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[14] In accordance with Eq. (1.3), the left-hand side of (3.17) is written in partial differential notation, i.e., $\partial p/\partial z$, because the variation of pressure with height is taken with other independent variables held constant.

The geopotential $\Phi(z)$ at height z is thus given by

$$\Phi(z) = \int_0^z g\,dz \qquad (3.21)$$

where the geopotential $\Phi(0)$ at sea level ($z = 0$) has, by convention, been taken as zero. The geopotential at a particular point in the atmosphere depends only on the height of that point and not on the path through which the unit mass is taken in reaching that point. The work done in taking a mass of 1 kg from point A with geopotential Φ_A to point B with geopotential Φ_B is $\Phi_B - \Phi_A$.

We can also define a quantity called the *geopotential height Z* as

$$Z \equiv \frac{\Phi(z)}{g_0} = \frac{1}{g_0}\int_0^z g\,dz \qquad (3.22)$$

where g_0 is the globally averaged acceleration due to gravity at the Earth's surface (taken as 9.81 m s^{-2}). Geopotential height is used as the vertical coordinate in most atmospheric applications in which energy plays an important role (e.g., in large-scale atmospheric motions). It can be seen from Table 3.1 that the values of z and Z are almost the same in the lower atmosphere where $g_0 \simeq g$.

In meteorological practice it is not convenient to deal with the density of a gas, ρ, the value of which is generally not measured. By making use of (3.2) or (3.15) to eliminate ρ in (3.17), we obtain

$$\frac{\partial p}{\partial z} = -\frac{pg}{RT} = -\frac{pg}{R_d T_v}$$

Rearranging the last expression and using (3.20) yields

$$d\Phi = g\,dz = -RT\frac{dp}{p} = -R_d T_v \frac{dp}{p} \qquad (3.23)$$

Table 3.1 Values of geopotential height (Z) and acceleration due to gravity (g) at 40° latitude for geometric height (z)

z (km)	Z (km)	g (m s⁻²)
0	0	9.81
1	1.00	9.80
10	9.99	9.77
100	98.47	9.50
500	463.6	8.43

If we now integrate between pressure levels p_1 and p_2, with geopotentials Φ_1 and Φ_2, respectively,

$$\int_{\Phi_1}^{\Phi_2} d\Phi = -\int_{p_1}^{p_2} R_d T_v \frac{dp}{p}$$

or

$$\Phi_2 - \Phi_1 = -R_d \int_{p_1}^{p_2} T_v \frac{dp}{p}$$

Dividing both sides of the last equation by g_0 and reversing the limits of integration yields

$$Z_2 - Z_1 = \frac{R_d}{g_0}\int_{p_2}^{p_1} T_v \frac{dp}{p} \qquad (3.24)$$

This difference $Z_2 - Z_1$ is referred to as the (geopotential) *thickness* of the layer between pressure levels p_1 and p_2.

3.2.2 Scale Height and the Hypsometric Equation

For an *isothermal* atmosphere (i.e., temperature constant with height), if the virtual temperature correction is neglected, (3.24) becomes

$$Z_2 - Z_1 = H \ln(p_1/p_2) \qquad (3.25)$$

or

$$p_2 = p_1 \exp\left[-\frac{(Z_2 - Z_1)}{H}\right] \qquad (3.26)$$

where

$$H \equiv \frac{RT}{g_0} = 29.3T \qquad (3.27)$$

H is the scale height as discussed in Section 1.3.4.

Because the atmosphere is well mixed below the turbopause (about 105 km), the pressures and densities of the individual gases decrease with altitude at the same rate and with a scale height proportional to the gas constant R (and therefore inversely proportional to the apparent molecular weight of the mixture). If we take a value for T_v of 255 K (the approximate mean value for the troposphere and stratosphere), the scale height H for air in the atmosphere is found from (3.27) to be about 7.5 km.

Above the turbopause the vertical distribution of gases is largely controlled by molecular diffusion and a scale height may then be defined for each of the individual gases in air. Because for each gas the scale height is proportional to the gas constant for a unit mass of the gas, which varies inversely as the molecular weight of the gas [see, for example (3.13)], the pressures (and densities) of heavier gases fall off more rapidly with height above the turbopause than those of lighter gases.

Exercise 3.2 If the ratio of the number density of oxygen atoms to the number density of hydrogen atoms at a geopotential height of 200 km above the Earth's surface is 10^5, calculate the ratio of the number densities of these two constituents at a geopotential height of 1400 km. Assume an isothermal atmosphere between 200 and 1400 km with a temperature of 2000 K.

Solution: At these altitudes, the distribution of the individual gases is determined by diffusion and therefore by (3.26). Also, at constant temperature, the ratio of the number densities of two gases is equal to the ratio of their pressures. From (3.26)

$$\frac{(p_{1400 \text{ km}})_{\text{oxy}}}{(p_{1400 \text{ km}})_{\text{hyd}}}$$

$$= \frac{(p_{200 \text{ km}})_{\text{oxy}} \exp[-1200 \text{ km}/H_{\text{oxy}} \text{ (km)}]}{(p_{200 \text{ km}})_{\text{hyd}} \exp[-1200 \text{ km}/H_{\text{hyd}} \text{ (km)}]}$$

$$= 10^5 \exp\left[-1200 \text{ km} \left(\frac{1}{H_{\text{oxy}}} - \frac{1}{H_{\text{hyd}}}\right)\right]$$

From the definition of scale height (3.27) and analogous expressions to (3.11) for oxygen and hydrogen atoms and the fact that the atomic weights of oxygen and hydrogen are 16 and 1, respectively, we have at 2000 K

$$H_{\text{oxy}} = \frac{1000R^*}{16} \frac{2000}{9.81} \text{ m} = \frac{8.3145}{16} \frac{2 \times 10^6}{9.81} \text{ m}$$

$$= 0.106 \times 10^6 \text{ m}$$

and

$$H_{\text{hyd}} = \frac{1000R^*}{1} \frac{2000}{9.81} \text{ m} = 8.3145 \frac{2 \times 10^6}{9.81} \text{ m}$$

$$= 1.695 \times 10^6 \text{ m}$$

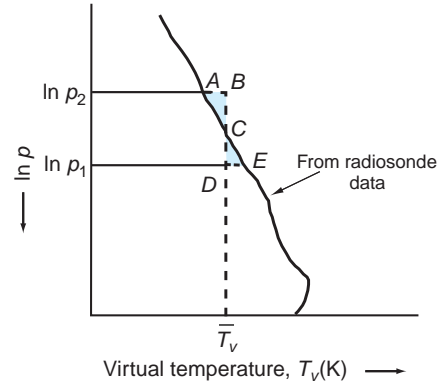**Fig. 3.2** Vertical profile, or sounding, of virtual temperature. If area ABC = area CDE, $\overline{T_v}$ is the mean virtual temperature with respect to $\ln p$ between the pressure levels p_1 and p_2.

Therefore,

$$\frac{1}{H_{\text{oxy}}} - \frac{1}{H_{\text{hyd}}} = 8.84 \times 10^{-6} \text{ m}^{-1}$$

$$= 8.84 \times 10^{-3} \text{ km}^{-1}$$

and

$$\frac{(p_{1400 \text{ km}})_{\text{oxy}}}{(p_{1400 \text{ km}})_{\text{hyd}}} = 10^5 \exp(-10.6) = 2.5$$

Hence, the ratio of the number densities of oxygen to hydrogen atoms at a geopotential height of 1400 km is 2.5. ∎

The temperature of the atmosphere generally varies with height and the virtual temeprature correction cannot always be neglected. In this more general case (3.24) may be integrated if we define a mean virtual temperature $\overline{T_v}$ with respect to p as shown in Fig. 3.2. That is,

$$\overline{T_v} \equiv \frac{\int_{p_2}^{p_1} T_v \, d(\ln p)}{\int_{p_2}^{p_1} d(\ln p)} = \frac{\int_{p_2}^{p_1} T_v \frac{dp}{p}}{\ln\left(\frac{p_1}{p_2}\right)} \quad (3.28)$$

Then, from (3.24) and (3.28),

$$Z_2 - Z_1 = \overline{H} \ln\left(\frac{p_1}{p_2}\right) = \frac{R_d \overline{T_v}}{g_0} \ln\left(\frac{p_1}{p_2}\right) \quad (3.29)$$

Equation (3.29) is called the *hypsometric equation*.

Exercise 3.3 Calculate the geopotential height of the 1000-hPa pressure surface when the pressure at sea level is 1014 hPa. The scale height of the atmosphere may be taken as 8 km.

Solution: From the hypsometric equation (3.29)

$$Z_{1000 \text{ hPa}} - Z_{\text{sea level}} = \overline{H} \ln \left(\frac{p_0}{1000} \right)$$

$$= \overline{H} \ln \left(1 + \frac{p_0 - 1000}{1000} \right) \simeq \overline{H} \left(\frac{p_0 - 1000}{1000} \right)$$

where p_0 is the sea-level pressure and the relationship $\ln (1 + x) \simeq x$ for $x \ll 1$ has been used. Substituting $\overline{H} \simeq 8000$ into this expression, and recalling that $Z_{\text{sea level}} = 0$ (Table 3.1), gives

$$Z_{1000 \text{ hPa}} \simeq 8 \, (p_0 - 1000)$$

Therefore, with $p_0 = 1014$ hPa, the geopotential height $Z_{1000 \text{ hPa}}$ of the 1000-hPa pressure surface is found to be 112 m above sea level. ∎

3.2.3 Thickness and Heights of Constant Pressure Surfaces

Because pressure decreases monotonically with height, pressure surfaces (i.e., imaginary surfaces on which pressure is constant) never intersect. It can be seen from (3.29) that the thickness of the layer between any two pressure surfaces p_2 and p_1 is proportional to the mean virtual temperature of the layer, $\overline{T}_v$. We can visualize that as $\overline{T}_v$ increases, the air between the two pressure levels expands and the layer becomes thicker.

Exercise 3.4 Calculate the thickness of the layer between the 1000- and 500-hPa pressure surfaces (a) at a point in the tropics where the mean virtual temperature of the layer is 15 °C and (b) at a point in the polar regions where the corresponding mean virtual temperature is −40 °C.

Solution: From (3.29)

$$\Delta Z = Z_{500 \text{ hPa}} - Z_{1000 \text{ hPa}} = \frac{R_d \overline{T}_v}{g_0} \ln \left(\frac{1000}{500} \right) = 20.3 \, \overline{T}_v \text{ m}$$

Therefore, for the tropics with $\overline{T}_v = 288$ K, $\Delta Z = 5846$ m. For polar regions with $\overline{T}_v = 233$ K, $\Delta Z = 4730$ m. In operational practice, thickness is rounded to the nearest 10 m and is expressed in decameters (dam). Hence, answers for this exercise would normally be expressed as 585 and 473 dam, respectively. ∎

Before the advent of remote sensing of the atmosphere by satellite-borne radiometers, thickness was evaluated almost exclusively from radiosonde data, which provide measurements of the pressure, temperature, and humidity at various levels in the atmosphere. The virtual temperature T_v at each level was calculated and mean values for various layers were estimated using the graphical method illustrated in Fig. 3.2. Using soundings from a network of stations, it was possible to construct topographical maps of the distribution of geopotential height on selected pressure surfaces. These calculations, which were first performed by observers working on site, are now incorporated into sophisticated data assimilation protocols, as described in the Appendix of Chapter 8 on the book Web site.

In moving from a given pressure surface to another pressure surface located above or below it, the change in the geopotential height is related geometrically to the thickness of the intervening layer, which, in turn, is directly proportional to the mean virtual temperature of the layer. Therefore, if the three-dimensional distribution of virtual temperature is known, together with the distribution of geopotential height on one pressure surface, it is possible to infer the distribution of geopotential height of any other pressure surface. The same hypsometric relationship between the three-dimensional temperature field and the shape of pressure surface can be used in a qualitative way to gain some useful insights into the three-dimensional structure of atmospheric disturbances, as illustrated by the following examples.

i. The air near the center of a hurricane is warmer than its surroundings. Consequently, the intensity of the storm (as measured by the depression of the isobaric surfaces) must decrease with height (Fig. 3.3a). The winds in such *warm core lows*
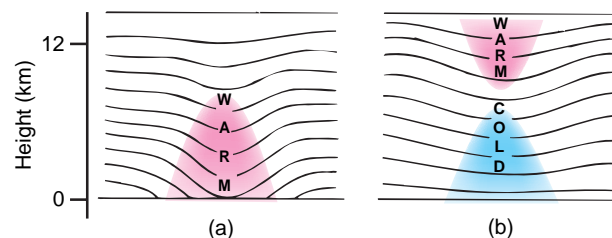

Fig. 3.3 Cross sections in the longitude–height plane. The solid lines indicate various constant pressure surfaces. The sections are drawn such that the thickness between adjacent pressure surfaces is smaller in the cold (blue) regions and larger in the warm (red) regions.

always exhibit their greatest intensity near the ground and diminish with increasing height above the ground.

ii. Some upper level lows do not extend downward to the ground, as indicated in Fig. 3.3b. It follows from the hypsometric equation that these lows must be *cold core* below the level at which they achieve their greatest intensity and *warm core* above that level, as shown in Fig. 3.3b.

3.2.4 Reduction of Pressure to Sea Level

In mountainous regions the difference in surface pressure from one observing station to another is largely due to differences in elevation. To isolate that part of the pressure field that is due to the passage of weather systems, it is necessary to reduce the pressures to a common reference level. For this purpose, sea level is normally used.

Let the subscripts g and 0 refer to conditions at the ground and at sea level ($Z = 0$), respectively. Then, for the layer between the Earth's surface and sea level, the hypsometric equation (3.29) assumes the form

$$Z_g = \overline{H} \ln \frac{p_0}{p_g} \qquad (3.30)$$

which can be solved to obtain the sea-level pressure

$$p_0 = p_g \exp\left(\frac{Z_g}{\overline{H}}\right) = p_g \exp\left(\frac{g_0 Z_g}{R_d \overline{T_v}}\right) \qquad (3.31)$$

If Z_g is small, the scale height $\overline{H}$ can be evaluated from the ground temperature. Also, if $Z_g / \overline{H} \ll 1$, the exponential in (3.31) can be approximated by $1 + Z_g / \overline{H}$, in which case (3.31) becomes

$$p_0 - p_g \simeq p_g \frac{Z_g}{\overline{H}} = p_g \left(\frac{g_0 Z_g}{R_d \overline{T_v}}\right) \qquad (3.32)$$

Because $p_g \simeq 1000$ hpa and $\overline{H} \approx 8000$ m, the pressure correction (in hPa) is roughly equal to Z_g (in meters) divided by 8. In other words, for altitudes up to a few hundred meters above (or below) sea level, the pressure decreases by about 1 hPa for every 8 m of vertical ascent.

3.3 The First Law of Thermodynamics[15]

In addition to the macroscopic kinetic and potential energy that a system as a whole may possess, it also contains *internal energy* due to the kinetic and potential energy of its molecules or atoms. Increases in internal kinetic energy in the form of molecular motions are manifested as increases in the temperature of the system, whereas changes in the potential energy of the molecules are caused by changes in their relative positions by virtue of any forces that act between the molecules.

Let us suppose that a closed system[16] of unit mass takes in a certain quantity of thermal energy q (measured in joules), which it can receive by thermal conduction and/or radiation. As a result the system may do a certain amount of *external work w* (also measured in joules). The excess of the energy supplied to the body over and above the external work done by the body is $q - w$. Therefore, if there is no change in the macroscopic kinetic and potential energy of the body, it follows from the principle of conservation of energy that the internal energy of the system must increase by $q - w$. That is,

$$q - w = u_2 - u_1 \qquad (3.33)$$

where u_1 and u_2 are the internal energies of the system before and after the change. In differential form (3.33) becomes

$$dq - dw = du \qquad (3.34)$$

where dq is the differential increment of heat added to the system, dw is the differential element

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[15] The first law of thermodynamics is a statement of the conservation of energy, taking into account the conversions between the various forms that it can assume and the exchanges of energy between a system and its environment that can take place through the transfer of heat and the performance of mechanical work. A general formulation of the first law of thermodynamics is beyond the scope of this text because it requires consideration of conservation laws, not only for energy, but also for momentum and mass. This section presents a simplified formulation that ignores the macroscopic kinetic and potential energy (i.e., the energy that air molecules possess by virtue of their height above sea level and their organized fluid motions). As it turns out, the expression for the first law of thermodynamics that emerges in this simplified treatment is identical to the one recovered from a more complete treatment of the conservation laws, as is done in J. R. Holton, *Introduction to Dynamic Meteorology*, 4th Edition, Academic Press, New York, 2004, pp. 146–149.

[16] A closed system is one in which the total amount of matter, which may be in the form of gas, liquid, solid or a mixture of these phases, is kept constant.