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of which are presented later. We must also discuss what happens when water vapor condenses in air.

### 3.5.1 Moisture Parameters

#### a. Mixing ratio and specific humidity

The amount of water vapor in a certain volume of air may be defined as the ratio of the mass  $m_v$  of water vapor to the mass of dry air; this is called the *mixing ratio*  $w$ . That is

$$w \equiv \frac{m_v}{m_d} \quad (3.57)$$

The mixing ratio is usually expressed in grams of water vapor per kilogram of dry air (but in solving numerical exercises  $w$  must be expressed as a dimensionless number, e.g., as kg of water vapor per kg of dry air). In the atmosphere, the magnitude of  $w$  typically ranges from a few grams per kilogram in middle latitudes to values of around 20 g kg<sup>-1</sup> in the tropics. If neither condensation nor evaporation takes place, the mixing ratio of an air parcel is constant (i.e., it is a conserved quantity).

The mass of water vapor  $m_v$  in a unit mass of air (dry air plus water vapor) is called the *specific humidity*  $q$ , that is

$$q \equiv \frac{m_v}{m_v + m_d} = \frac{w}{1 + w}$$

Because the magnitude of  $w$  is only a few percent, it follows that the numerical values of  $w$  and  $q$  are nearly equivalent.

**Exercise 3.6** If air contains water vapor with a mixing ratio of 5.5 g kg<sup>-1</sup> and the total pressure is 1026.8 hPa, calculate the vapor pressure  $e$ .

**Solution:** The partial pressure exerted by any constituent in a mixture of gases is proportional to the number of moles of the constituent in the mixture. Therefore, the pressure  $e$  due to water vapor in air is given by

$$e = \frac{n_v}{n_d + n_v} p = \frac{\frac{m_v}{M_w}}{\frac{m_d}{M_d} + \frac{m_v}{M_w}} p \quad (3.58)$$

$n_v$  and  $n_d$  are the number of moles of water vapor and dry air in the mixture, respectively,  $M_w$  is the molecular weight of water,  $M_d$  is the apparent molec-

ular weight of dry air, and  $p$  is the total pressure of the moist air. From (3.57) and (3.58) we obtain

$$e = \frac{w}{w + \varepsilon} p \quad (3.59)$$

where  $\varepsilon = 0.622$  is defined by (3.14). Substituting  $p = 1026.8$  hPa and  $w = 5.5 \times 10^{-3}$  kg kg<sup>-1</sup> into (3.59), we obtain  $e = 9.0$  hPa. ■

**Exercise 3.7** Calculate the virtual temperature correction for moist air at 30 °C that has a mixing ratio of 20 g kg<sup>-1</sup>.

**Solution:** Substituting  $e/p$  from (3.59) into (3.16) and simplifying

$$T_v = T \frac{w + \varepsilon}{\varepsilon(1 + w)}$$

Dividing the denominator into the numerator in this expression and neglecting terms in  $w^2$  and higher orders of  $w$ , we obtain

$$T_v - T \approx \frac{1 - \varepsilon}{\varepsilon} w T$$

or, substituting  $\varepsilon = 0.622$  and rearranging,

$$T_v \approx T(1 + 0.61w) \quad (3.60)$$

With  $T = 303$  K and  $w = 20 \times 10^{-3}$  kg kg<sup>-1</sup>, Eq. (3.60) gives  $T_v = 306.7$  K. Therefore, the virtual temperature correction is  $T_v - T = 3.7$  degrees (K or °C). Note that (3.60) is a useful expression for obtaining  $T_v$  from  $T$  and the moisture parameter  $w$ . ■

#### b. Saturation vapor pressures

Consider a small closed box, the floor of which is covered with pure water at temperature  $T$ . Initially assume that the air is completely dry. Water will begin to evaporate and, as it does, the number of water molecules in the box, and therefore the water vapor pressure, will increase. As the water vapor pressure increases, so will the rate at which the water molecules condense from the vapor phase back to the liquid phase. If the rate of condensation is less than the rate of evaporation, the box is said to be *unsaturated* at temperature  $T$  (Fig. 3.8a). When the water vapor pressure in the box increases to the point that the rate of condensation is equal to the rate of evaporation (Fig. 3.8b), the air is said to be *saturated with respect*

### 3.3 Can Air Be Saturated with Water Vapor?<sup>26</sup>

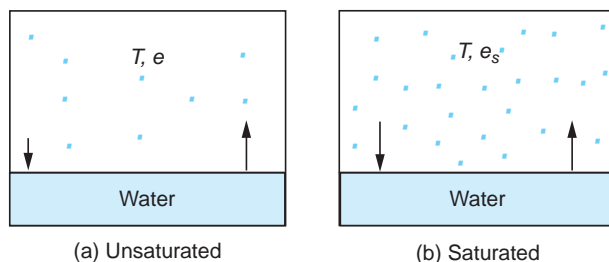
It is common to use phrases such as “the air is saturated with water vapor,” “the air can hold no more water vapor,” and “warm air can hold more water vapor than cold air.” These phrases, which suggest that air absorbs water vapor, rather like a sponge, are misleading. We have seen that the total pressure exerted by a mixture of gases is equal to the sum of the pressures that each gas would exert if it alone occupied the total volume of the mixture of gases (Dalton’s law of partial pressures). Hence, the exchange of water mole-

cules between its liquid and vapor phases is (essentially) independent of the presence of air. Strictly speaking, the pressure exerted by water vapor that is in equilibrium with water at a given temperature is referred more appropriately to as *equilibrium vapor pressure* rather than saturation vapor pressure at that temperature. However, the latter term, and the terms “unsaturated air” and “saturated air,” provide a convenient shorthand and are so deeply rooted that they will appear in this book.

to a plane surface of pure water at temperature  $T$ , and the pressure  $e_s$  that is then exerted by the water vapor is called the *saturation vapor pressure over a plane surface of pure water at temperature  $T$* .

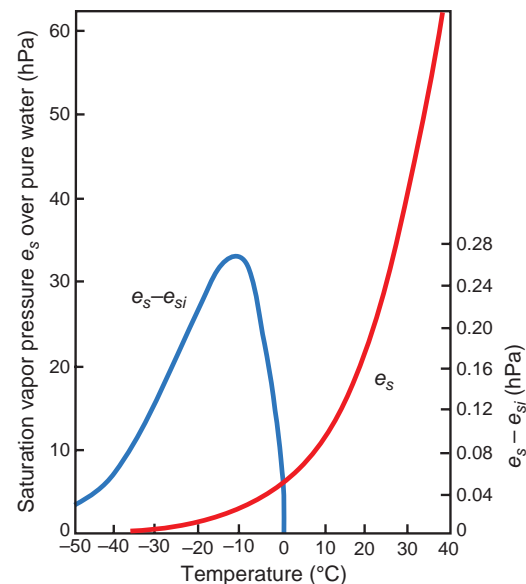
Similarly, if the water in Fig. 3.8 were replaced by a plane surface of pure ice at temperature  $T$  and the rate of condensation of water vapor were equal to the rate of evaporation of the ice, the pressure  $e_{si}$  exerted by the water vapor would be the saturation vapor pressure over a plane surface of pure ice at  $T$ . Because, at any given temperature, the rate of evaporation from ice is less than from water,  $e_s(T) > e_{si}(T)$ .

The rate at which water molecules evaporate from either water or ice increases with increasing temperature.<sup>27</sup> Consequently, both  $e_s$  and  $e_{si}$  increase with increasing temperature, and their magnitudes



**Fig. 3.8** A box (a) unsaturated and (b) saturated with respect to a plane surface of pure water at temperature  $T$ . Dots represent water molecules. Lengths of the arrows represent the relative rates of evaporation and condensation. The saturated (i.e., equilibrium) vapor pressure over a plane surface of pure water at temperature  $T$  is  $e_s$  as indicated in (b).

depend only on temperature. The variations with temperature of  $e_s$  and  $e_s - e_{si}$  are shown in Fig. 3.9, where it can be seen that the magnitude of  $e_s - e_{si}$  reaches a peak value at about  $-12^\circ\text{C}$ . It follows that if an ice particle is in water-saturated air it will grow due to the deposition of water vapor upon it. In Section 6.5.3 it is shown that this phenomenon



**Fig. 3.9** Variations with temperature of the saturation (i.e., equilibrium) vapor pressure  $e_s$  over a plane surface of pure water (red line, scale at left) and the difference between  $e_s$  and the saturation vapor pressure over a plane surface of ice  $e_{si}$  (blue line, scale at right).

<sup>26</sup> For further discussion of this and some other common misconceptions related to meteorology see C. F. Bohren’s *Clouds in a Glass of Beer*, Wiley and Sons, New York, 1987.

<sup>27</sup> As a rough rule of thumb, it is useful to bear in mind that the saturation vapor pressure roughly doubles for a  $10^\circ\text{C}$  increase in temperature.

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plays a role in the initial growth of precipitable particles in some clouds.

### c. Saturation mixing ratios

The *saturation mixing ratio*  $w_s$  with respect to water is defined as the ratio of the mass  $m_{vs}$  of water vapor in a given volume of air that is saturated with respect to a plane surface of pure water to the mass  $m_d$  of the dry air. That is

$$w_s \equiv \frac{m_{vs}}{m_d} \quad (3.61)$$

Because water vapor and dry air both obey the ideal gas equation

$$w_s = \frac{\rho'_{vs}}{\rho'_d} = \frac{e_s}{(R_v T)} \bigg/ \frac{(p - e_s)}{(R_d T)} \quad (3.62)$$

where  $\rho'_{vs}$  is the partial density of water vapor required to saturate air with respect to water at temperature  $T$ ,  $\rho'_d$  is the partial density of the dry air (see Section 3.1.1), and  $p$  is the total pressure. Combining (3.62) with (3.14), we obtain

$$w_s = 0.622 \frac{e_s}{p - e_s}$$

For the range of temperatures observed in the Earth's atmosphere,  $p \gg e_s$ ; therefore

$$w_s \approx 0.622 \frac{e_s}{p} \quad (3.63)$$

Hence, at a given temperature, the saturation mixing ratio is inversely proportional to the total pressure.

Because  $e_s$  depends only on temperature, it follows from (3.63) that  $w_s$  is a function of temperature and pressure. Lines of constant saturation mixing ratio are printed as dashed green lines on the skew  $T - \ln p$  chart and are labeled with the value of  $w_s$  in grams of water vapor per kilogram of dry air. It is apparent from the slope of these lines that at constant pressure  $w_s$  increases with increasing temperature, and at constant temperature  $w_s$  increases with decreasing pressure.

### d. Relative humidity; dew point and frost point

The relative humidity (RH) with respect to water is the ratio (expressed as a percentage) of the actual

mixing ratio  $w$  of the air to the saturation mixing ratio  $w_s$  with respect to a plane surface of pure water at the same temperature and pressure. That is

$$RH \equiv 100 \frac{w}{w_s} \approx 100 \frac{e}{e_s} \quad (3.64)$$

The *dew point*  $T_d$  is the temperature to which air must be cooled at constant pressure for it to become saturated with respect to a plane surface of pure water. In other words, the dew point is the temperature at which the saturation mixing ratio  $w_s$  with respect to liquid water becomes equal to the actual mixing ratio  $w$ . It follows that the relative humidity at temperature  $T$  and pressure  $p$  is given by

$$RH = 100 \frac{w_s(\text{at temperature } T_d \text{ and pressure } p)}{w_s(\text{at temperature } T \text{ and pressure } p)} \quad (3.65)$$

A simple rule of thumb for converting RH to a dew point depression ( $T - T_d$ ) for moist air ( $RH > 50\%$ ) is that  $T_d$  decreases by  $\sim 1^\circ\text{C}$  for every 5% decrease in RH (starting at  $T_d =$  dry bulb temperature ( $T$ ), where  $RH = 100\%$ ). For example, if the RH is 85%,  $T_d = T - \left(\frac{100 - 85}{5}\right)$  and the dew point depression is  $T - T_d = 3^\circ\text{C}$ .

The *frost point* is defined as the temperature to which air must be cooled at constant pressure to saturate it with respect to a plane surface of pure ice. Saturation mixing ratios and relative humidities with respect to ice may be defined in analogous ways to their definitions with respect to liquid water. When the terms mixing ratio and relative humidity are used without qualification they are with respect to liquid water.

**Exercise 3.8** Air at 1000 hPa and  $18^\circ\text{C}$  has a mixing ratio of  $6 \text{ g kg}^{-1}$ . What are the relative humidity and dew point of the air?

**Solution:** This exercise may be solved using a skew  $T - \ln p$  chart. The students should duplicate the following steps. First locate the point with pressure 1000 hPa and temperature  $18^\circ\text{C}$ . We see from the chart that the saturation mixing ratio for this state is  $\sim 13 \text{ g kg}^{-1}$ . Since the air specified in the problem has a mixing ratio of only  $6 \text{ g kg}^{-1}$ , it is unsaturated and its relative humidity is, from (3.64),  $100 \times 6/13 = 46\%$ . To find the dew point we move from right to