CMSC 451: Polynomial-time Reductions & NP-completeness

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The Class NP

We will define two classes of problems called **NP** and **NP**-complete.

We need some new ideas.

Certificate

Recall the independent set problem (decision version):

Independent Set

Given a graph G, is there set S of size $\geq k$ such that no two nodes in S are connected by an edge?

Finding the set S is hard (we will see).

But if I give you a set S^*, checking whether S^* is the answer is easy: check that $|S| \geq k$ and no edges go between 2 nodes in S^*.

S^* acts as a certificate that $\langle G, k \rangle$ is a `yes` instance of Independent Set.

Efficient Certification

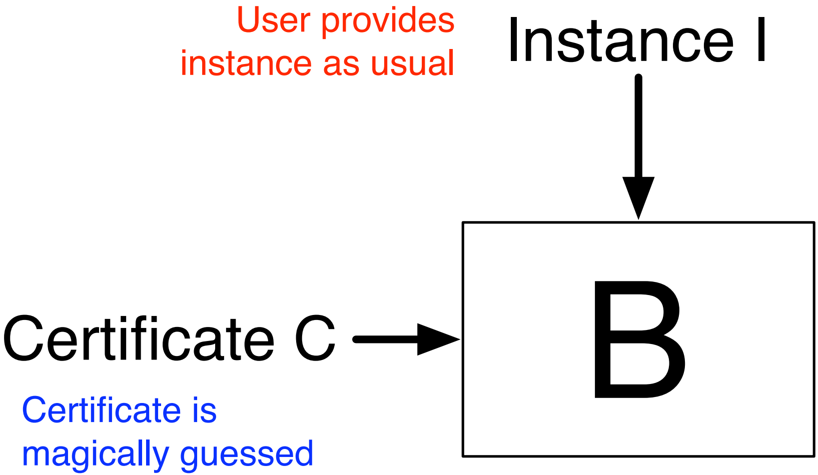
Def. An algorithm B is an efficient certifier for problem X if:

1. B is a polynomial time algorithm that takes two input strings I (instance of X) and C (a certificate).
2. B outputs either `yes` or `no`.
3. There is a polynomial $p(n)$ such that for every string I:

   $I \in X$ if and only if there exists string C of length $\leq p(|I|)$ such that $B(I, C) =$ `yes`.

B is an algorithm that can decide whether an instance I is a yes instance if it is given some "help" in the form of a polynomially long certificate.

CERTIFIERS


User provides
instance as usual
Instance I
B
Certificate C
Certificate is
magically guessed

The class NP

NP is the set of languages for which there exists an efficient certifier.

The class NP

NP is the set of languages for which there exists an efficient certifier.

P is the set of languages for which there exists an efficient certifier that ignores the certificate.

That's the difference: A problem is in **P** if we can decided them in polynomial time. It is in **NP** if we can decide them in polynomial time, if we are given the right certificate.

$\mathbf{P} \subseteq \mathbf{NP}$

Theorem

$\mathbf{P} \subseteq \mathbf{NP}$

Proof. Suppose $X \in \mathbf{P}$. Then there is a polynomial time algorithm A for X.

To show that $X \in \mathbf{NP}$, we need to design an efficient certifier $B(I, C)$.

Just take $B(I, C) = A(I)$. $\square$

Every problem with a polynomial time algorithm is in **NP**.

$\mathbf{P} \neq \mathbf{NP}$?

The big question:

$$\mathbf{P} = \mathbf{NP}?$$

We know $\mathbf{P} \subseteq \mathbf{NP}$. So the question is:

Is there some problem in **NP** that is not in **P**?

Seems like the power of the certificate would help a lot.
But no one knows. . . .

Reductions as tool for hardness

We want prove some problems are computationally difficult.

As a first step, we settle for relative judgements:

Problem X is at least as hard as problem Y

To prove such a statement, we reduce problem Y to problem X:

> *If you had a black box that can solve instances of problem X, can you solve any instance of Y using polynomial number of steps, plus a polynomial number of calls to the black box that solves X?*

Polynomial Reductions

- If problem Y can be reduced to problem X, we denote this by $Y \leq_P X$.
- This means "Y is polynomal-time reducible to X."
- It also means that X is at least as hard as Y because if you can solve X, you can solve Y.
- Note: We reduce *to* the problem we want to show is the harder problem.

Polynomial Problems

Suppose:

- $Y \leq_P X$, and
- there is an polynomial time algorithm for X.

Then, there is a polynomial time algorithm for Y.

Why?

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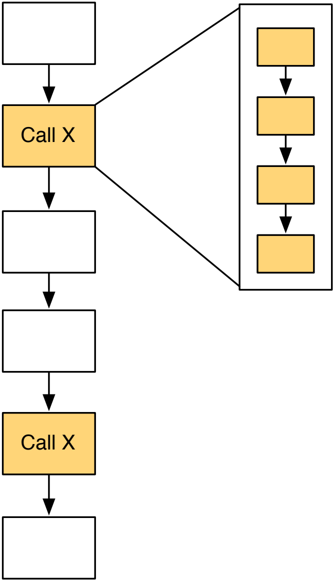

Reductions for Hardness

Theorem

If $Y \leq_P X$ and Y cannot be solved in polynomial time, then X cannot be solved in polynomial time.

Why? If we *could* solve X in polynomial time, then we'd be able to solve Y in polynomial time, contradicting the assumption.

So: If we could find one hard problem Y, we could prove that another problem X is hard by reducing Y to X.

Vertex Cover

Def. A vertex cover of a graph is a set S of nodes such that every edge has at least one endpoint in S.

In other words, we try to "cover" each of the edges by choosing at least one of its vertices.

(Yes, "Vertex Cover" is a horrible name: we're covering *edges* with vertices. There's no hope to change this now.)

Vertex Cover

Given a graph G and a number k, does G contain a vertex cover of size at most k.

Independent Set to Vertex Cover

Independent Set

Given graph G and a number k, does G contain a set of at least k independent vertices?

Can we reduce independent set to vertex cover?

Vertex Cover

Given a graph G and a number k, does G contain a vertex cover of size at most k.

Relation btwn Vertex Cover and Indep. Set

Theorem

If $G = (V, E)$ is a graph, then S is an independent set $\iff$ $V - S$ is a vertex cover.

Proof. $\implies$ Suppose S is an independent set, and let $e = (u, v)$ be some edge. Only one of u, v can be in S. Hence, at least one of u, v is in $V - S$. So, $V - S$ is a vertex cover.

$\impliedby$ Suppose $V - S$ is a vertex cover, and let $u, v \in S$. There can't be an edge between u and v (otherwise, that edge wouldn't be covered in $V - S$). So, S is an independent set. $\square$

Independent Set $\leq_P$ Vertex Cover

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To show this, we change any instance of Independent Set into an instance of Vertex Cover.

Proof.

- Given an instance of Independent Set $\langle G, k \rangle$, with $|G| = n$
- we ask our Vertex Cover black box if there is a vertex cover of with $n - k$ vertices.

By our previous theorem, S is an independent set iff $V - S$ is a vertex cover.

So: G has a independent set of size k iff G has a vertex cover of size $n - k$.

Vertex Cover $\leq_P$ Independent Set

Actually, we also have:

Vertex Cover $\leq_P$ Independent Set

Proof. To decide if G has an vertex cover of size k, we ask if it has a independent set of size $n - k$. $\square$

So: Vertex Cover and Independent Set are equivalently difficult.

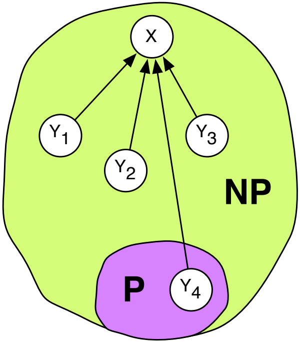
NP-completeness

Def. We say X is NP-complete if:

- $X \in \mathbf{NP}$
- for all $Y \in \mathbf{NP}$, $Y \leq_P X$.

If these hold, then X can be used to solve every problem in **NP**.

Therefore, X is definitely at least as hard as every problem in **NP**.



NP-completeness and P=NP

Theorem

If X is NP-complete, then X is solvable in polynomial time if and only if $\mathbf{P} = \mathbf{NP}$.

Proof. If $\mathbf{P} = \mathbf{NP}$, then X can be solved in polytime.

Suppose X is solvable in polytime, and let Y be any problem in **NP**. We can solve Y in polynomial time: reduce it to X.

Therefore, every problem in **NP** has a polytime algorithm and $\mathbf{P} = \mathbf{NP}$.

Reductions and NP-completeness

Theorem

If Y is NP-complete, and

1. *X is in NP*
2. *$Y \leq_P X$*

then X is NP-complete.

In other words, we can prove a new problem in NP-complete by reducing some other NP-complete problem to it.

Proof. Let Z be any problem in **NP**. Since Y is NP-complete, $Z \leq_P Y$. By assumption, $Y \leq_P X$. Therefore: $Z \leq_P Y \leq_P X$. $\square$

Boolean Formulas

Boolean Formulas:

VARIABLES: x_1, x_2, x_3 (can be either **true** or **false**)

TERMS: $t_1, t_2, \ldots, t_\ell$: t_j is either x_i or $\bar{x}_i$
(meaning either x_i or **not** x_i).

CLAUSES: $t_1 \vee t_2 \vee \cdots \vee t_\ell$ ($\vee$ stands for "OR")
A clause is **true** if any term in it is **true**.

Example 1: $(x_1 \vee \bar{x}_2), (\bar{x}_1 \vee \bar{x}_3), (x_2 \vee \bar{v}_3)$

Example 2: $(x_1 \vee x_2 \vee \bar{x}_3), (\bar{x}_2 \vee x_1)$

Boolean Formulas

Def. A truth assignment is a choice of **true** or **false** for each variable, ie, a function $v : \{x_1, \ldots, x_n\} \to \{\textbf{true}, \textbf{false}\}$.

Def. A CNF formula is a conjunction of clauses:

$$C_1 \wedge C_2, \wedge \cdots \wedge C_k$$

Example: $(x_1 \vee \bar{x_2}) \wedge (\bar{x_1} \vee \bar{x_3}) \wedge (x_2 \vee \bar{v_3})$

Def. A truth assignment is a satisfying assignment for such a formula if it makes every clause **true**.

SAT and 3-SAT

Satisfiability (SAT)

Given a set of clauses $C_1, \ldots, C_k$ over variables $X = \{x_1, \ldots, x_n\}$ is there a satisfying assignment?

Satisfiability (3-SAT)

Given a set of clauses $C_1, \ldots, C_k$, each of length 3, over variables $X = \{x_1, \ldots, x_n\}$ is there a satisfying assignment?

Cook-Levin Theorem

Theorem (Cook-Levin)

3-SAT is NP-complete.

Proven in early 1970s by Cook. Slightly different proof by Levin independently.

Idea of the proof: encode the workings of a Nondeterministic Turing machine for and instance I of problem $X \in \mathbf{NP}$ as a SAT formula so that the formula is satisfiable if and only if the nondeterministic Turing machine would accept instance I.

We won't have time to prove this, but it gives us our first hard problem.

Reducing 3-SAT to Independent Set

Thm. 3-SAT $\leq_P$ Independent Set

Proof. Suppose we have an algorithm to solve Independent Set, how can we use it to solve 3-SAT?

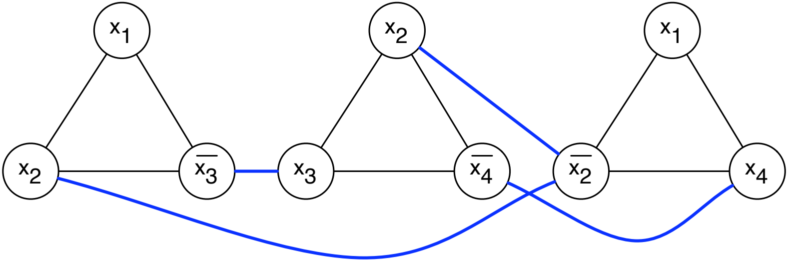
To solve 3-SAT,

- you have to choose a term from each clause to set to **true**,
- but you can't set both x_i and $\bar{x}_i$ to **true**.

How do we do the reduction?

3-SAT $\leq_P$ Independent Set

$$(x_1 \vee x_2 \vee \overline{x_3}) \wedge (x_2 \vee x_3 \vee \overline{x_4}) \wedge (x_1 \vee \overline{x_2} \vee x_4)$$



Proof

Theorem

This graph has an independent set of size k iff the formula is satisfiable.

Proof. $\Longrightarrow$ If the formula is satisfiable, there is at least one true literal in each clause. Let S be a set of one such true literal from each clause. $|S| = k$ and no two nodes in S are connected by an edge.

$\Longrightarrow$ If the graph has an independent set S of size k, we know that it has one node from each "clause triangle." Set those terms to true. This is possible because no two are negations of each other.
□

General Proof Strategy

General Strategy for Proving Something is NP-complete:

1. Must show that $X \in \mathbf{NP}$. Do this by showing there is an certificate that can be efficiently checked.

2. Look at all the problems that are known to be NP-complete (there are thousands), and choose one Y that seems "similar" to your problem in some way.

3. Show that $Y \leq_P X$.