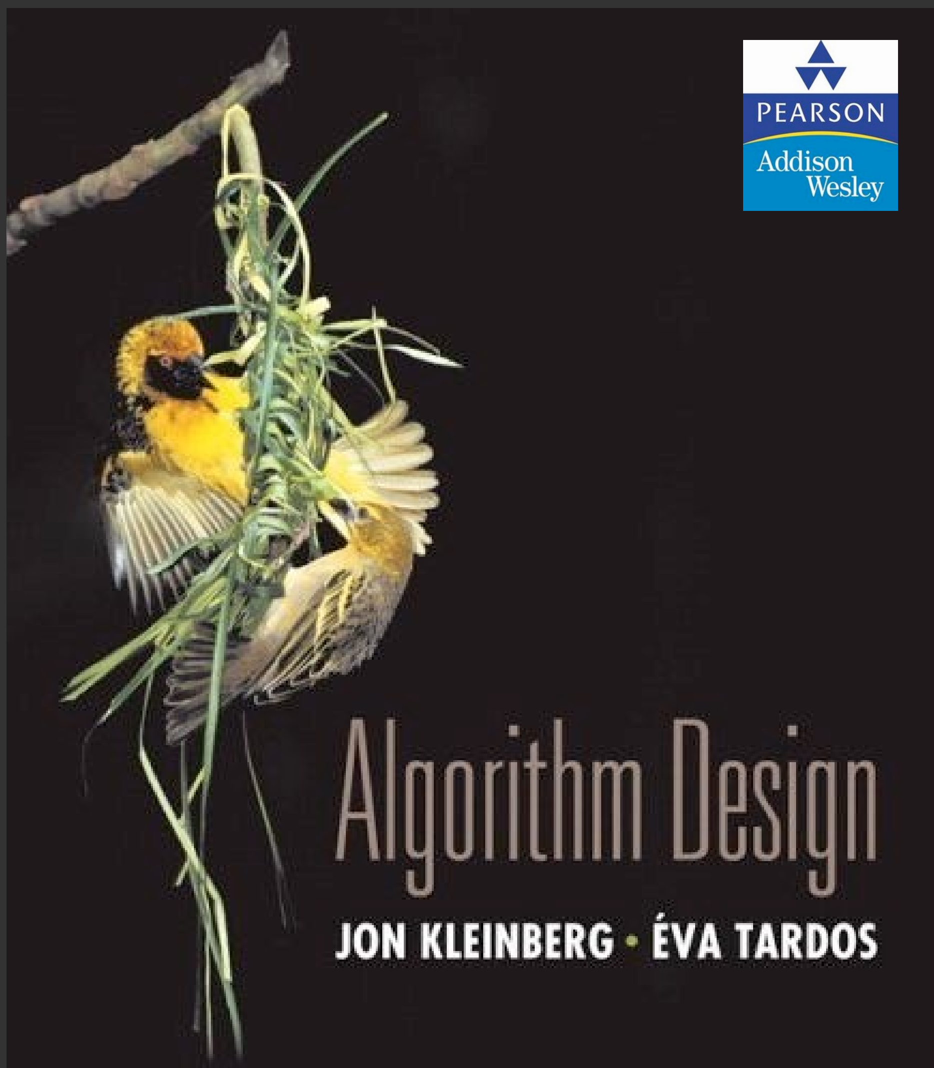
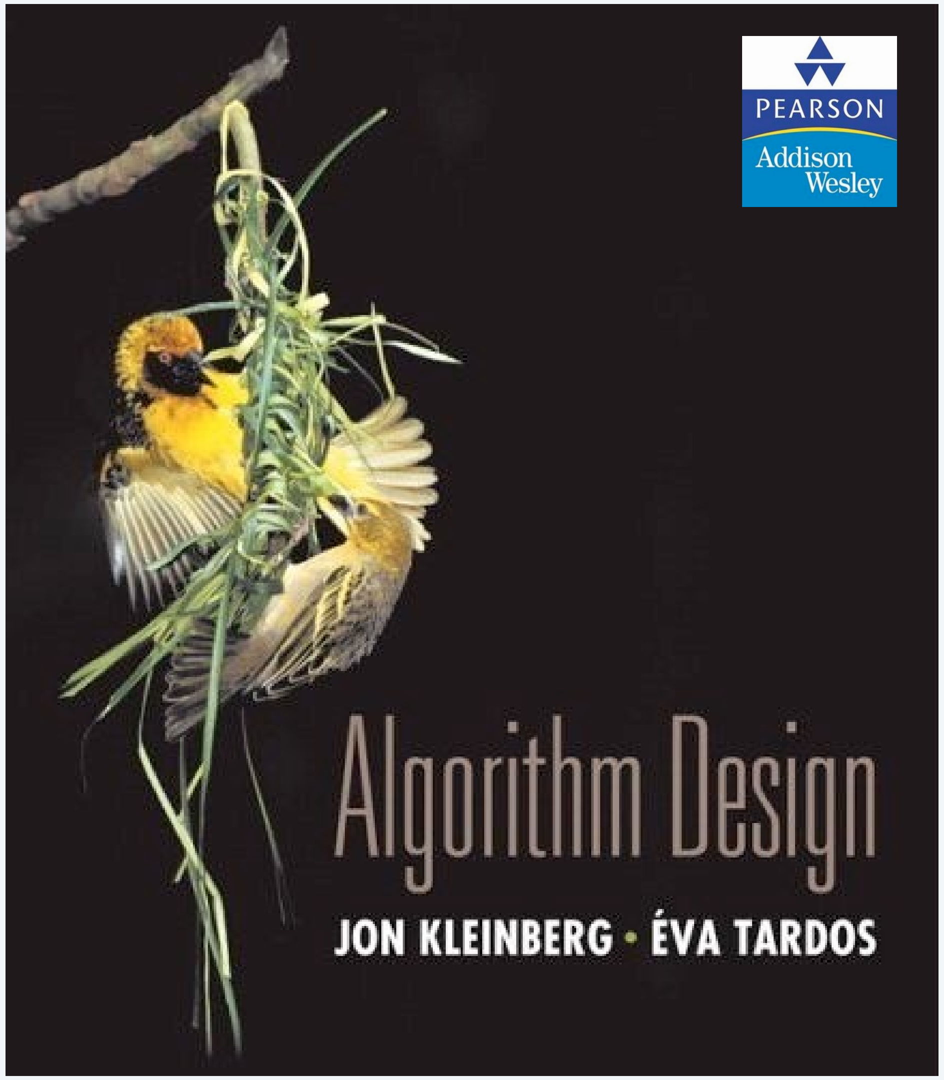


3. GRAPHS

- *basic definitions and applications*
- *graph connectivity and graph traversal*
- *testing bipartiteness*
- *connectivity in directed graphs*
- *DAGs and topological ordering*

Lecture slides by Kevin Wayne

Last updated on 8/10/17 2:08 PM



3. GRAPHS

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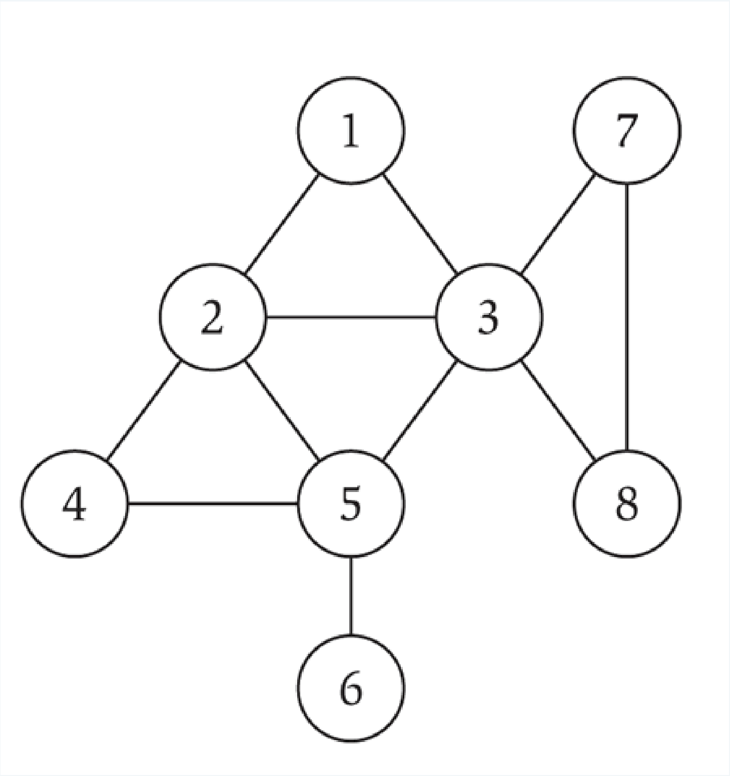
Undirected graphs

Notation. $G = (V, E)$

- V = nodes (or vertices).
- E = edges (or arcs) between pairs of nodes.
- Captures pairwise relationship between objects.
- Graph size parameters: $n = |V|$, $m = |E|$.

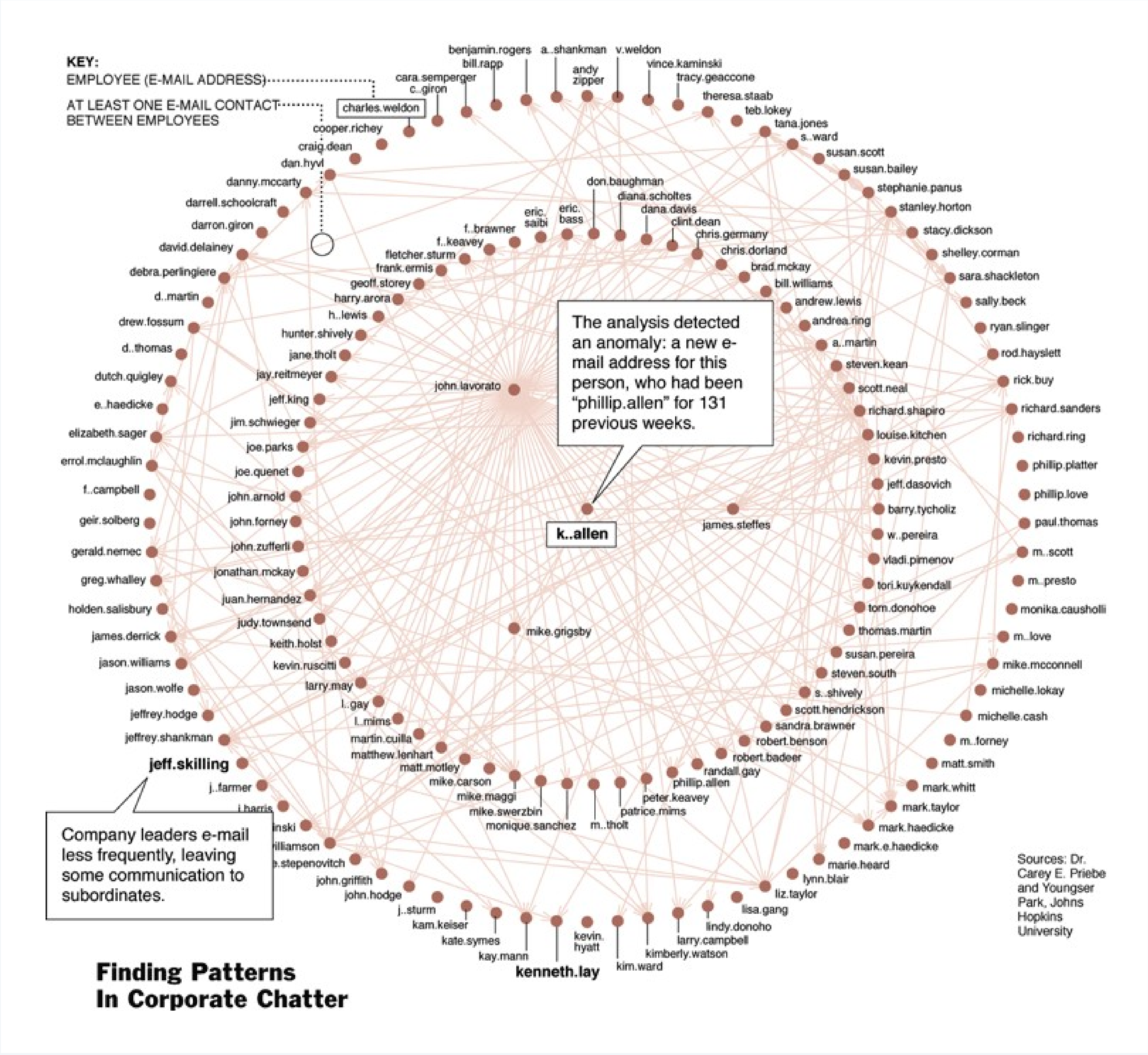


$V = \{ 1, 2, 3, 4, 5, 6, 7, 8 \}$

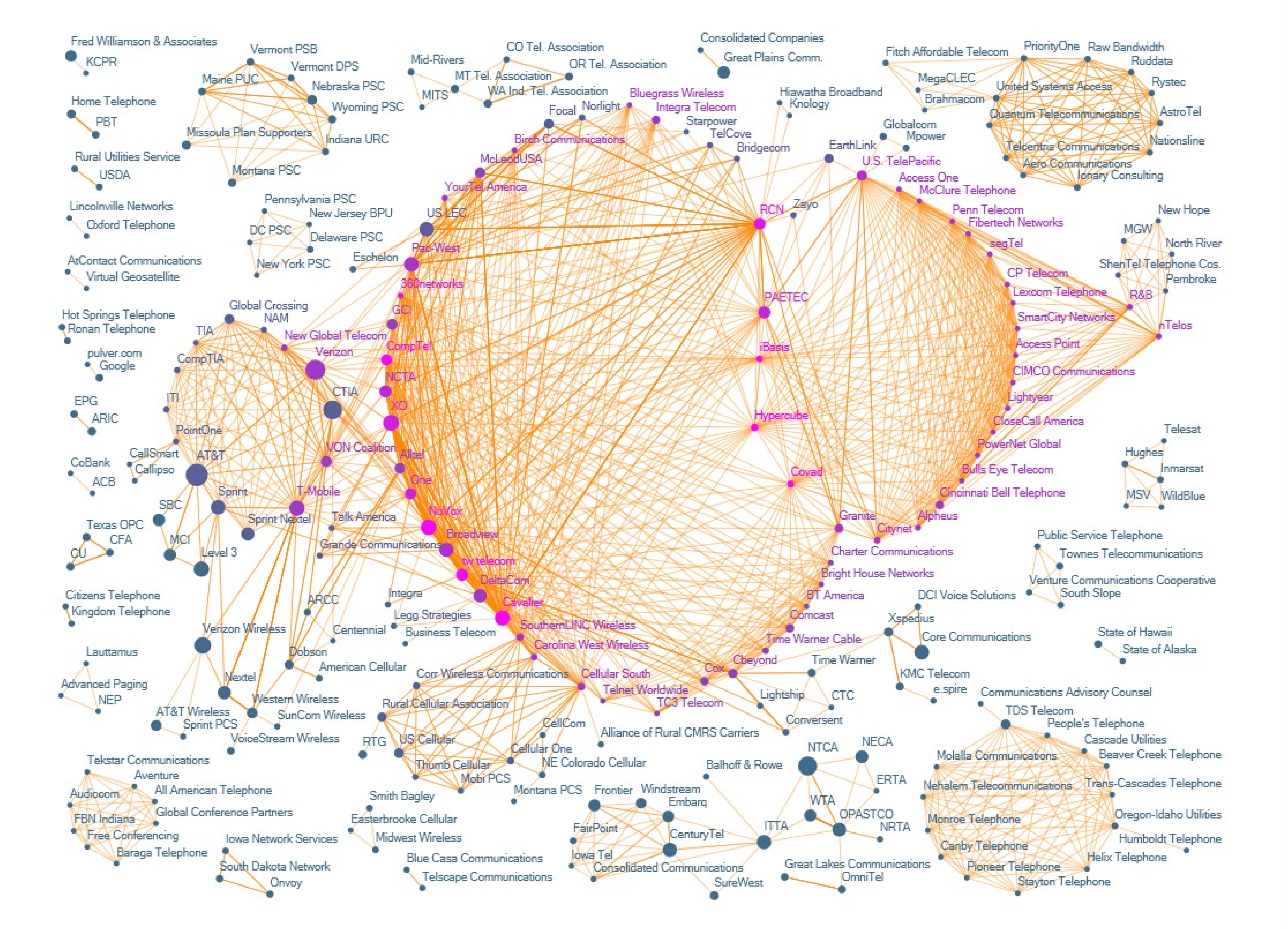
$E = \{ 1–2, 1–3, 2–3, 2–4, 2–5, 3–5, 3–7, 3–8, 4–5, 5–6, 7–8 \}$

$m = 11$, $n = 8$

One week of Enron emails



The evolution of FCC lobbying coalitions



"The Evolution of FCC Lobbying Coalitions" by Pierre de Vries in JoSS Visualization Symposium 2010

Framingham heart study

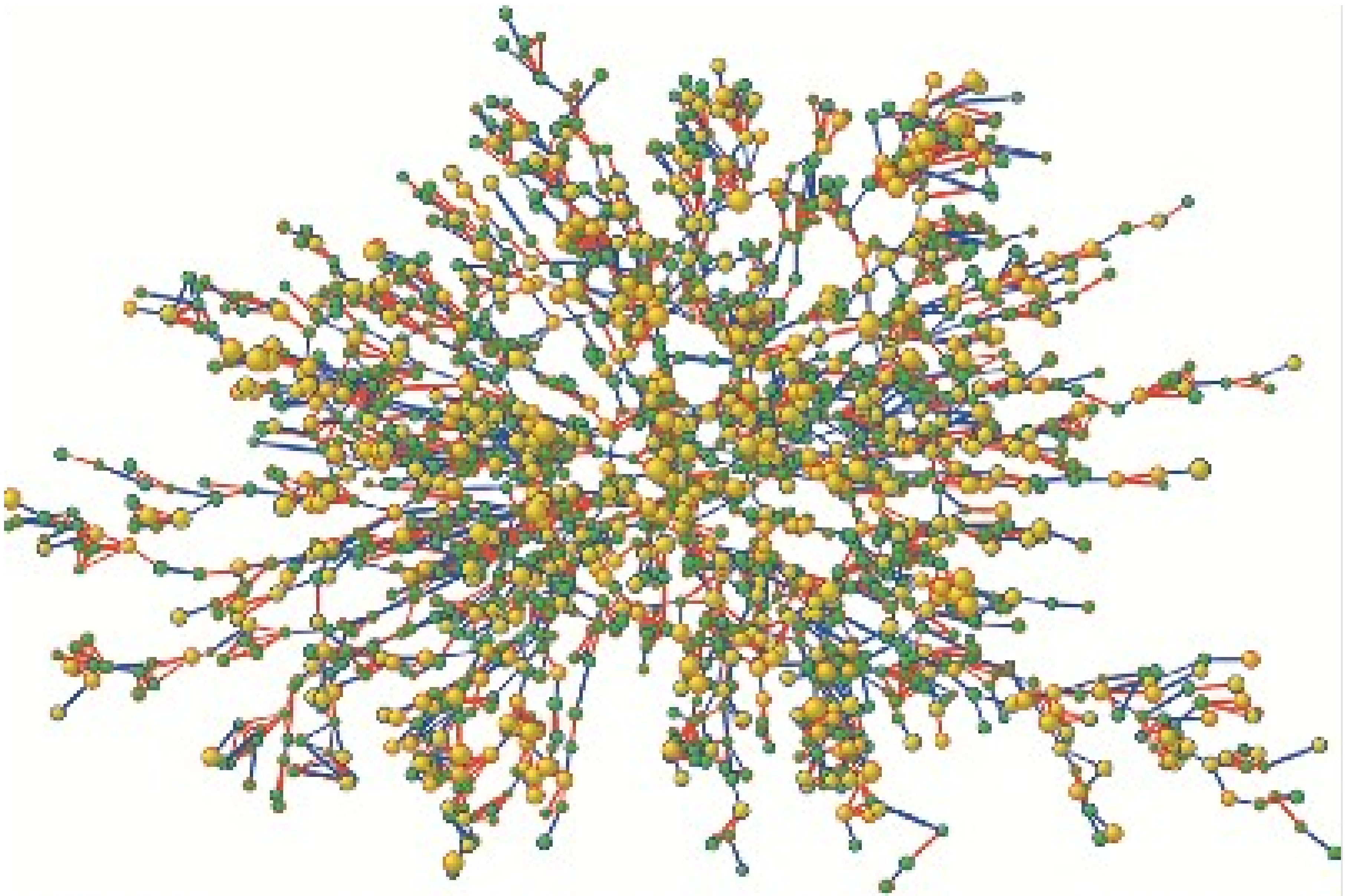


Figure 1. Largest Connected Subcomponent of the Social Network in the Framingham Heart Study in the Year 2000.

Each circle (node) represents one person in the data set. There are 2200 persons in this subcomponent of the social network. Circles with red borders denote women, and circles with blue borders denote men. The size of each circle is proportional to the person's body-mass index. The interior color of the circles indicates the person's obesity status: yellow denotes an obese person (body-mass index, ≥30) and green denotes a nonobese person. The colors of the ties between the nodes indicate the relationship between them: purple denotes a friendship or marital tie and orange denotes a familial tie.

"[The Spread of Obesity in a Large Social Network over 32 Years]" by Christakis and Fowler in New England Journal of Medicine, 2007

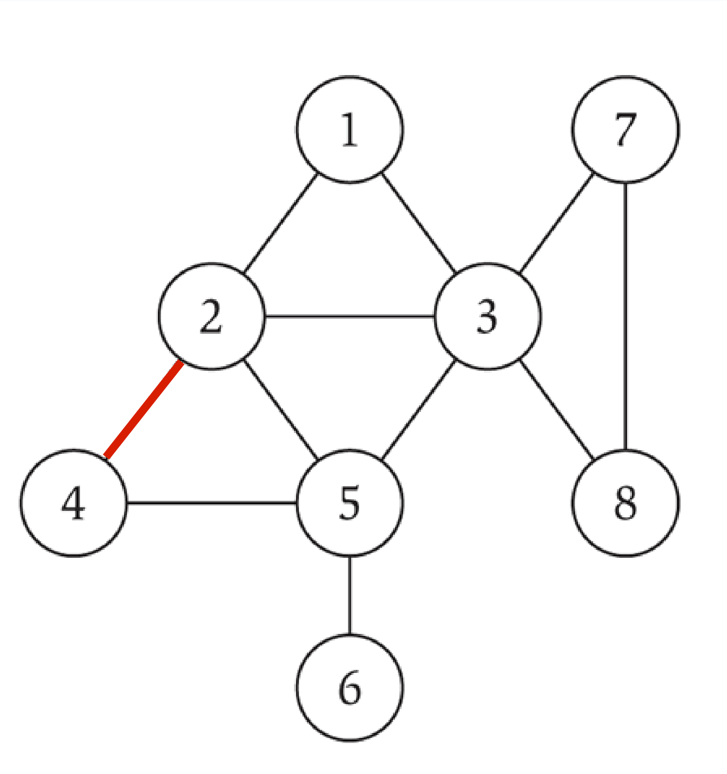
Some graph applications

graph	node	edge
communication	telephone, computer	fiber optic cable
circuit	gate, register, processor	wire
mechanical	joint	rod, beam, spring
financial	stock, currency	transactions
transportation	street intersection, airport	highway, airway route
internet	class C network	connection
game	board position	legal move
social relationship	person, actor	friendship, movie cast
neural network	neuron	synapse
protein network	protein	protein-protein interaction
molecule	atom	bond

Graph representation: adjacency matrix

Adjacency matrix. n-by-n matrix with $A_{uv} = 1$ if (u, v) is an edge.

- Two representations of each edge.
- Space proportional to n^2.
- Checking if (u, v) is an edge takes $\Theta(1)$ time.
- Identifying all edges takes $\Theta(n^2)$ time.



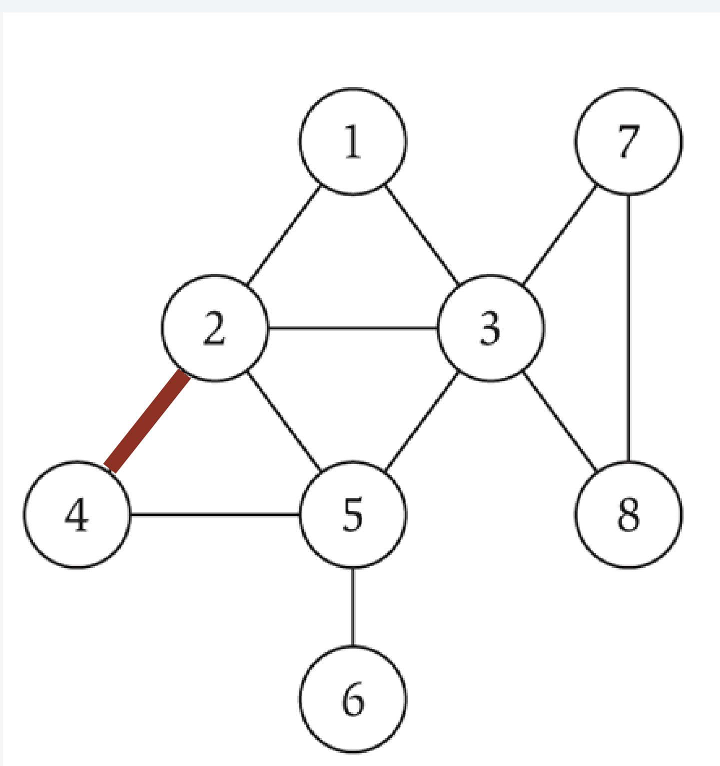
	1	2	3	4	5	6	7	8
1	0	1	1	0	0	0	0	0
2	1	0	1	1	1	0	0	0
3	1	1	0	0	1	0	1	1
4	0	1	0	0	1	0	0	0
5	0	1	1	1	0	1	0	0
6	0	0	0	0	1	0	0	0
7	0	0	1	0	0	0	0	1
8	0	0	1	0	0	0	1	0

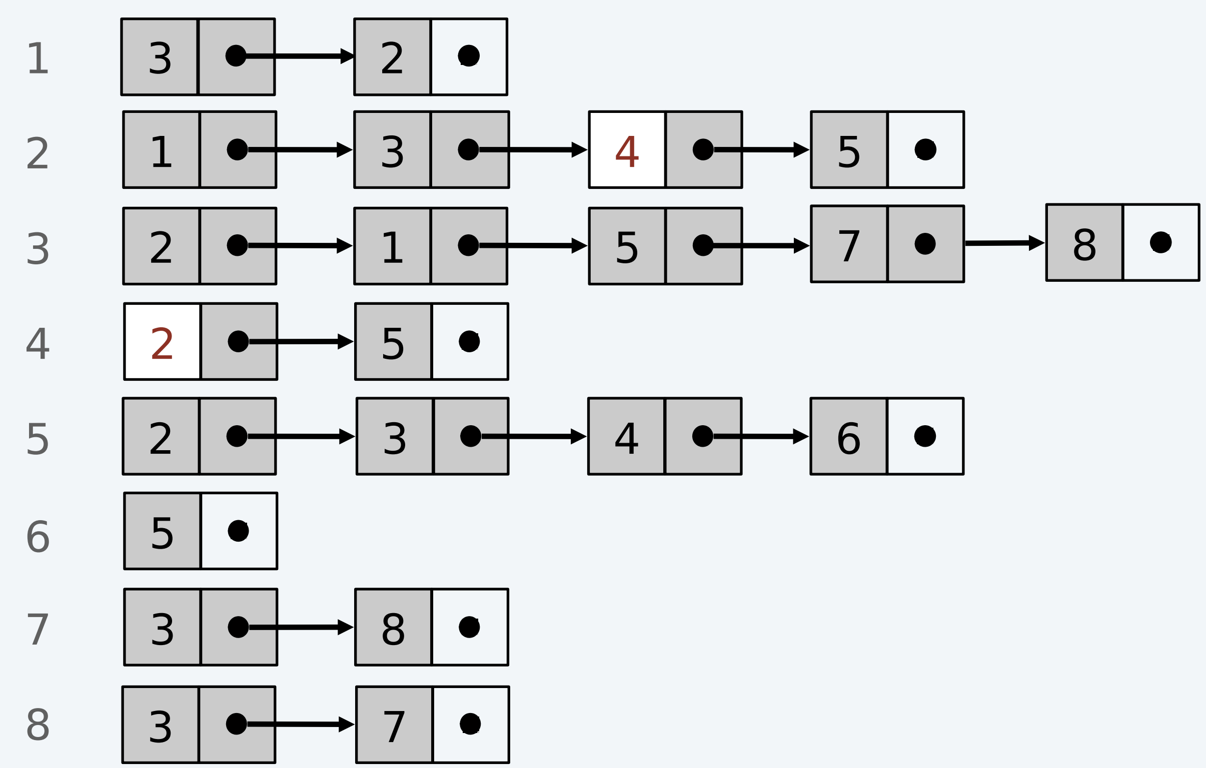
Graph representation: adjacency lists

Adjacency lists. Node-indexed array of lists.

- Two representations of each edge.
- Space is $\Theta(m + n)$.
- Checking if (u, v) is an edge takes $O(degree(u))$ time.
- Identifying all edges takes $\Theta(m + n)$ time.

degree = number of neighbors of u



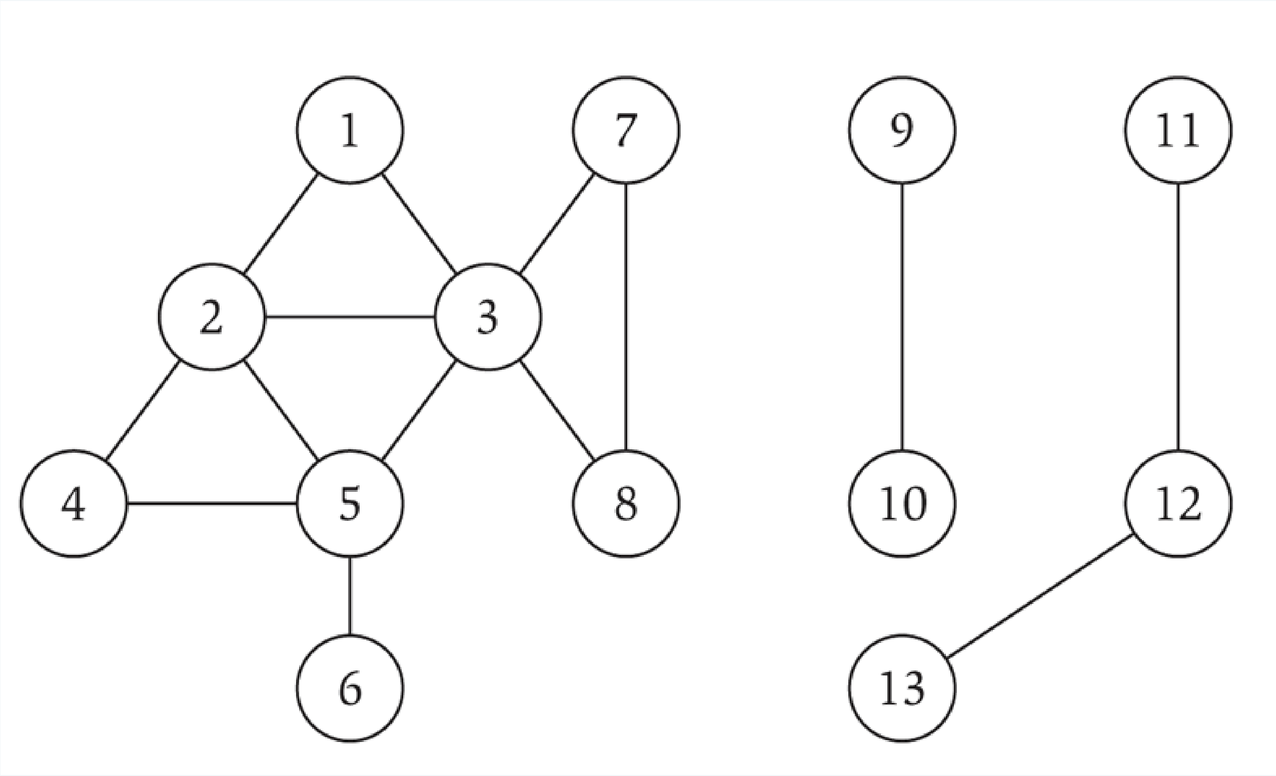


Paths and connectivity

Def. A path in an undirected graph $G = (V, E)$ is a sequence of nodes $v_1, v_2, \ldots, v_k$ with the property that each consecutive pair v_{i-1}, v_i is joined by an edge in E.

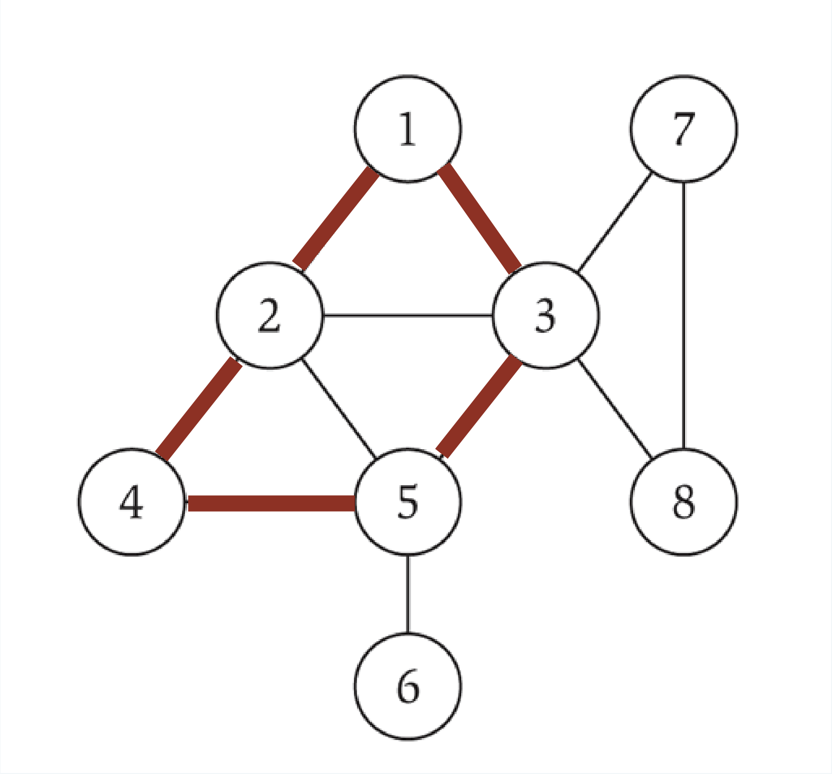
Def. A path is simple if all nodes are distinct.

Def. An undirected graph is connected if for every pair of nodes u and v, there is a path between u and v.



Cycles

Def. A cycle is a path $v_1, v_2, \ldots, v_k$ in which $v_1 = v_k$, $k > 2$, and the first $k - 1$ nodes are all distinct.



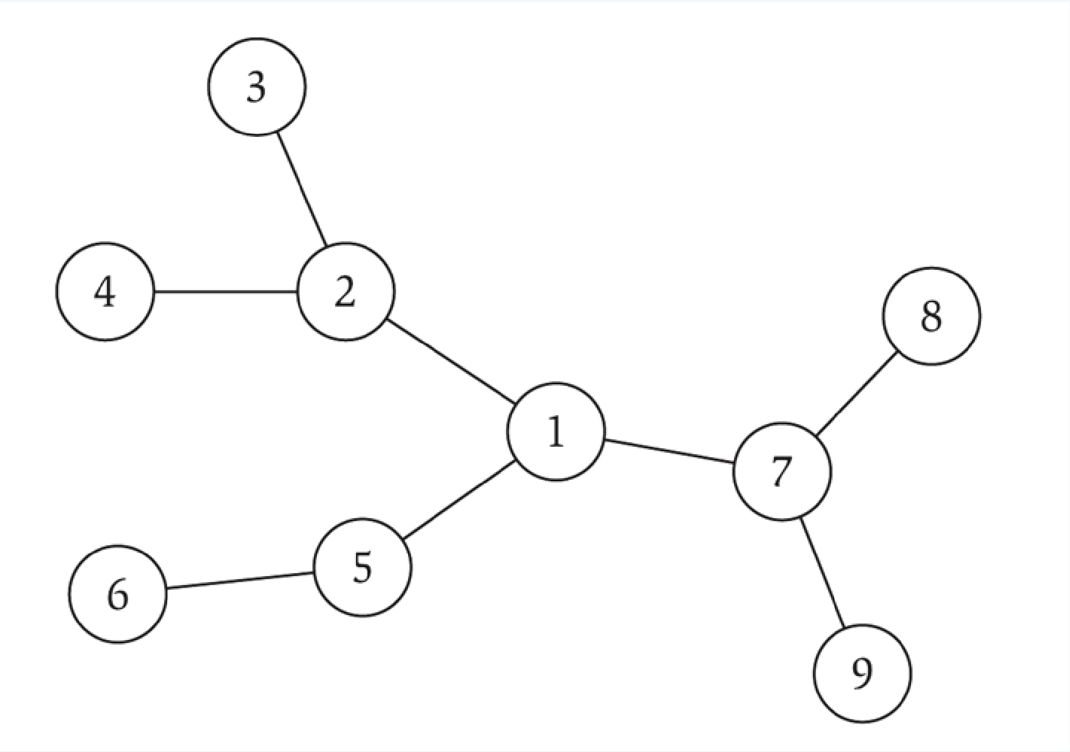
cycle C = 1-2-4-5-3-1

Trees

Def. An undirected graph is a tree if it is connected and does not contain a cycle.

Theorem. Let G be an undirected graph on n nodes. Any two of the following statements imply the third:

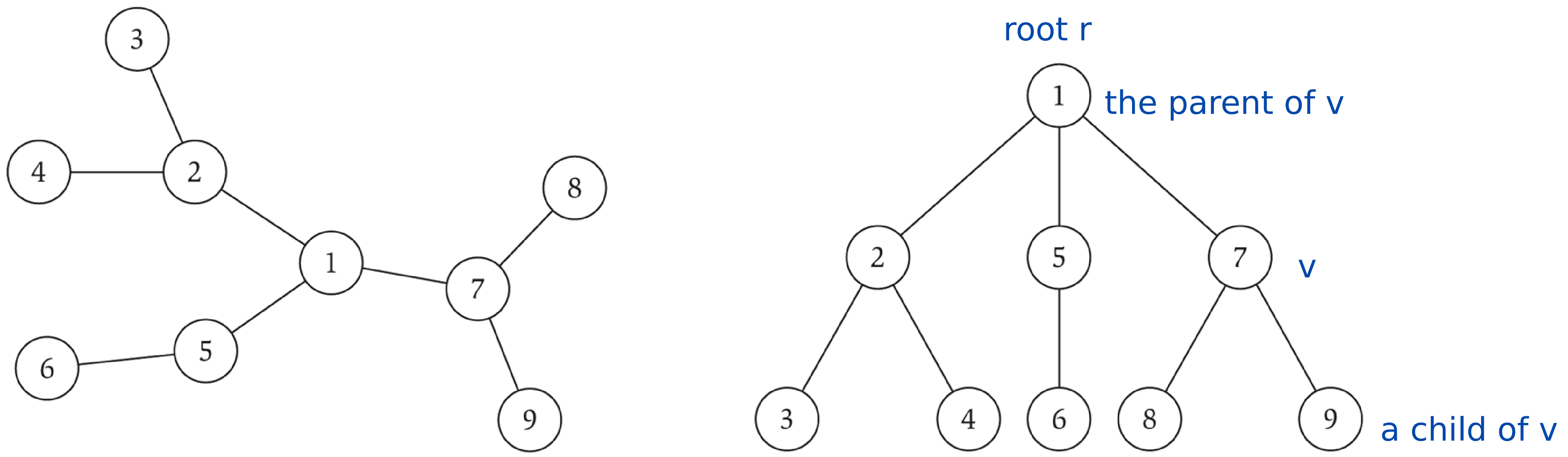
- G is connected.
- G does not contain a cycle.
- G has $n-1$ edges.



Rooted trees

Given a tree T, choose a root node r and orient each edge away from r.

Importance. Models hierarchical structure.

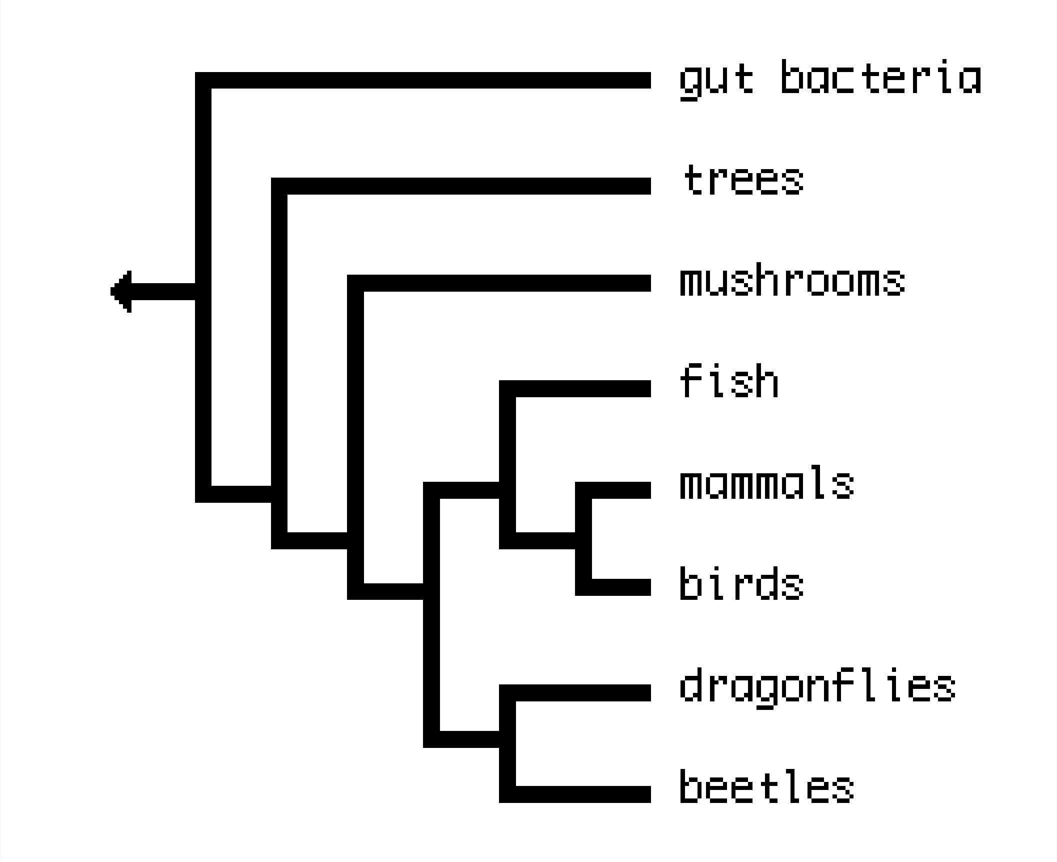


a tree

the same tree, rooted at 1

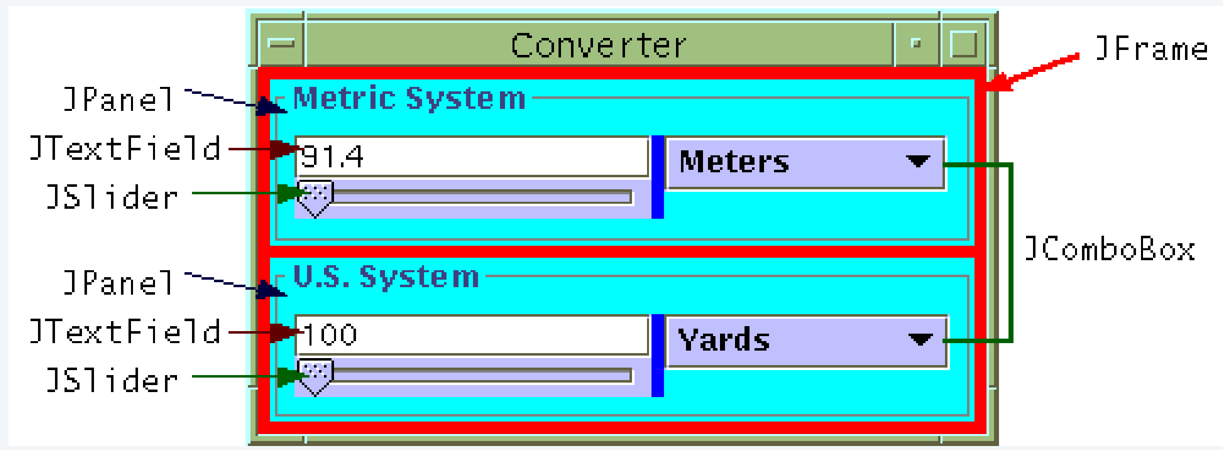
Phylogeny trees

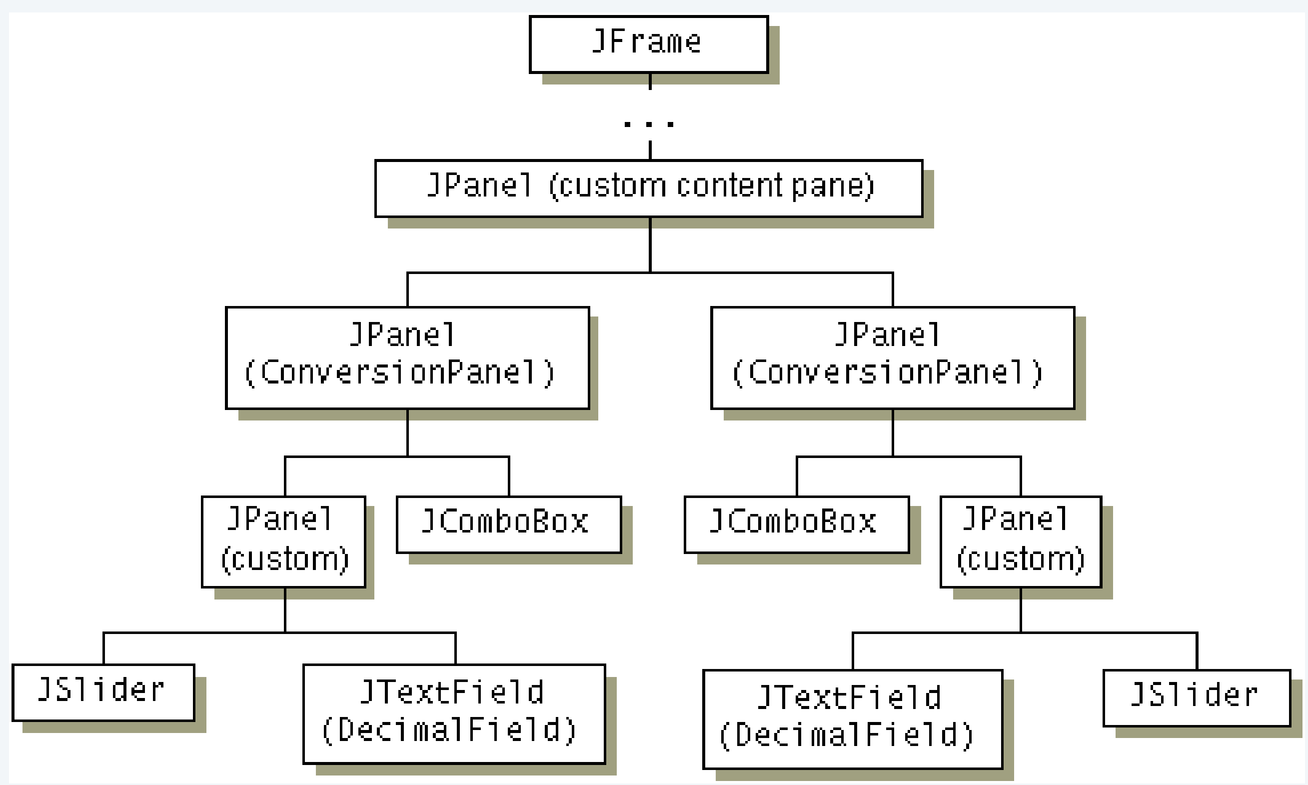
Describe evolutionary history of species.



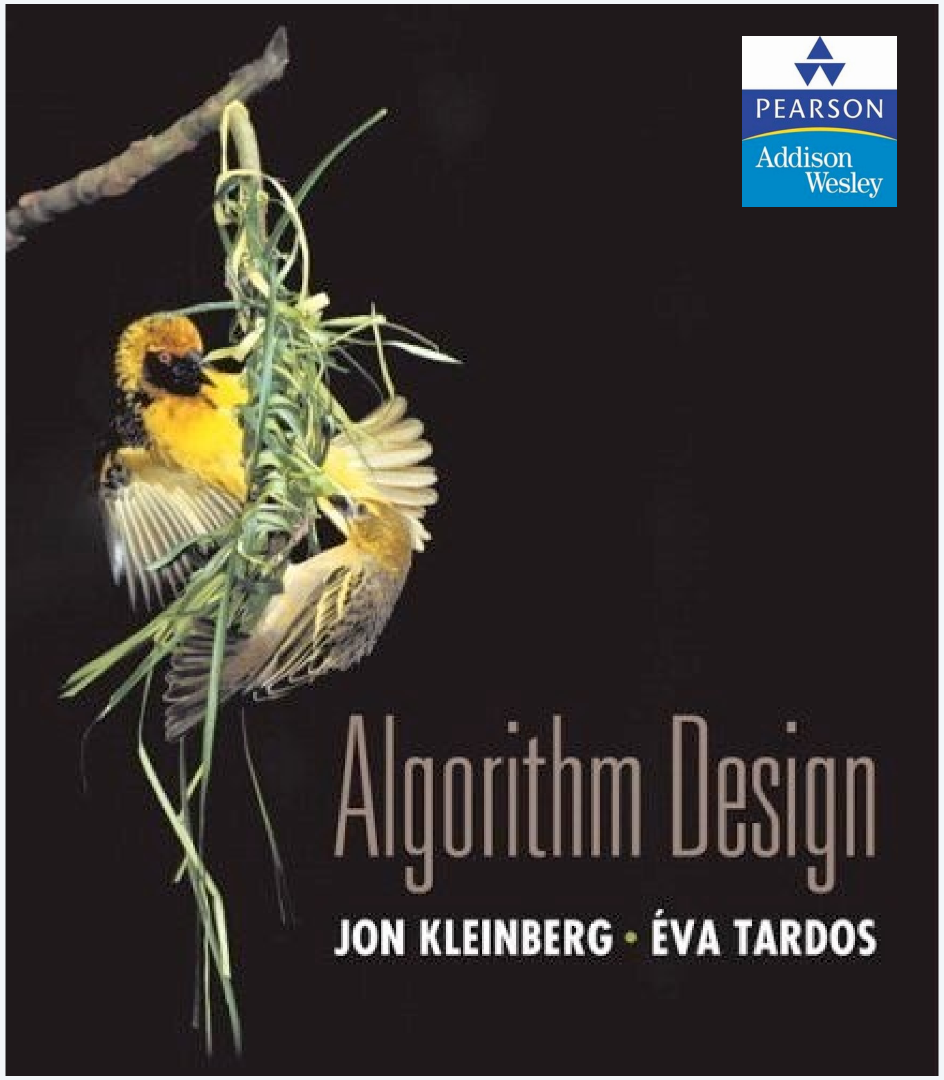
GUI containment hierarchy

Describe organization of GUI widgets.





http://java.sun.com/docs/books/tutorial/uiswing/overview/anatomy.html



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Connectivity

s-t connectivity problem. Given two nodes s and t, is there a path between
s and t ?

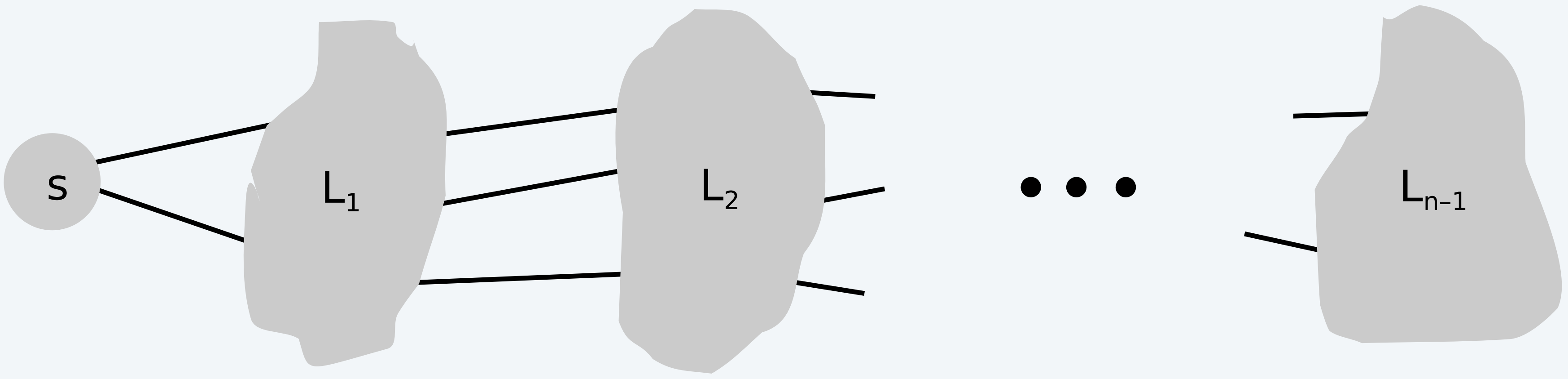
s-t shortest path problem. Given two nodes s and t, what is the length of a shortest path between s and t ?

Applications.

- Friendster.
- Maze traversal.
- Kevin Bacon number.
- Fewest hops in a communication network.

Breadth-first search

BFS intuition. Explore outward from s in all possible directions, adding nodes one "layer" at a time.



BFS algorithm.

- $L_0 = \{ s \}$.
- L_1 = all neighbors of L_0.
- L_2 = all nodes that do not belong to L_0 or L_1, and that have an edge to a node in L_1.
- L_{i+1} = all nodes that do not belong to an earlier layer, and that have an edge to a node in L_i.

Breadth-first search

```
BFS(s):
  Set Discovered[s] = true and Discovered[v] = false for all other v
  Initialize L[0] to consist of the single element s
  Set the layer counter i = 0
  Set the current BFS tree T = ∅
  While L[i] is not empty
    Initialize an empty list L[i + 1]
    For each node u ∈ L[i]
      Consider each edge (u, v) incident to u
      If Discovered[v] = false then
        Set Discovered[v] = true
        Add edge (u, v) to the tree T
        Add v to the list L[i + 1]
      Endif
    Endfor
    Increment the layer counter i by one
  Endwhile
```

Breadth-first search: analysis

Theorem. The above implementation of BFS runs in $O(m + n)$ time if the graph is given by its adjacency representation.

Pf.

- Easy to prove $O(n^2)$ running time:
  - at most n lists $L[i]$
  - each node occurs on at most one list; for loop runs $\leq n$ times
  - when we consider node u, there are $\leq n$ incident edges (u, v), and we spend $O(1)$ processing each edge

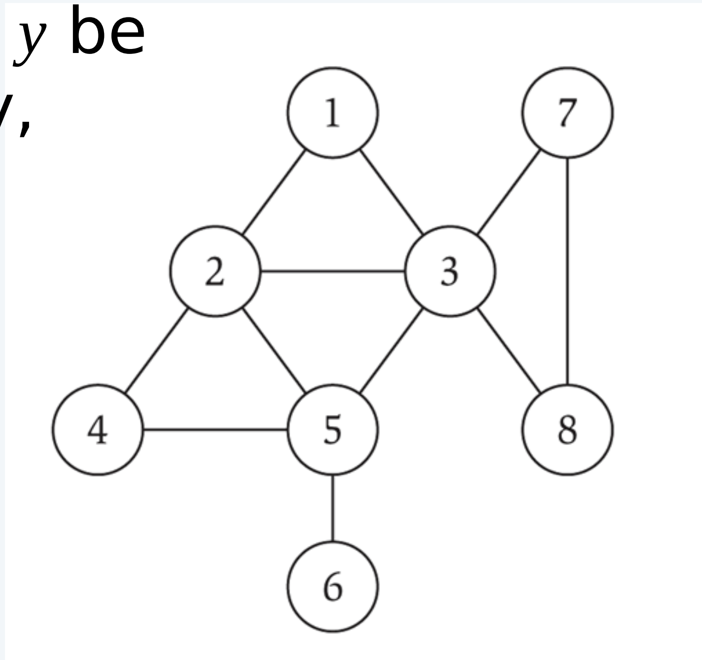
- Actually runs in $O(m + n)$ time:
  - when we consider node u, there are $degree(u)$ incident edges (u, v)
  - total time processing edges is $\Sigma_{u \in V}\ degree(u) = 2m$. ■

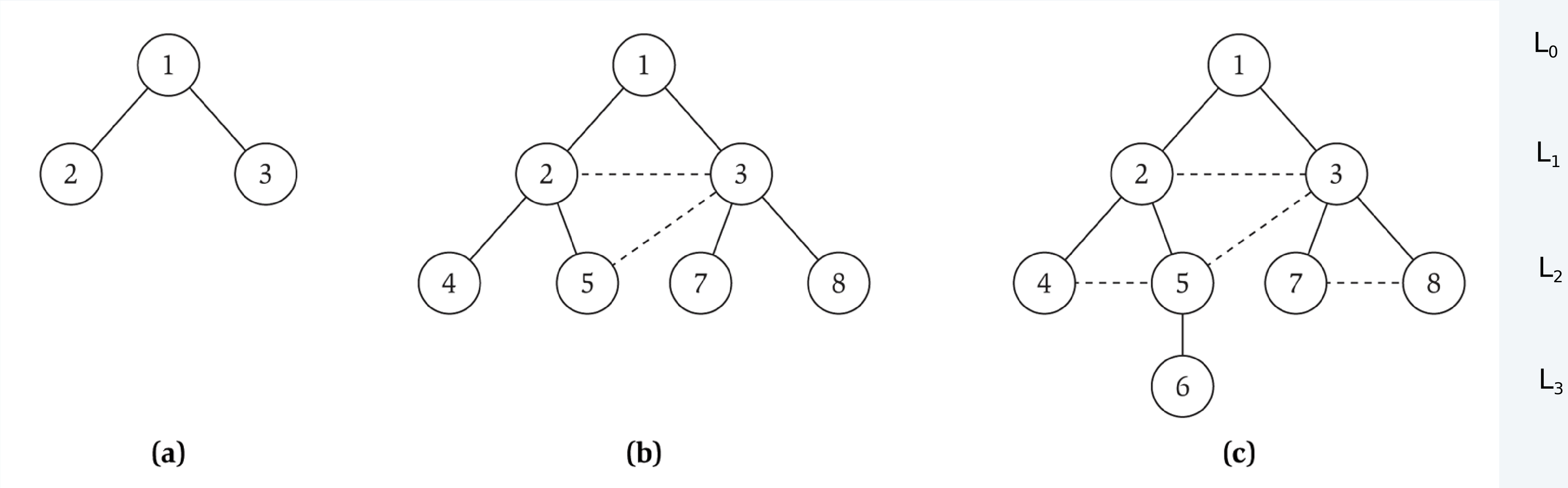
each edge (u, v) is counted exactly twice
in sum: once in degree(u) and once in degree(v)

Breadth-first search

Theorem. For each i, L_i consists of all nodes at distance exactly i from s. There is a path from s to t iff t appears in some layer.

Property. Let T be a BFS tree of $G = (V, E)$, let x and y be nodes in T belonging to layers L_i and L_j respectively, and let (x, y) be an edge of G.
Then, the levels of x and y differ by at most 1.





Depth-First Search

You might take if the graph G were truly a maze of interconnected rooms and you were walking around in it.

```
DFS(u):
  Mark u as "Explored" and add u to R
  For each edge (u, v) incident to u
    If v is not marked "Explored" then
      Recursively invoke DFS(v)
    Endif
  Endfor
```

Depth-First Search

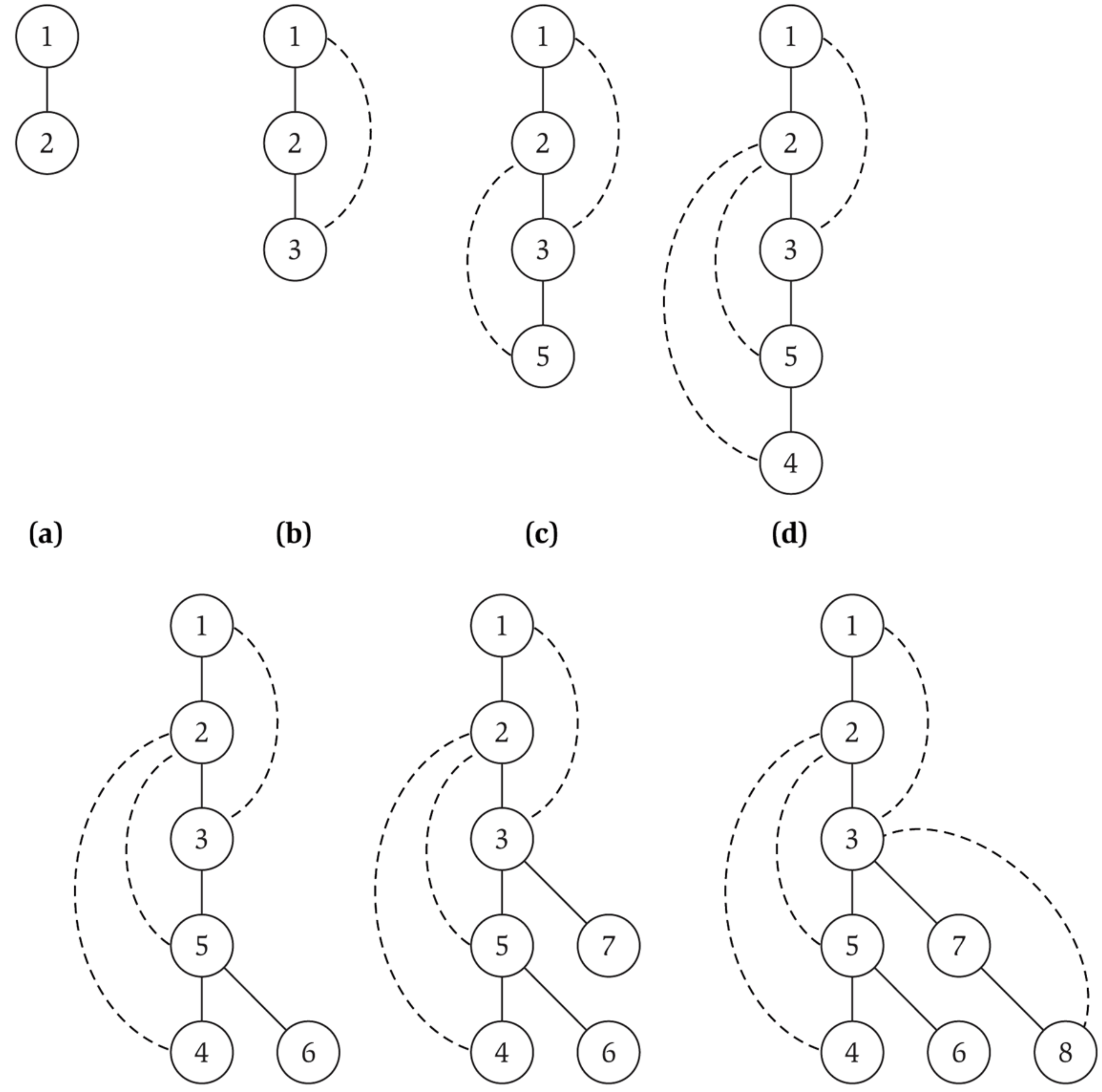


Figure 3.5 The construction of a depth-first search tree T for the graph in Figure 3.2, with (a) through (g) depicting the nodes as they are discovered in sequence. The solid edges are the edges of T; the dotted edges are edges of G that do not belong to T.

Implementing Depth-First Search

```
DFS(s):
  Initialize S to be a stack with one element s
  While S is not empty
    Take a node u from S
    If Explored[u] = false then
       Set Explored[u] = true
       For each edge (u, v) incident to u
         Add v to the stack S
       Endfor
    Endif
  Endwhile
```

- implements DFS, in the sense that it visits the nodes in exactly the same order as the recursive DFS procedure in the previous section (except that each adjacency list is processed in reverse order).
- runs in time O(m + n), if the graph is given by the adjacency list representation.

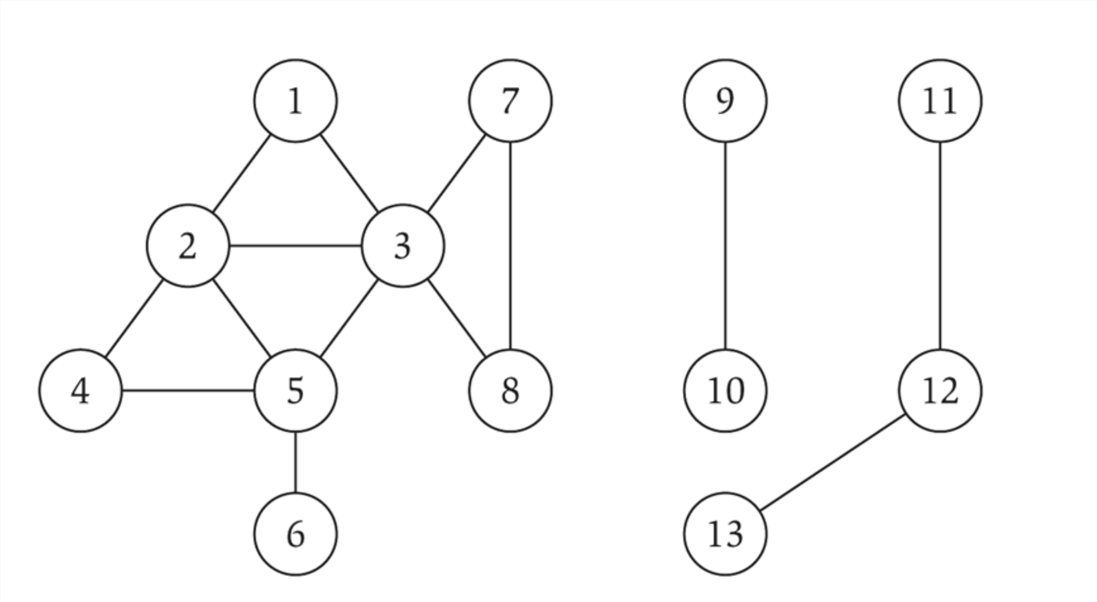
Depth-First Search

Property. For a given recursive call *DFS*(u), all nodes that are marked "Explored" between the invocation and end of this recursive call are descendants of u in T.

Property. Let T be a depth-first search tree, let x and y be nodes in T, and let (x, y) be an edge of G that is not an edge of T. Then one of x or y is an ancestor of the other.

Connected component

Connected component. Find all nodes reachable from s.



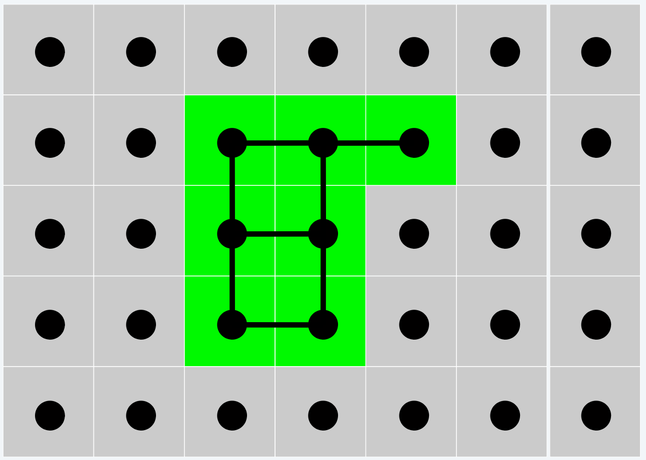
Connected component containing node $1 = \{ 1, 2, 3, 4, 5, 6, 7, 8 \}$.

Flood fill

Flood fill. Given lime green pixel in an image, change color of entire blob of neighboring lime pixels to blue.

- Node: pixel.
- Edge: two neighboring lime pixels.
- Blob: connected component of lime pixels.

recolor lime green blob to blue

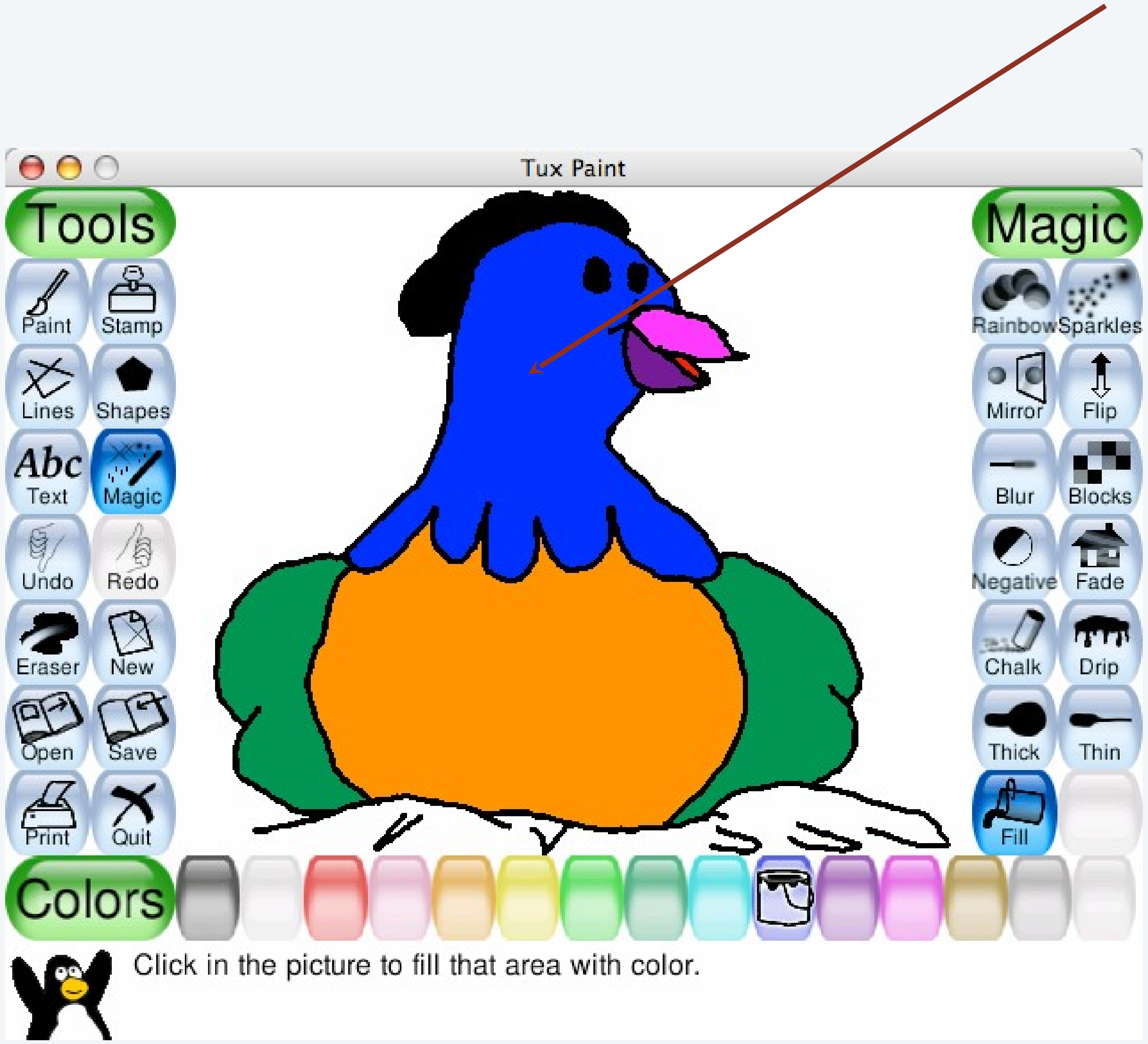


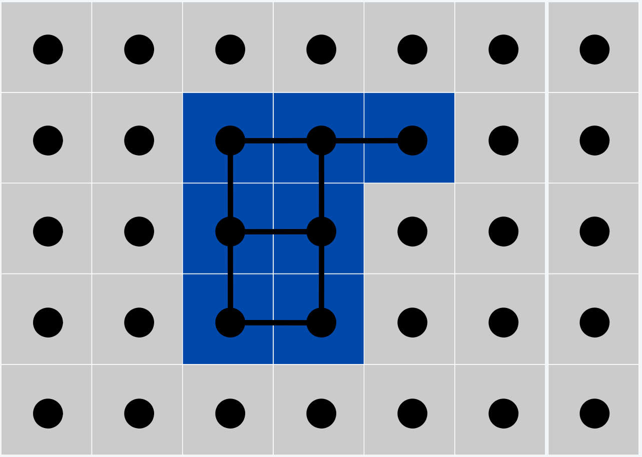
Flood fill

Flood fill. Given lime green pixel in an image, change color of entire blob of neighboring lime pixels to blue.

- Node: pixel.
- Edge: two neighboring lime pixels.
- Blob: connected component of lime pixels.

recolor lime green blob to blue

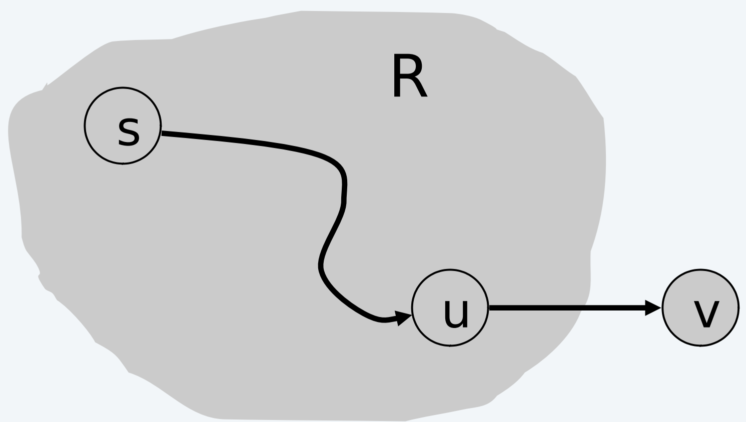




Connected component

Connected component. Find all nodes reachable from s.

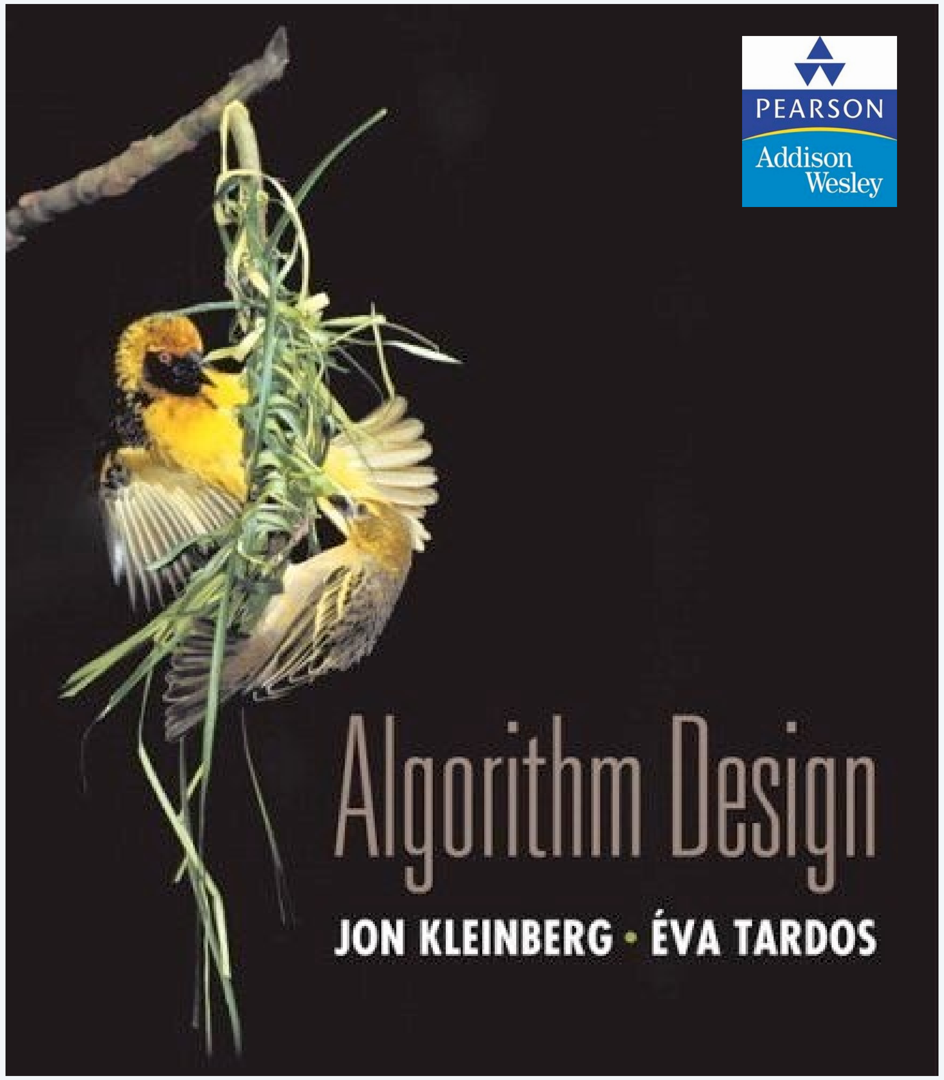
```
R will consist of nodes to which s has a path
Initially R = {s}
While there is an edge (u, v) where u ∈ R and v ∉ R
  Add v to R
Endwhile
```



it's safe to add v

Theorem. Upon termination, R is the connected component containing s.

Theorem. For any two nodes s and t in a graph, their connected components are either identical o disjoint.



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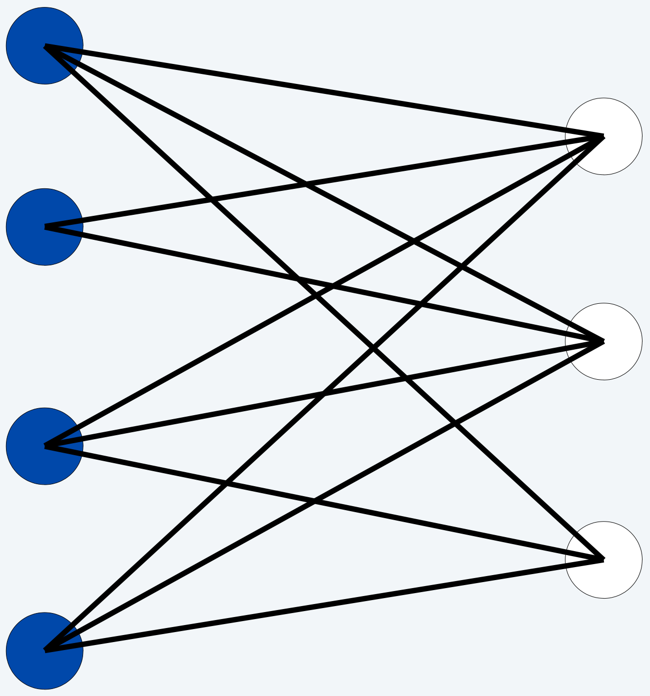
Bipartite graphs

Def. An undirected graph $G = (V, E)$ is bipartite if the nodes can be colored blue or white such that every edge has one white and one blue end.

Applications.

- Stable matching: med-school residents = blue, hospitals = white.
- Scheduling: machines = blue, jobs = white.



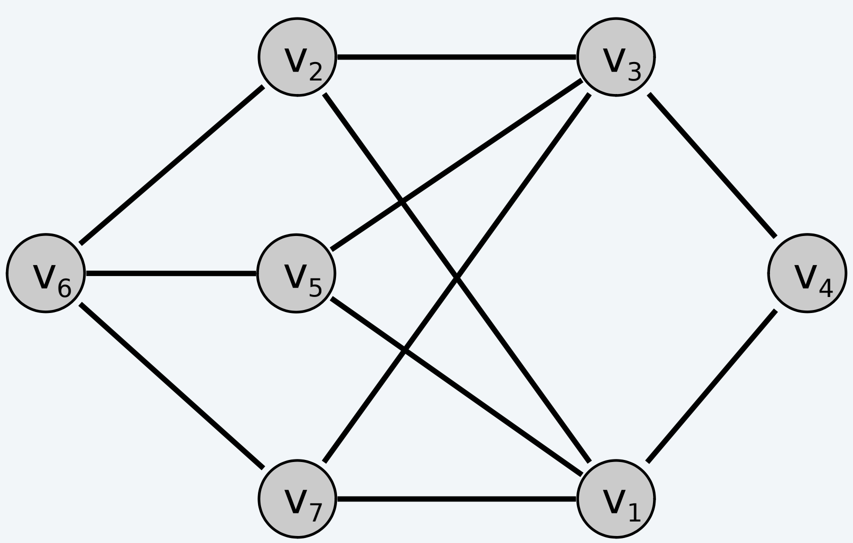
a bipartite graph

Testing bipartiteness

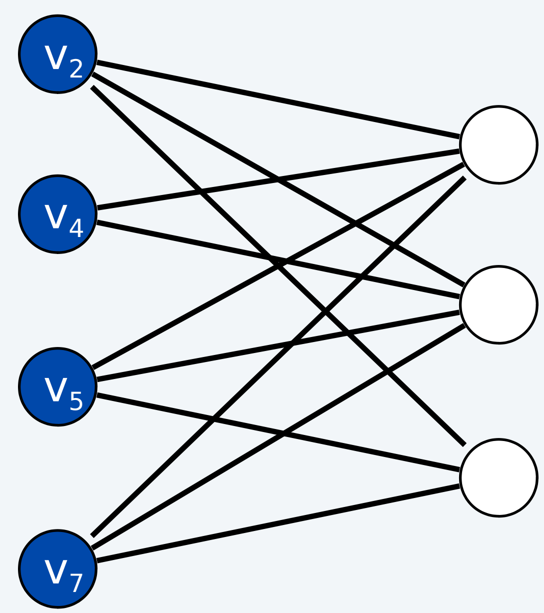
Many graph problems become:

- Easier if the underlying graph is bipartite (matching).
- Tractable if the underlying graph is bipartite (independent set).

Before attempting to design an algorithm, we need to understand structure of bipartite graphs.



a bipartite graph G

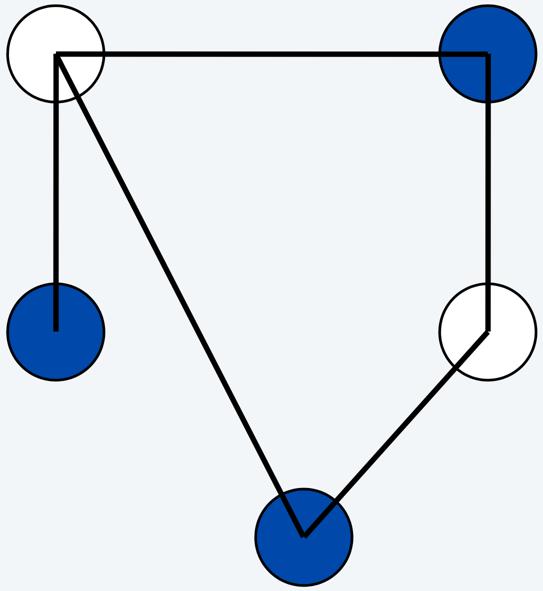


another drawing of G

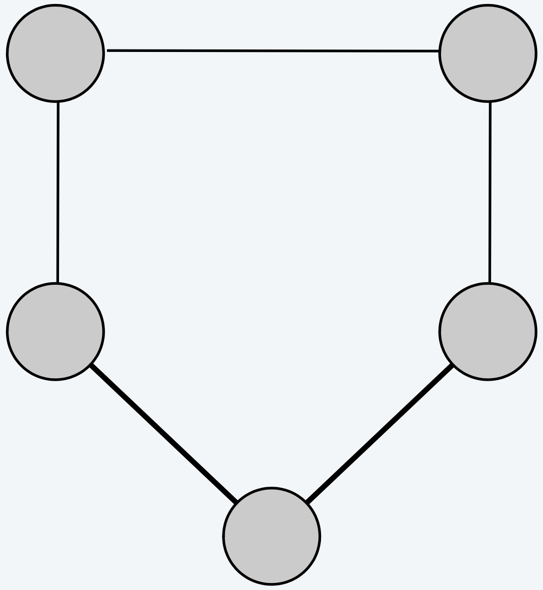
An obstruction to bipartiteness

Lemma. If a graph G is bipartite, it cannot contain an odd-length cycle.

Pf. Not possible to 2-color the odd-length cycle, let alone G.



bipartite
(2-colorable)



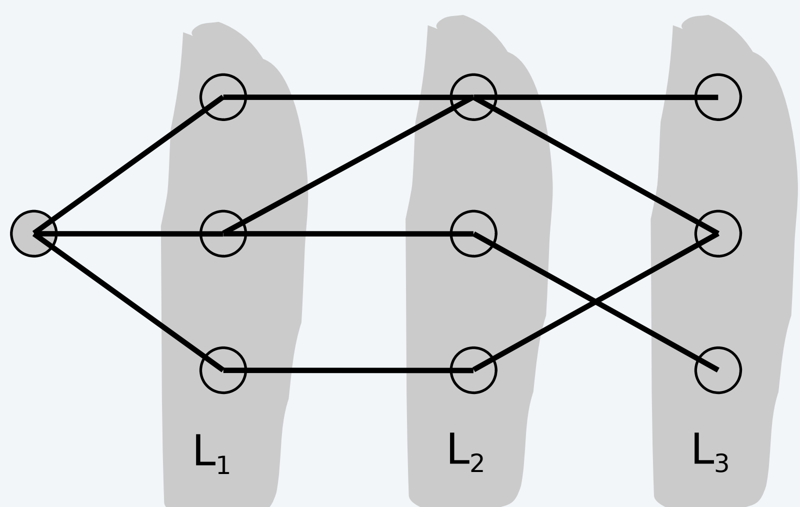
not bipartite
(not 2-colorable)

Bipartite graphs

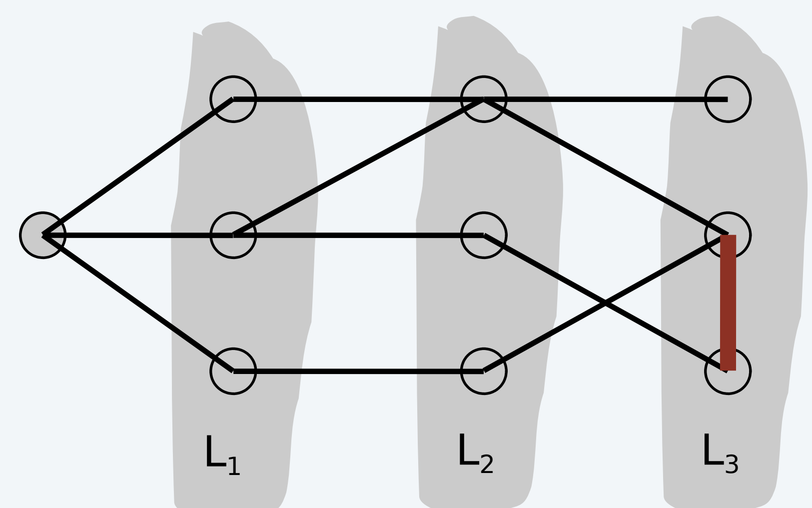
Lemma. Let G be a connected graph, and let $L_0, \ldots, L_k$ be the layers produced by BFS starting at node s. Exactly one of the following holds.

(i) No edge of G joins two nodes of the same layer, and G is bipartite.

(ii) An edge of G joins two nodes of the same layer, and G contains an odd-length cycle (and hence is not bipartite).



Case (i)

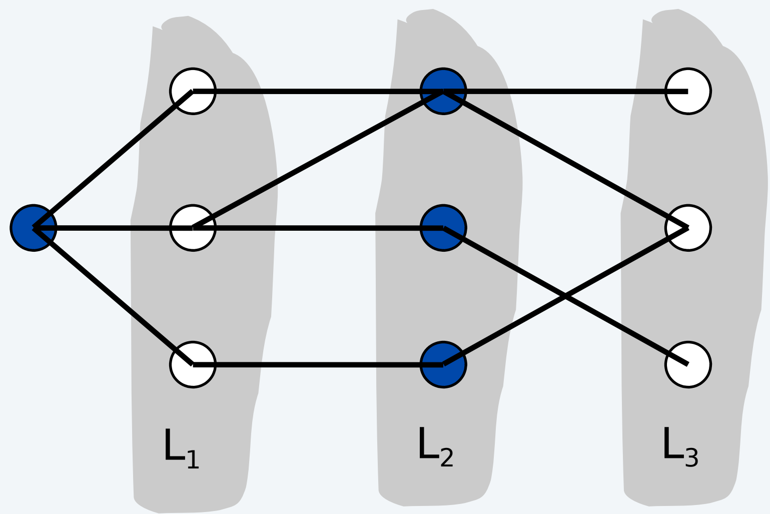


Case (ii)

Bipartite graphs

Lemma. Let G be a connected graph, and let $L_0, \ldots, L_k$ be the layers produced by BFS starting at node s. Exactly one of the following holds.
(i) No edge of G joins two nodes of the same layer, and G is bipartite.
(ii) An edge of G joins two nodes of the same layer, and G contains an odd-length cycle (and hence is not bipartite).

Pf. (i)
- Suppose no edge joins two nodes in same layer.
- By BFS property, each edge joins two nodes in adjacent levels.
- Bipartition: white = nodes on odd levels, blue = nodes on even levels.



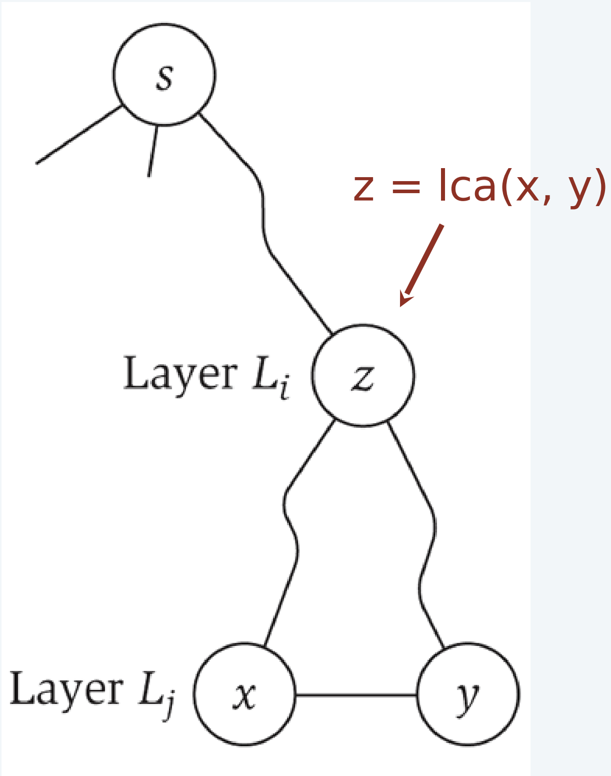
Case (i)

Bipartite graphs

Lemma. Let G be a connected graph, and let $L_0, \ldots, L_k$ be the layers produced by BFS starting at node s. Exactly one of the following holds.
(i) No edge of G joins two nodes of the same layer, and G is bipartite.
(ii) An edge of G joins two nodes of the same layer, and G contains an odd-length cycle (and hence is not bipartite).

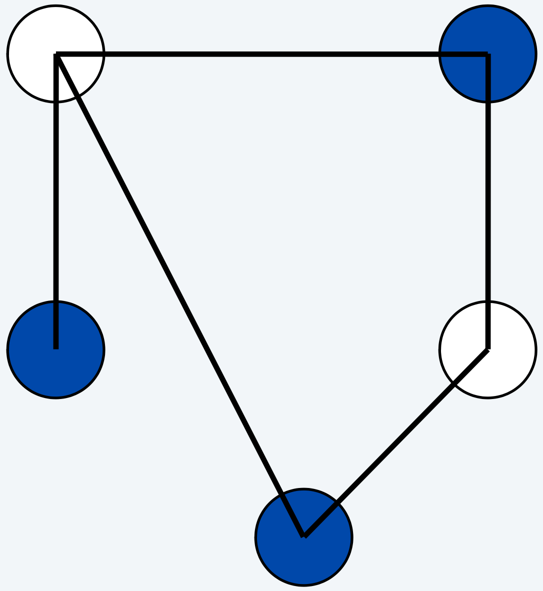
Pf. (ii)

- Suppose (x, y) is an edge with x, y in same level L_j.
- Let $z = lca(x, y)$ = lowest common ancestor.
- Let L_i be level containing z.
- Consider cycle that takes edge from x to y, then path from y to z, then path from z to x.
- Its length is $\underbrace{1}_{(x,\,y)} + \underbrace{(j-i)}_{\text{path from } y \text{ to } z} + \underbrace{(j-i)}_{\text{path from } z \text{ to } x}$, which is odd. ■

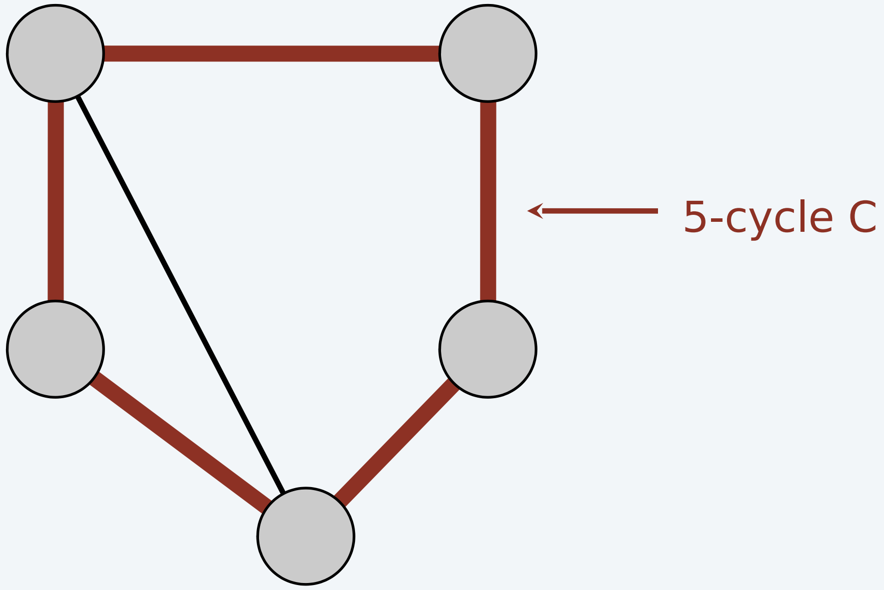


The only obstruction to bipartiteness

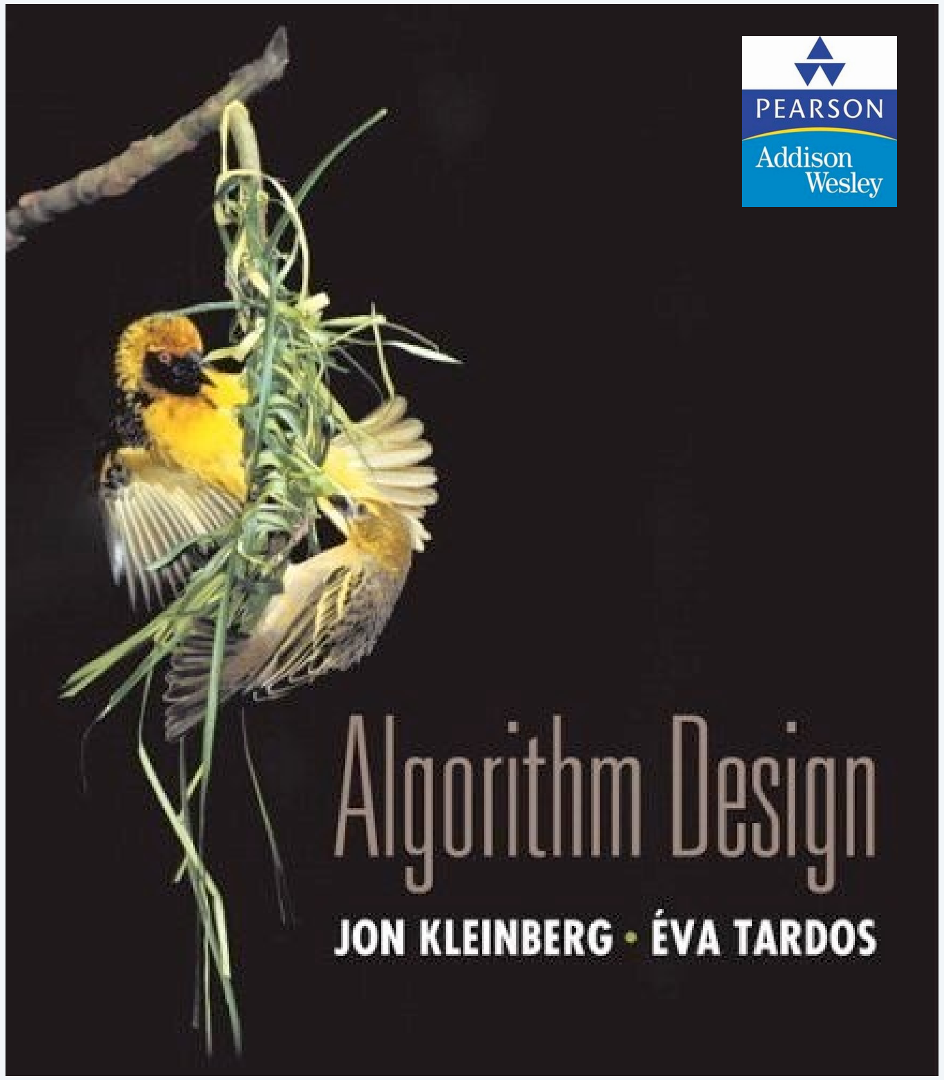
Corollary. A graph G is bipartite iff it contains no odd-length cycle.



bipartite (2-colorable)



not bipartite (not 2-colorable)



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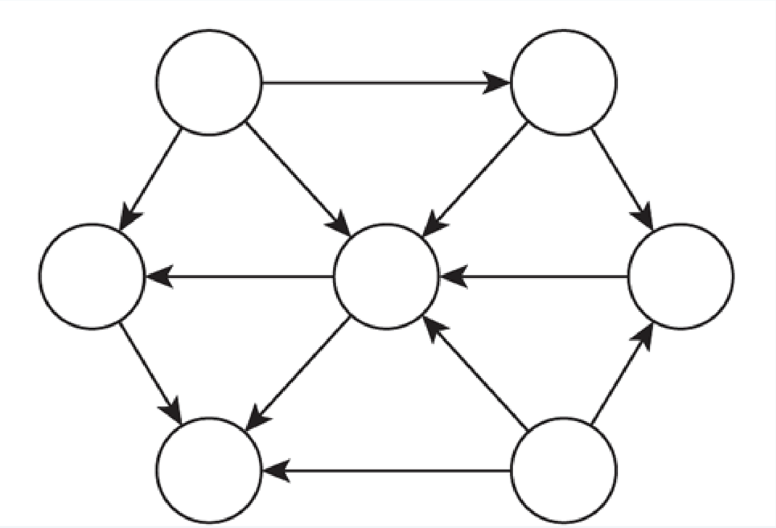
Directed graphs

Notation. $G = (V, E)$.

- Edge (u, v) leaves node u and enters node v.



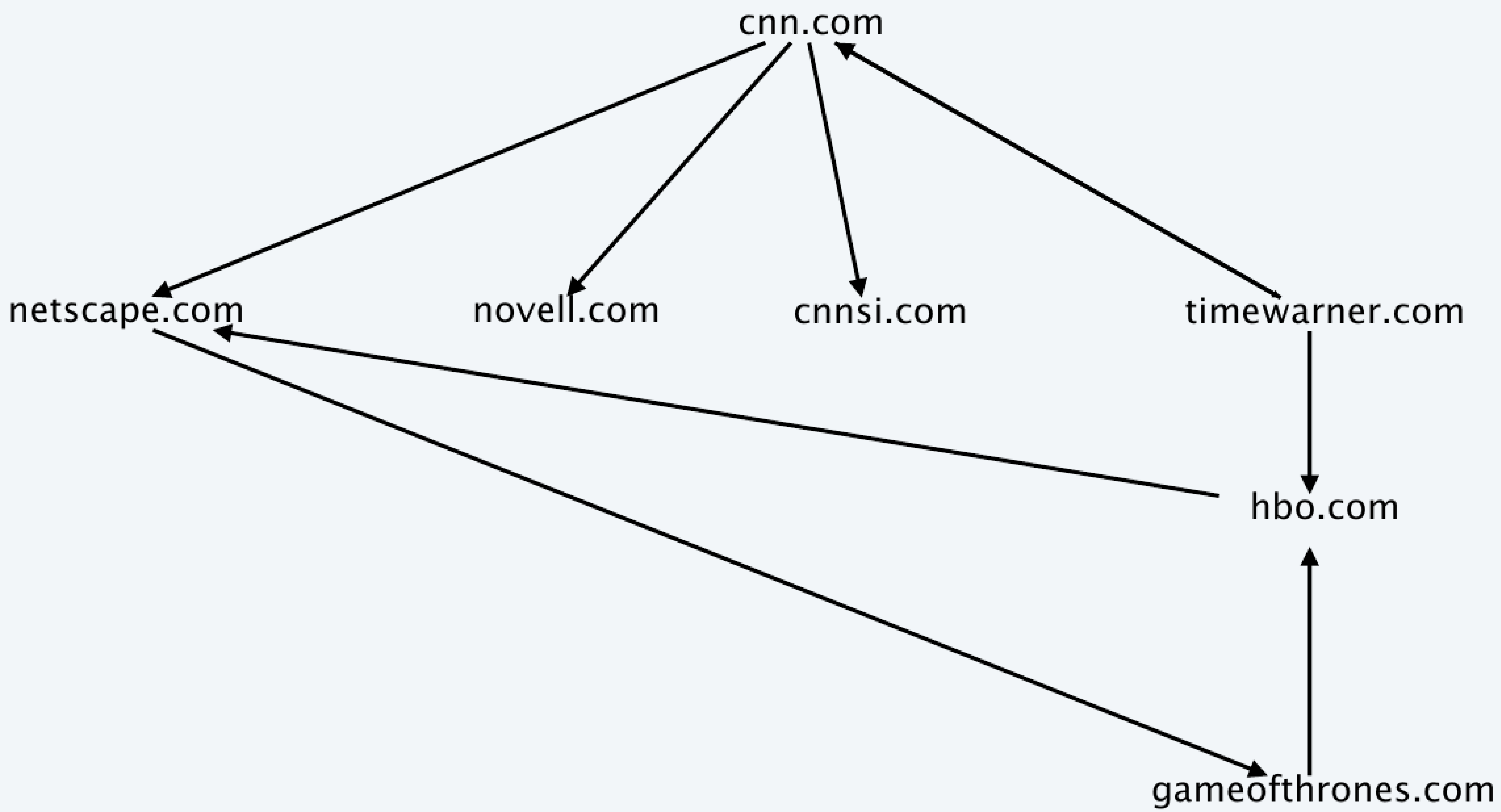
Ex. Web graph: hyperlink points from one web page to another.

- Orientation of edges is crucial.
- Modern web search engines exploit hyperlink structure to rank web pages by importance.

World wide web

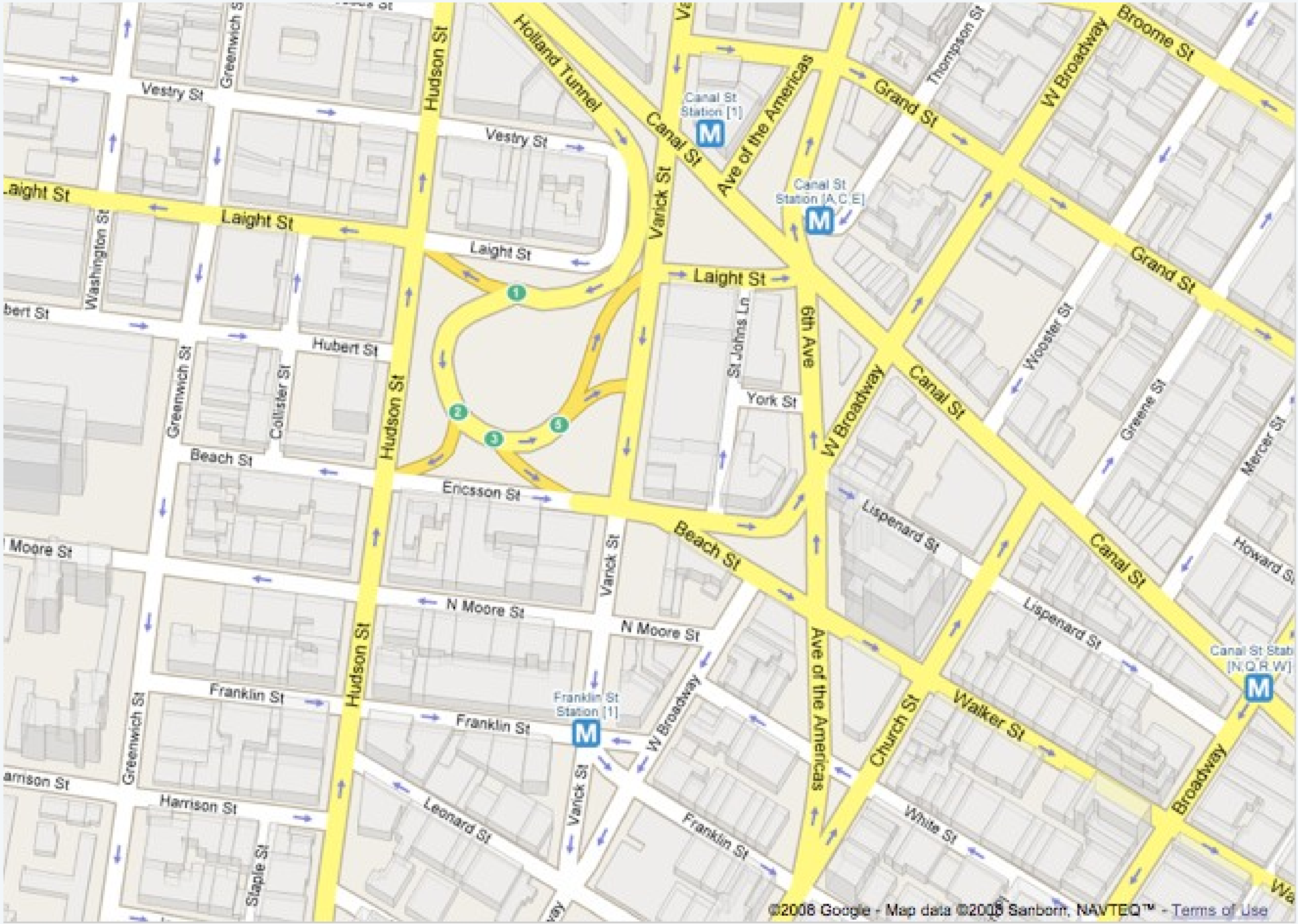
Web graph.

- Node: web page.
- Edge: hyperlink from one page to another (orientation is crucial).
- Modern search engines exploit hyperlink structure to rank web pages



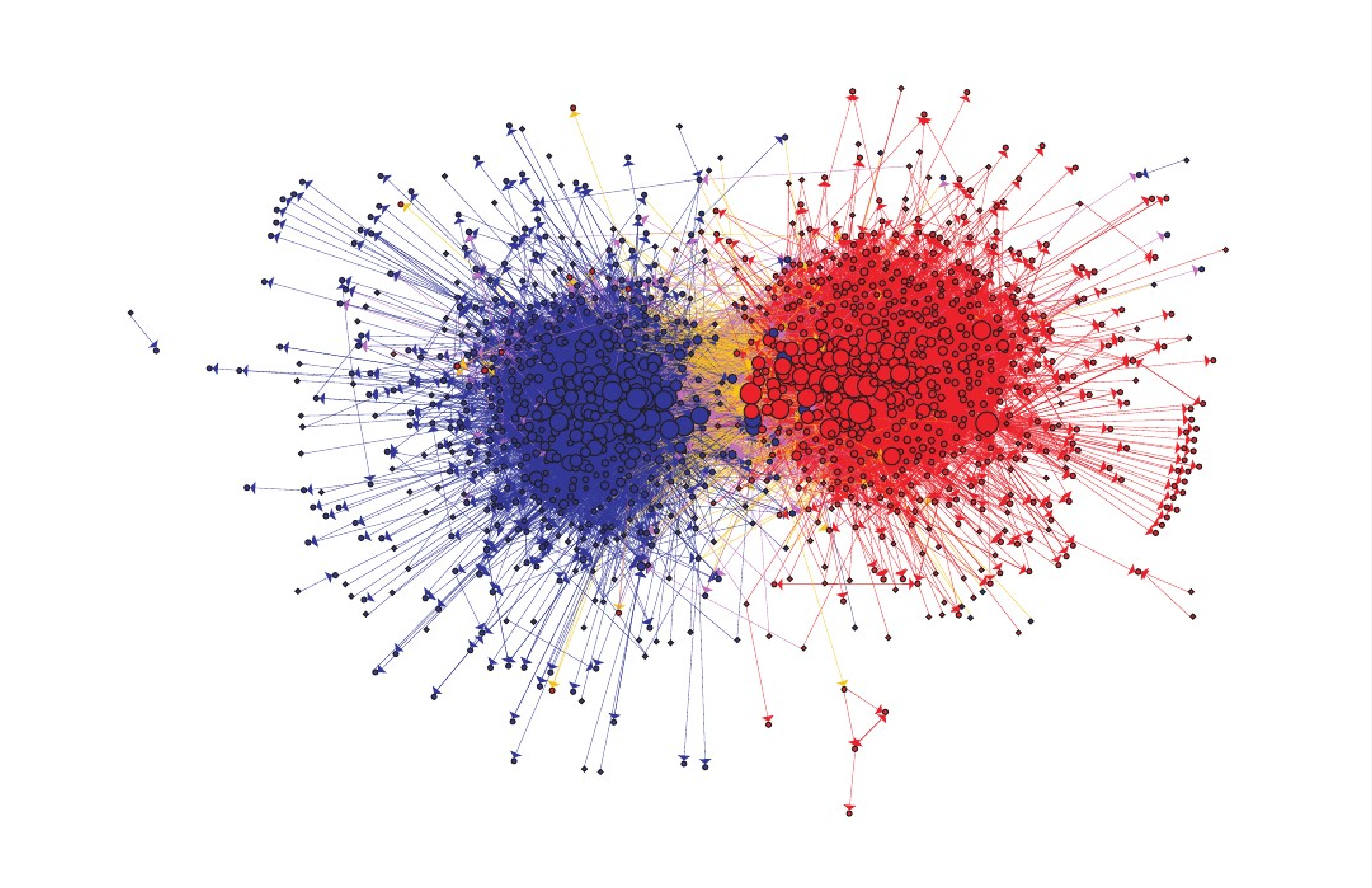
Road network

Node = intersection; edge = one-way street.



Political blogosphere graph

Node = political blog; edge = link.

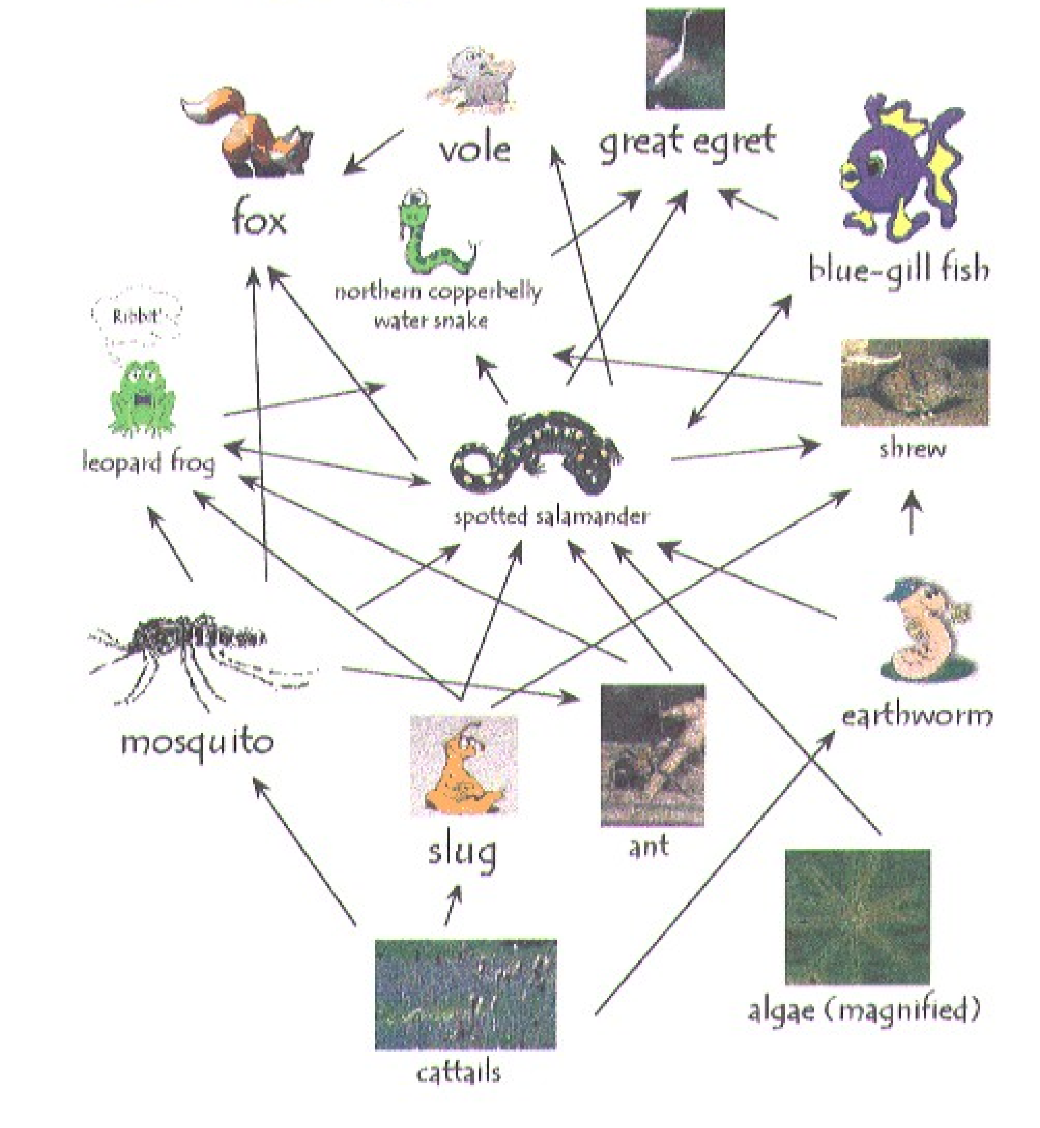


The Political Blogosphere and the 2004 U.S. Election: Divided They Blog, Adamic and Glance, 2005

Ecological food web

Food web graph.

- Node = species.
- Edge = from prey to predator.



Reference: http://www.twingroves.district96.k12.il.us/Wetlands/Salamander/SalGraphics/salfoodweb.giff

Some directed graph applications

directed graph	node	directed edge
transportation	street intersection	one-way street
web	web page	hyperlink
food web	species	predator-prey relationship
WordNet	synset	hypernym
scheduling	task	precedence constraint
financial	bank	transaction
cell phone	person	placed call
infectious disease	person	infection
game	board position	legal move
citation	journal article	citation
object graph	object	pointer
inheritance hierarchy	class	inherits from
control flow	code block	jump

Graph search

Directed reachability. Given a node s, find all nodes reachable from s.

Directed s–t shortest path problem. Given two nodes s and t, what is the length of a shortest path from s to t ?

Graph search. BFS extends naturally to directed graphs.

Web crawler. Start from web page s. Find all web pages linked from s, either directly or indirectly.

Strong connectivity

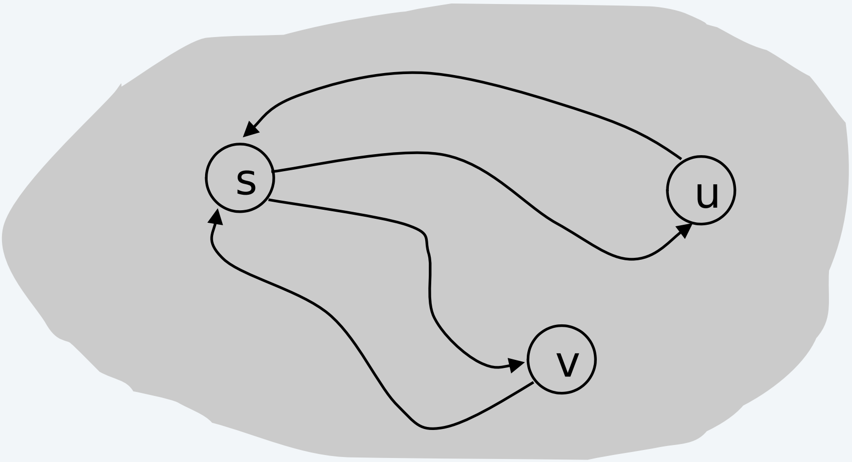
Def. Nodes u and v are **mutually reachable** if there is both a path from u to v and also a path from v to u.

Def. A graph is **strongly connected** if every pair of nodes is mutually reachable.

Lemma. Let s be any node. G is strongly connected iff every node is reachable from s, and s is reachable from every node.

Pf. $\Rightarrow$ Follows from definition.
Pf. $\Leftarrow$ Path from u to v: concatenate $u \leadsto s$ path with $s \leadsto v$ path.
Path from v to u: concatenate $v \leadsto s$ path with $s \leadsto u$ path. ▪

ok if paths overlap

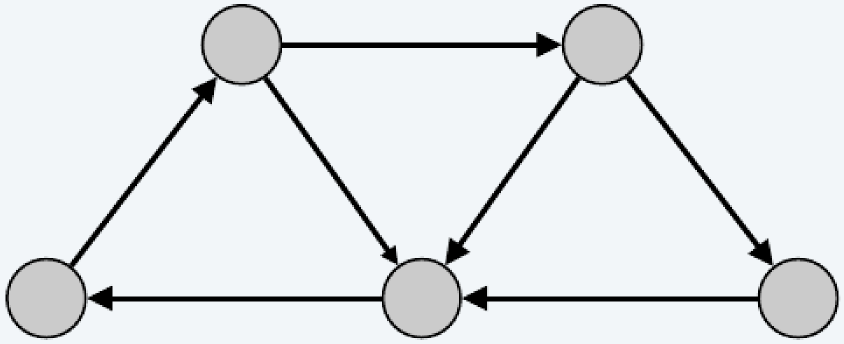


Strong connectivity: algorithm

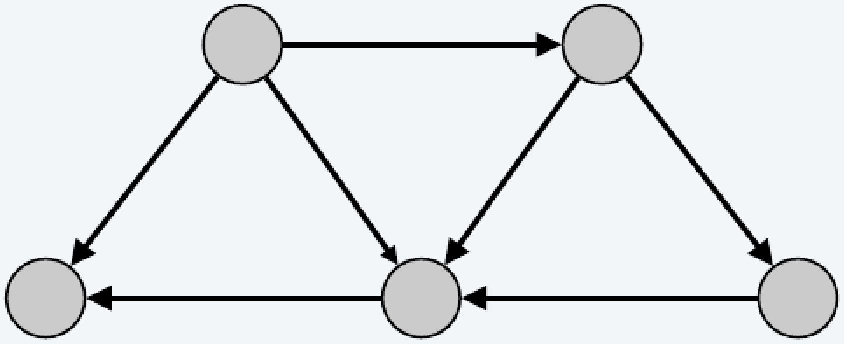
Theorem. Can determine if G is strongly connected in $O(m + n)$ time.

Pf.

- Pick any node s.
- Run BFS from s in G.
- Run BFS from s in $G^{reverse}$. (reverse orientation of every edge in G)
- Return true iff all nodes reached in both BFS executions.
- Correctness follows immediately from previous lemma. ▪

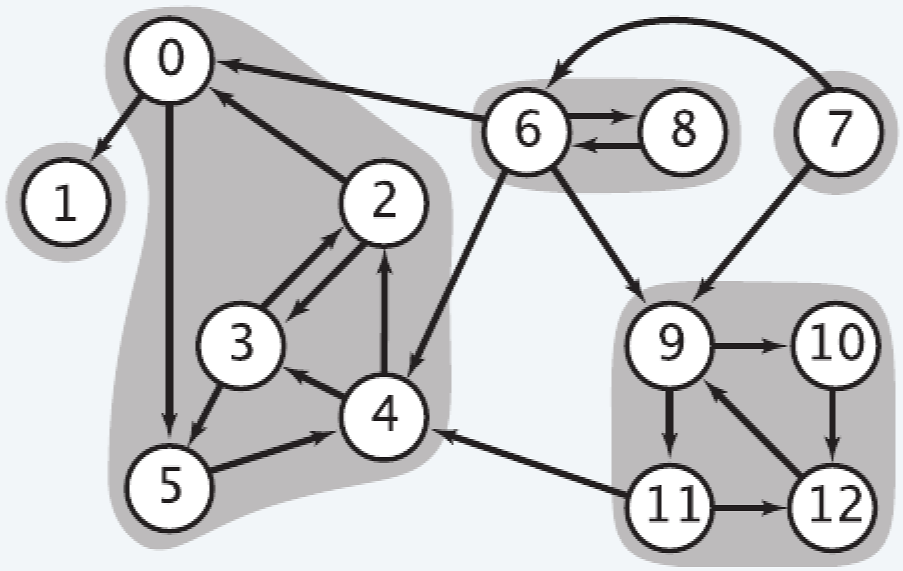


strongly connected

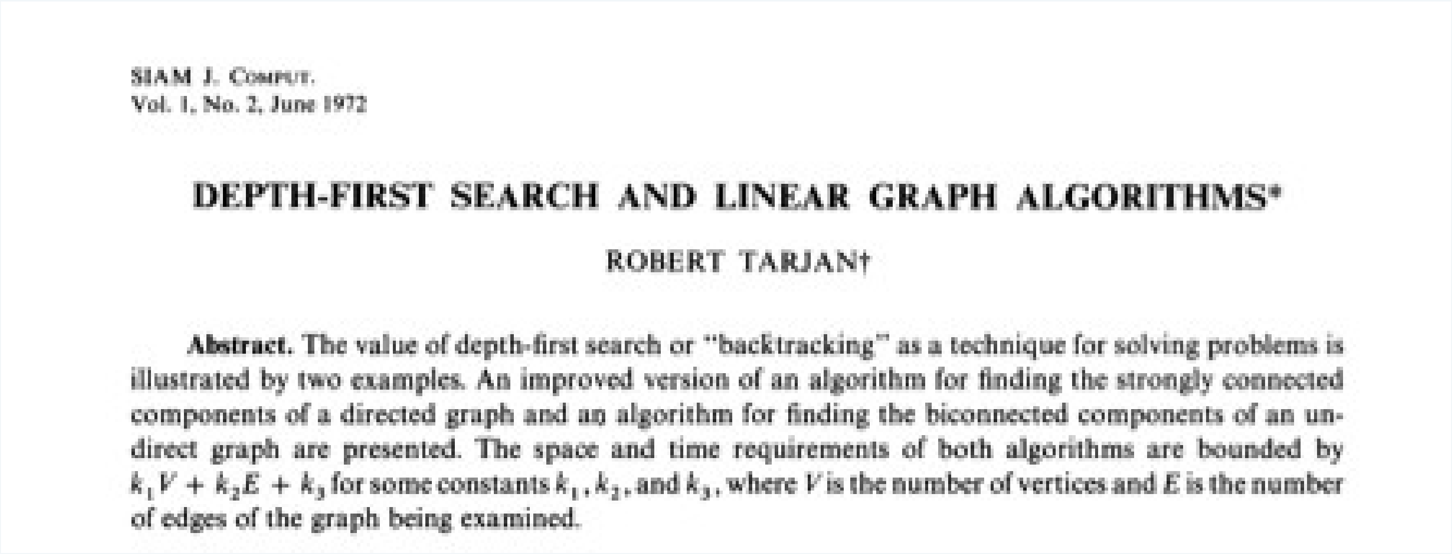


not strongly connected

Strong components



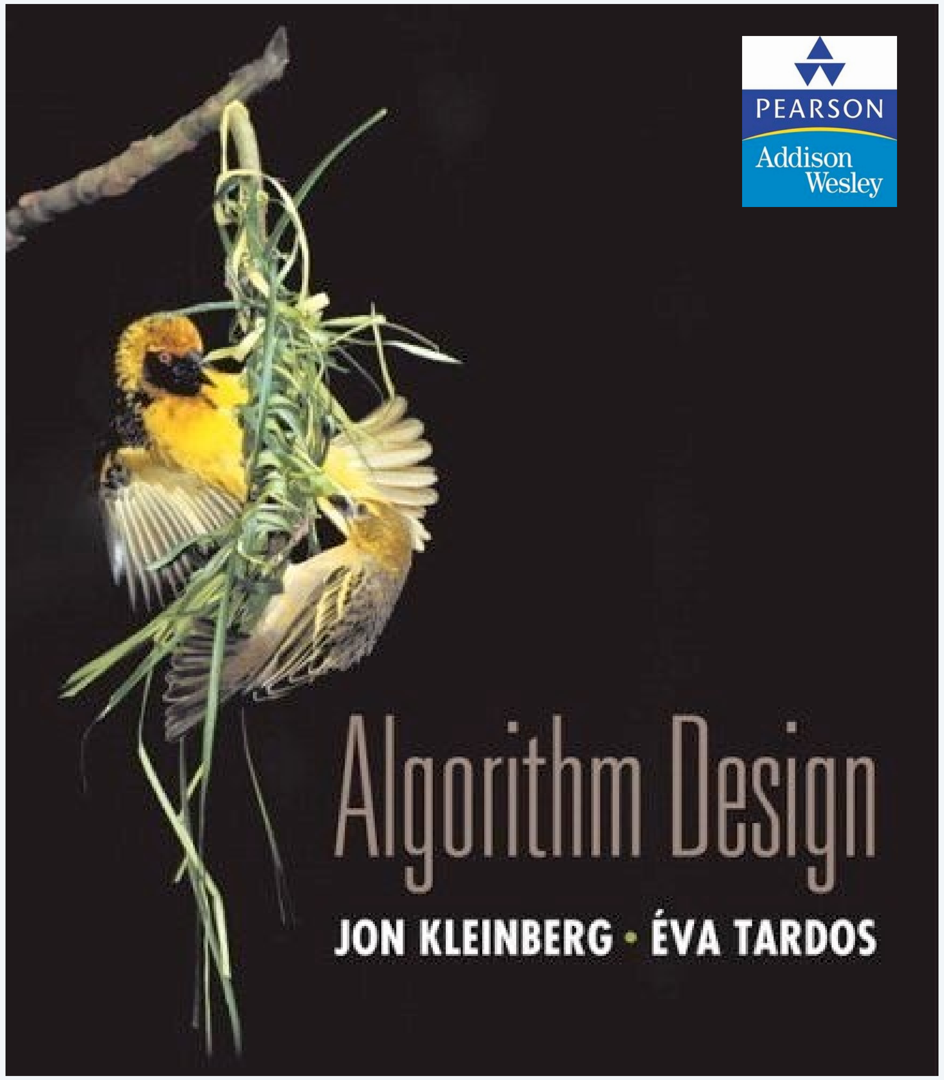
Theorem. [Tarjan 1972] Can find all strong components in $O(m+n)$ time.


SIAM J. Comput.
Vol. 1, No. 2, June 1972

DEPTH-FIRST SEARCH AND LINEAR GRAPH ALGORITHMS*

ROBERT TARJAN†

Abstract. The value of depth-first search or "backtracking" as a technique for solving problems is illustrated by two examples. An improved version of an algorithm for finding the strongly connected components of a directed graph and an algorithm for finding the biconnected components of an undirect graph are presented. The space and time requirements of both algorithms are bounded by $k_1V + k_2E + k_3$ for some constants k_1, k_2, and k_3, where V is the number of vertices and E is the number of edges of the graph being examined.



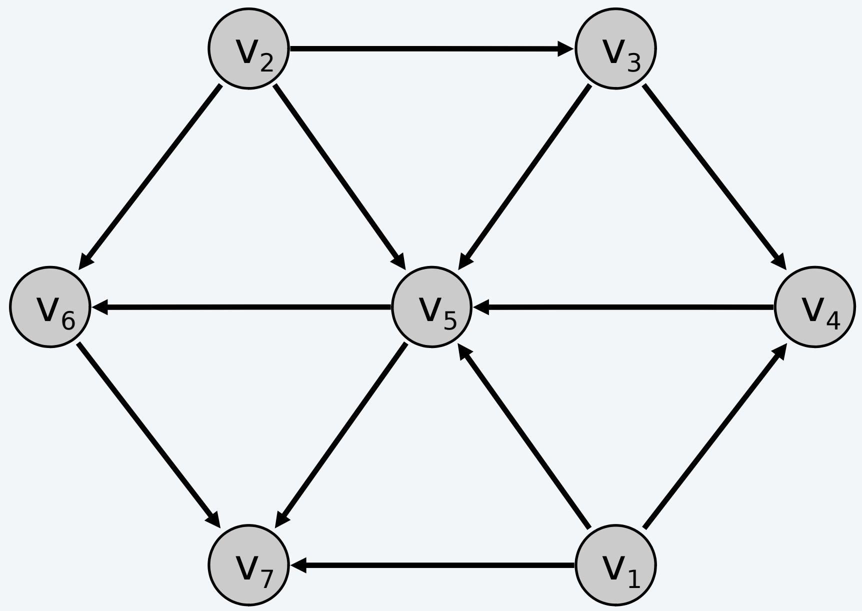
3. GRAPHS

- *basic definitions and applications*
- *graph connectivity and graph traversal*
- *testing bipartiteness*
- *connectivity in directed graphs*
- *DAGs and topological ordering*

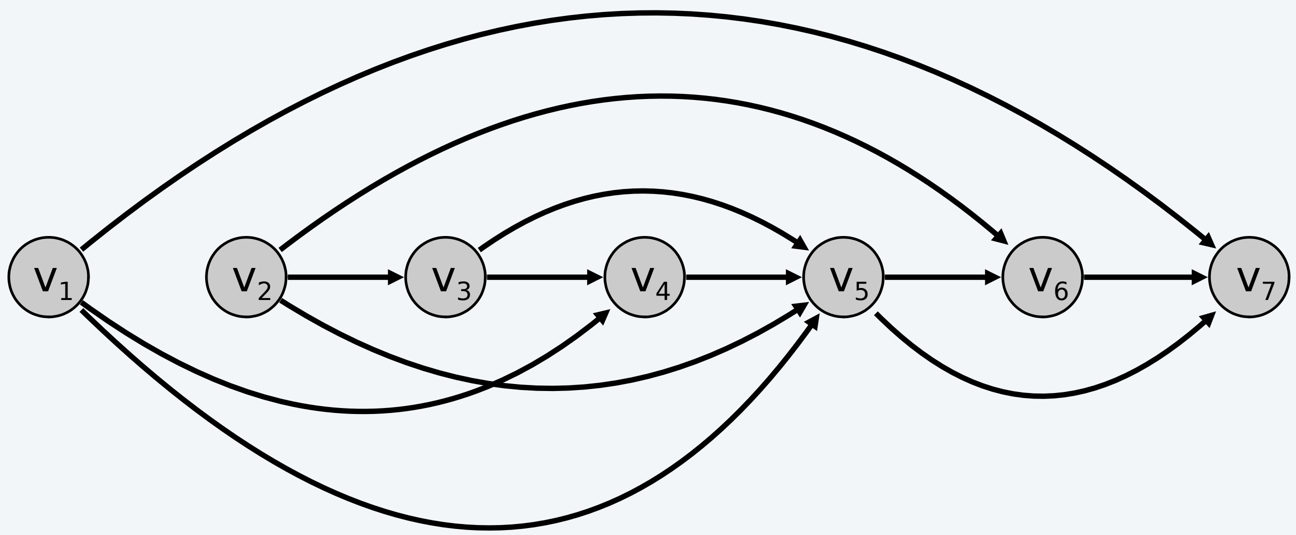
Directed acyclic graphs

Def. A DAG is a directed graph that contains no directed cycles.

Def. A topological order of a directed graph $G = (V, E)$ is an ordering of its nodes as $v_1, v_2, \ldots, v_n$ so that for every edge (v_i, v_j) we have $i < j$.



a DAG



a topological ordering

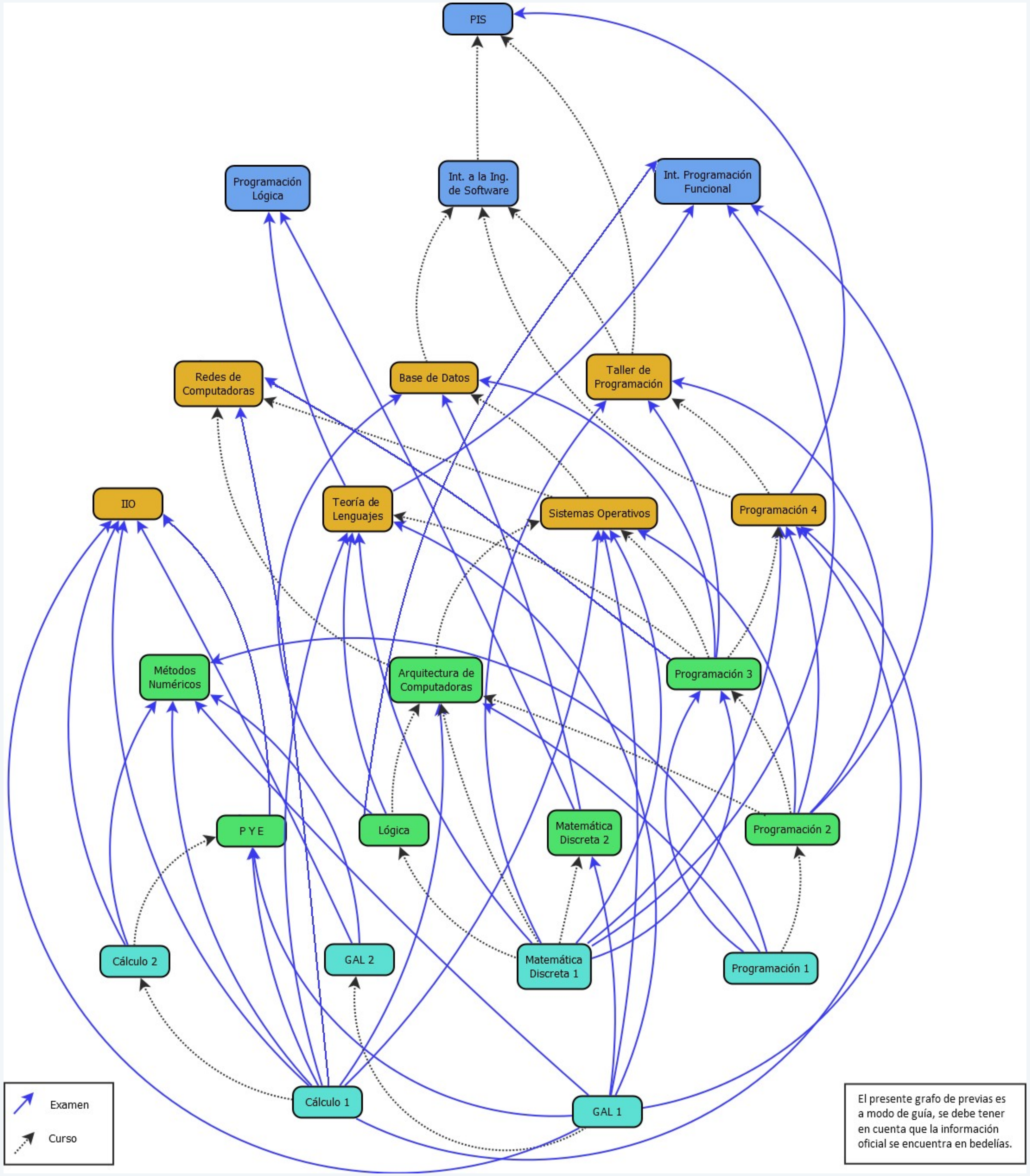
Precedence constraints

Precedence constraints. Edge (v_i, v_j) means task v_i must occur before v_j.

Applications.

- Course prerequisite graph: course v_i must be taken before v_j.
- Compilation: module v_i must be compiled before v_j.
- Pipeline of computing jobs: output of job v_i needed to determine input of job v_j.

Precedence constraints

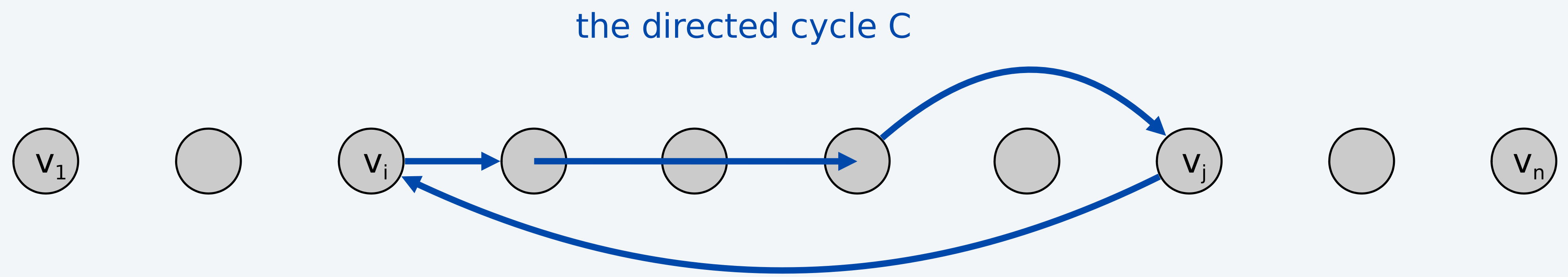


Directed acyclic graphs

Lemma. If G has a topological order, then G is a DAG.

Pf. [by contradiction]

- Suppose that G has a topological order $v_1, v_2, \ldots, v_n$ and that G also has a directed cycle C. Let's see what happens.
- Let v_i be the lowest-indexed node in C, and let v_j be the node just before v_i; thus (v_j, v_i) is an edge.
- By our choice of i, we have $i < j$.
- On the other hand, since (v_j, v_i) is an edge and $v_1, v_2, \ldots, v_n$ is a topological order, we must have $j < i$, a contradiction. ▪



the supposed topological order: $v_1, \ldots, v_n$

Directed acyclic graphs

Lemma. If G has a topological order, then G is a DAG.

Q. Does every DAG have a topological ordering?

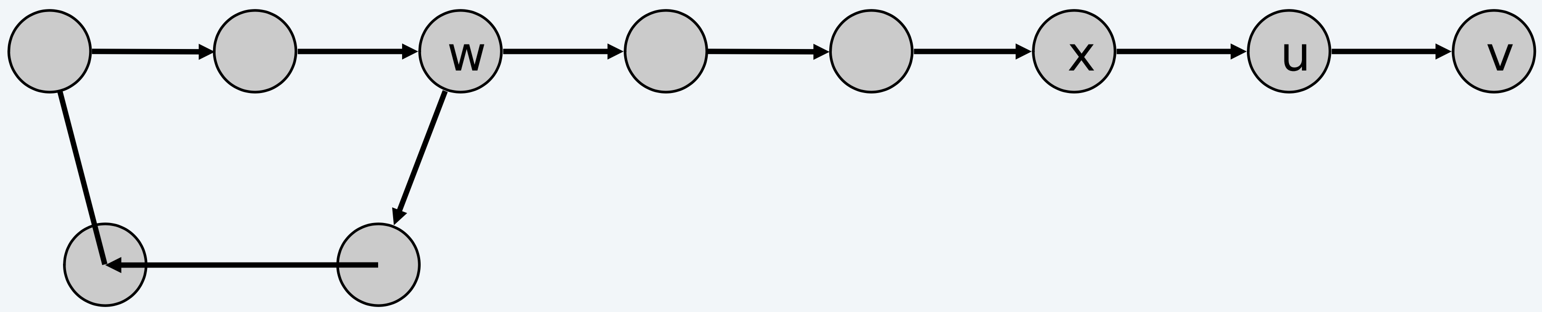
Q. If so, how do we compute one?

Directed acyclic graphs

Lemma. If G is a DAG, then G has a node with no entering edges.

Pf. [by contradiction]

- Suppose that G is a DAG and every node has at least one entering edge. Let's see what happens.
- Pick any node v, and begin following edges backward from v. Since v has at least one entering edge (u, v) we can walk backward to u.
- Then, since u has at least one entering edge (x, u), we can walk backward to x.
- Repeat until we visit a node, say w, twice.
- Let C denote the sequence of nodes encountered between successive visits to w. C is a cycle. ▪



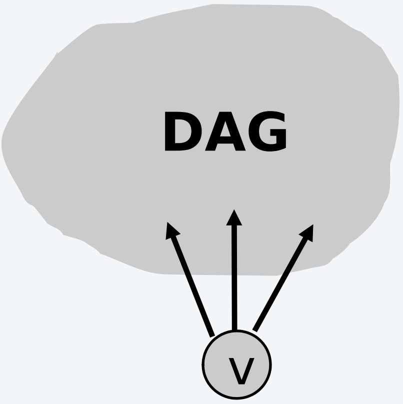
Directed acyclic graphs

Lemma. If G is a DAG, then G has a topological ordering.

Pf. [by induction on n]

- Base case: true if $n = 1$.
- Given DAG on $n > 1$ nodes, find a node v with no entering edges.
- $G - \{ v \}$ is a DAG, since deleting v cannot create cycles.
- By inductive hypothesis, $G - \{ v \}$ has a topological ordering.
- Place v first in topological ordering; then append nodes of $G - \{ v \}$ in topological order. This is valid since v has no entering edges. ■

```
To compute a topological ordering of G:
Find a node v with no incoming edges and order it first
Delete v from G
Recursively compute a topological ordering of G−{v}
  and append this order after v
```



Directed acyclic graphs

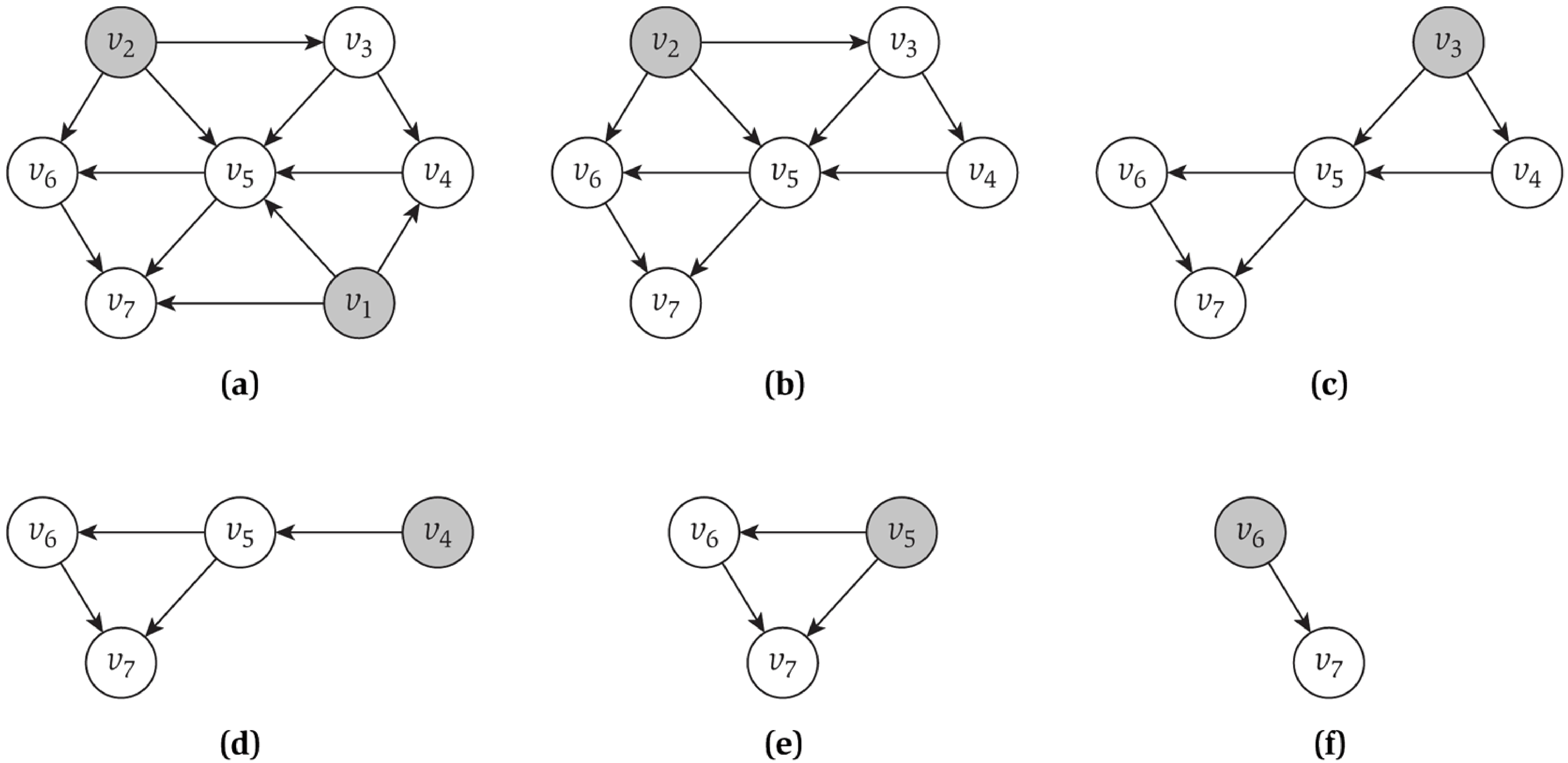


Figure 3.8 Starting from the graph in Figure 3.7, nodes are deleted one by one so as to be added to a topological ordering. The shaded nodes are those with no incoming edges; note that there is always at least one such edge at every stage of the algorithm's execution.

Topological sorting algorithm: running time

Theorem. Algorithm finds a topological order in $O(m + n)$ time.

Pf.

- Maintain the following information:
  - $count(w)$ = remaining number of incoming edges
  - S = set of remaining nodes with no incoming edges
- Initialization: $O(m + n)$ via single scan through graph.
- Update: to delete v
  - remove v from S
  - decrement $count(w)$ for all edges from v to w; and add w to S if $count(w)$ hits 0
  - this is $O(1)$ per edge ▪

Solved Exercise 1

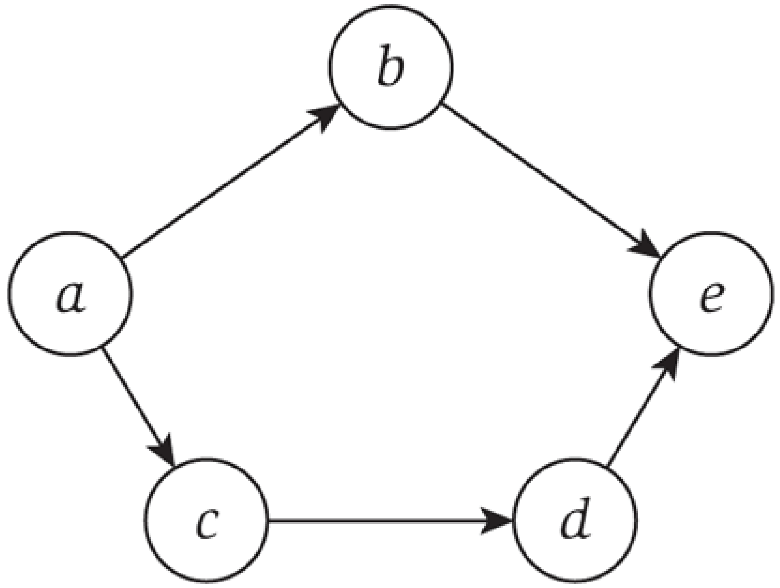


Figure 3.9 How many topological orderings does this graph have?

Solved Exercise 1

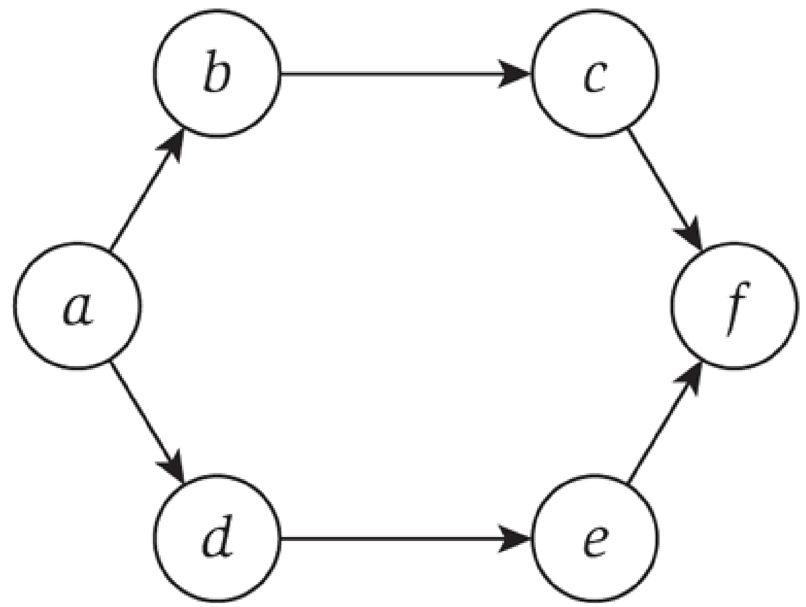


Figure 3.10 How many topological orderings does this graph have?