# Introducción a la teoría de juegos
# Lista final de ejercicios

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## 1 Normal form games

**Exercise 1** *The game of Three Fingers is played as followed: Alice and Bob simultaneously hold up one, two, or three fingers. Alice wins in case of a match (both players show the same number of fingers), and Bob wins if there is a nonmatch. If Alice wins, Bob must give her an amount equal to the total number of fingers held up. Otherwise, Alice must give Bob an amount equal to the number of fingers that he held up.*

1. *Write this game in normal form.*
2. *Does this game have a Nash equilibrium (of pure strategies) ?*

**Exercise 2 (Coordination game)** *Two operators can choose to invest in two technologies. Operator 1 already runs WiMAX, Operator 2 already runs LTE. But if different technologies are chosen, the gain of operators is null (customers refuse to choose a new technology if no norm has emerged). Operators have to coordinate on the same technology given that no one manages the same. We assume the following matrix of gains::*

| | | *Operator II* | |
|---|---|---|---|
| | | *WiMAX* | *LTE* |
| *Operator* | *WiMAX* | $(3, 1)$ | $(0, 0)$ |
| *I* | *LTE* | $(0, 0)$ | $(1, 4)$ |

1. *Compute the Nash equilibrium (equilibria) if any*
2. *What will be the outcome?*
3. *Is there an equilibrium in mixed strategy where Operator 1 invests with probability $x$ on WiMAX and with probability $(1 - x)$ on LTE, and Operator II invests with probability $y$ on WiMAX and with $(1 - y)$ on LTE?*
4. *What are the corresponding expected gains?*

**Exercise 3** *We consider the following game under strategic form:*

| | | *Player 2* | |
|---|---|---|---|
| | | *L* | *R* |
| *Player 1* | *T* | *(0,2)* | *(3,0)* |
| | *B* | *(2,1)* | *(1,3)* |

1. *Express the best-reply correspondences of each player.*

2. *Show that there exists a unique Nash equilibrium in mixed strategies. Compute the expected payment of each player at that equilibrium*

**Exercise 4** *(Transportation - Choosing a route) Four people must drive from A to B at the same time. Two routes are available, one via X and one via Y. (See Fig. 1a.) The roads from A to X, and from Y to B are both short and narrow; in each case, one car takes 6 minutes, and each additional car increases the travel time per car by 3 minutes. (If two cars drive from A to X, for example, each car takes 9 minutes.) The roads from A to Y, and from X to B are long and wide; on A to Y one car takes 20 minutes, and each additional car increases the travel time per car by 1 minute; on X to B one car takes 20 minutes, and each additional car increases the travel time per car by 0.9 minutes.*

*Formulate this situation as a strategic game and find the Nash equilibria. (If all four people take one of the routes, can any of them do better by taking the other route? What if three take one route and one takes the other route, or if two take each route?)*

**Exercise 5** *(Baraess paradox illustration) In continuation to the previous exercise. Now suppose that a relatively short, wide road is built from X to Y, giving each person four options for travel from A to B: A-X-B, A-Y-B, A-X-Y-B, and A-Y-X-B. Assume that a person who takes A-X-Y-B travels the A-X portion at the same time as someone who takes A-X-B, and the Y-B portion at the same time as someone who takes A-Y-B. (Think of there being constant flows of traffic.) On the road between X and Y, one car takes 7 minutes and each additional car increases the travel time per car by 1 minute. (See Fig. 1b).*

*Find the Nash equilibria in this new situation. Compare each person's travel time with the travel time experienced in the equilibrium before the road from X to Y was built (i.e. in previous exercise).*

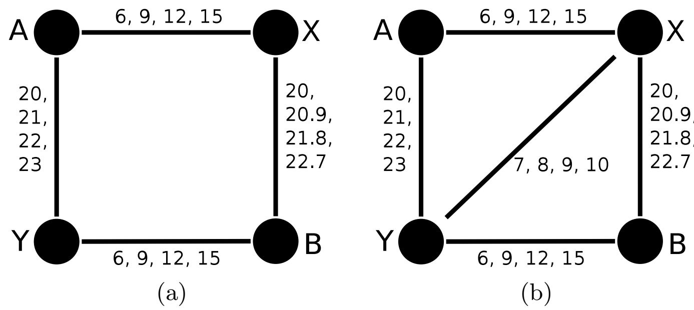


Figure 1: Illustration of Braess Paradox, exercises 4 and 5. Labels indicate the delay of traversing each road, for 1, 2, 3 and 4 vehicles.

## 2 Extensive form Games

**Exercise 6 (Ultimatum game)** *Consider the ultimatum game with two players and one step. In this game there is 1 dollar to split among two players, player 1 proposes a split, following, player 2 can accept or reject such split. If accept, the payoffs are as proposed by player 1; if reject, both get 0.*

1. *Find the values of $x$ for which there is a Nash equilibrium of the ultimatum game in which person 1 offers $x$.*
2. *Find the subgame perfect equilibria of a variant of the ultimatum game in which the amount of money is available only in multiples of a cent.*

## 3 Bayesian Games

**Exercise 7** *Two people are involved in a dispute. Person 1 does not know whether person 2 is strong or weak; she assigns probability $p$ to person 2's being weak. Person 2 is fully informed. Each person can either fight or yield. Each person's preferences are represented by the expected value of a payoff function that assigns the payoff of 0 if she yields (regardless of the other person's action) and a payoff of 1 if she fights and her opponent yields; if both people fight then their payoffs are (-1, 1) if person 2 is strong and (1, -1) if person 2 is weak. Formulate this situation as a Bayesian game and find its Nash equilibria if $p < 1/2$ and if $p > 1/2$ .*

**Exercise 8** *Consider the following static game of incomplete information. Nature selects the type (c) of player 1, where* $c = 2$ *with probability* $2/3$ *and* $c = 0$ *with probability* $1/3$*. Player 1 observes c (he knows his own type), but player 2 does not observe c. Then players make symultaneous and independent choices and receive payoffs as described by the following matrix.*

| | | *Player 2* | |
|---|---|---|---|
| | | *X* | *Y* |
| *Player 1* | *A* | *(0,1)* | *(1,0)* |
| | *B* | *(1,0)* | *(c,1)* |

*a. Draw the Bayesian normal form matrix of the game*

*b. Compute the Bayesian Nash equilibrium*

# 4 Auctions

**Exercise 9** *Consider an auction with five participants, each of them with the following (privately observed) valuation of the object for sale: Person A (\$10), Person B (\$6), Person C (\$45), Person D (\$81), and Person E (\$62).*

1. *If the seller organizes a second-price auction, who will be the winner? What will be his winning bid? What price he will pay for the object?*
2. *Suppose now that bidders can observe each other's valuations, but the seller cannot. The seller, however, only knows that bidder's valuations are in the range* $\{0, 1, \ldots, \$90\}$*. If players participate in a first-price auction, how will be the winner? What is his winning bid?*

**Exercise 10** *Consider a third-price auction, where the winner is the bidder who submits the highest bid, but he/she only pays the third highest bid. Assume that you compete against two other bidders, whose valuations you are unable to observe, and that your valuation for the object is \$10. Show that bidding above your valuation (with a bid of, for instance, \$15) can be a best response to the other bidder's bid, while submitting a bid that coincides with your valuation (\$10) might not be a best response to your opponent's bids.*

# 5 Cooperative Games

**Exercise 11** *(Voting) A legislature with three parties decides, by majority vote, the fate of three bills, A, B, and C. Each party's P1, P2 and P3,*

| | A | B | C |
|---|---|---|---|
| P1 | 2 | -1 | 1 |
| P2 | 1 | 2 | -1 |
| P3 | -1 | 1 | 2 |

Table 1: Parties' preferences for exercise 11.

*preferences are represented by the sum of the values they attach to the bills that pass. The value attached by each party to each bill is indicated in Table 1. For example, if bills A and B pass and C fails, then the three parties payoffs are 1, 3, and 0 respectively. Each majority coalition can achieve the passage of any set of bills, whereas each minority is powerless.*

1. *Model the problem as a coalitional game and find the core*
2. *We consider a variation where parties attach to the payoff of each bill differ from those in Table 1 only in that P3 values the passage of bill C at 0. Find the core in this new game.*
3. *We consider now that the values the parties attach to the payoff of each bill differ from those in Table 1 only in that each 1 is replaced by -1. Find the core of this new game.*

**Exercise 12 (Shapley value)** *Three friends (A,B,C) rent a house for one month. The pricing of the house is as follows: 200 the weekend, 400 a five-day week (without the weekend), 500 a 7-day week, 1500 a month. A wants to occupy the house the first two weeks, B the second two weeks, but not the weekends, C wants to go only the last two weekends. We assume a month of 4 weeks with four weekends.*

1. *Define a coalitional game modeling the situation*
2. *Compute how much should each one pay if using Shapley value for sharing cost*
3. *With these definitions, would you say that the coalition is stable?*

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