Introduction to Game Theory

Isabel Amigo

IMT Atlantique

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Notes:

Agenda

Notes:

Notes:

- the term cooperative may sometimes be misleading, it does not mean that we are modeling competing or non competing situations; it can model both
- coalitional game theory its a synonym
- the term coalitional means that the unit we are modeling are coalitions (i.e. groups of players) we are not modeling player's action but groups of players' actions.
- the problem we care about is how to split the revenue/cost resulting from the interaction, among the players

Cooperative games with transferable utility

Some application examples

- **Transportation** Share highways costs, Share Airport fees
- **P2P networks, WIFI share (e.g. fonera)** Give a share (i.e. fee reduction) to those users who collaborate with storage capacity or WIFI coverage
- **Electric Vehicles** Share the revenues of roaming among all recharging operators. Give a share (diminish his/her fee) to a user that provides energy to the network (V2G)

Notes:

Cooperative games with transferable utility

Motivation, simple example: *Connection to the power grid*

- Intuitively it makes sense to collaborate
- Cost function: $f(distance)$
- How to split the cost?

Notes:

Upcoming

- How is the revenue of the grand coalition split among its members? → revenue/cost sharing
- Is it interesting for a player to take part of a coalition? → stability

Some solution concepts: the core, the nucleolus, the τ-value, the Shapley value

- Establish revenue/cost shares
- Provide different properties

Notes:

Sharing Rules are Diverse and Provide Different Shares

Example: the contested garment.

- 2 persons A,B. One piece of fabric
- A wants half of the piece, B wants the whole piece

Which is the fairest way to share it?
Ideas:

- Proportional to the demand
- Answer provided in the Talmud
- GT?

Notes:

- The Talmud (central text of Rabbinic Judaism) provides the answer to both examples, without explaining how the calculations are done. For 2k years mathematicians and economist had not found such explanation. Game theory does.
- Lets start by defining a quite natural rule, the proportional rule: $\text{share}_i = \frac{\text{demand}_i}{\sum_{j \in N} \text{demand}_j}$
- Proportional rule yields $\text{share}_A = \frac{1/2}{1/2+1} = 1/3$, $\text{share}_B = \frac{1}{1/2+1} = 2/3$
- the Talmud law: A wants only half of it, thus only the other half of the fabric is contested. Share in equal parts the contested piece. $\text{share}_A = \frac{1/2}{2} = 1/4$, $\text{share}_B = 1/2 + \frac{1/2}{2} = 3/4$
- we will model this situation using GT

Another Talmud example

The bankrupt problem: One man dies with a wealth e and three debts d_1, d_2,d_3. How should e be split among the 3 debts claimers?

Claims d_1, d_2, d_3	Total (e)	Talmudic Law	Proportional
(100,200,300)	100	$(\frac{100}{3}, \frac{100}{3}, \frac{100}{3})$	$(\frac{50}{3}, \frac{100}{3}, 50)$
	200	(50,75,75)	$(\frac{100}{3}, \frac{200}{3}, 100)$
	300	(50,100,150)	(50,100,150)

Table: The bankrupt problem example.

Notes:

- Game theory also allows to explain this result
- Would you say is the same rule as the garment?

Definition

Coalitional Game with TU
Mathematically defined by:
· a set of players
· A value function, mapping every possible sub-coalition into a real value

Notes:

- $G = (N, v)$, where
- N is a set of players (usually referred as coalition or grand coalition)
- v is usually called (the worth or revenue of the coalition)
- $v : 2^{|N|} \to \mathbb{R}$

Vocabulary

superadditve game
A game where the outcome of being in the coalition is more interesting than the outcome of being alone

pre-imputation
A vector specifying the shares of the players, such that the sum of all shares equals the total revenue of the coalition

Notes:

- We will care about situations where the outcome of being in the coalition is more interesting than the outcome of being alone.
- These games are called superadditve games
- A superadditve game, is a cooperative game (N,v) such that for all sub coalitions $S, T \subset N$ such that $S \bigcap T = 0$ then $v(S \bigcup T) \geq v(S) + v(T)$
- A sharing rule is a function mapping a coalitional game into a real vector of dimension $|N|$: $\phi : N \times \mathbb{R}^{2^{|N|}} \to \mathbb{R}^{|N|}$
- For short we shall admit notation $x = \{x_i\}_{i \in N} = \phi(N, v)$, where x_i is the share for player i.
- A pre-imputation is a sharing vector $x = \{x_i\}_{i \in N}$ such that $\sum_{i} \in N \, x_i = v(N)$

Vocabulary

Marginal contribution
Contribution of player i to a sub-coalition is the difference between the worth of that subcoalition with i and the worth of that subcoalition when i is not there.

Notes:

- Marginal contribution of $i \in N$ to coalition $S \subset N$ is defined as $v(S) - v(S \setminus \{i\})$

Some desirable properties

- Efficiency
- Stability
- Fairness
- No free riders
- Monotonicity
  - Resource-
  - Population-

Notes:

- A rule is efficient if for every outcome x computed through such rule it renders $\sum_{n:n \in N} x_n = v(N)$
- Stability: incentives to remain in the coalition, we shall formalize it later on
- Fairness: not quite a consensus in the literature. Eg. that who contributes the most receives the most...
- Resource monotonicity: provide the right incentives to members to contribute to increase/decrease the revenue/cost of the coalition
- Many flavors of resource-monotonicity exist
- A population-monotonic revenue sharing rule guarantees that the entrance of a new member to the alliance does not reduce the revenue of each of the members already there

Notes:

- Photo Lloyd Shapley, Taken in 1980 by Konrad Jacobs, Erlangen, Copyright is with MFO, source Mathematisches Institut Oberwolfach (MFO), `http://owpdb.mfo.de/detail?photoID=3808`
- One of the most well known sharing rule is the Shapley value
- It was proposed by Lloyd Shapley in 1953 [10].
- Intuitively, its idea is to share the worth proportionally to the contribution of each player

The Shapley Value - Intuition

Example:

- Consider 2 players, $v(1) = 1$, $v(2) = 1$, $v(\{1, 2\}) = 3$.
- Then, each player's contribution is equal to 2 units

Notes:

- Is evident that we cant share the 3 units giving 2 units to each of them!
- The idea behind Shapley value is to share proportionally to each player's contribution to every subcoalition, but weighting the contribution.

Shapley Value - Definition

Shapley value (SV)

The SV for player i is the "average marginal contribution" of player i, averaging over all the different sequences according to which the grand coalition could be built up from the empty coalition.

Notes:

- That is, for player i, we consider all possible orderings for forming the grand coalition from the empty one, and we sum up the marginal contribution of player i at the moment of being added. We finally average by dividing by all the possible orderings for forming the grand coalition (i.e. by $|N|!$).
- Formally, Given a cooperative game (N,v) the SV for player $i \in N$ is given by
$\phi_i(N, v) = \frac{1}{|N|!} \sum_{S \subset N \setminus i} |S|!(|N| - |S| - 1)![v(S \cup \{i\}) - v(S)]$.

Example: Voting game

Set-up:

- Three parties, A, B, C
- Representatives per party 44, 39, 7
- They vote for approving a budget of 100 uruguayan pesos
- They need simple majority, i.e. 51 votes to approve the budget

Question: How should they split the budget among them?

Possible answer

. Define a coalitional game
. Compute the Shapley value

Notes:

- We define a coalitional game where N is the set of parties, and v is a function whose value is 100 for any subcoalition with more than 50 votes, and 0 for all the others.
- We compute the Shapley value for party A, using the intuitive interpretation
- We shall thus consider all possible orderings, sum up the marginal contribution of player A at the moment of being added, and divide by all the possible orderings.

Ordering	marginal contribution of A at the moment of adding A
A,B,C	$v(A) - V(\emptyset) = 0$
A,C,B	$v(A) - V(\emptyset) = 0$
B,A,C	$v(A, B) - V(B) = 100$
B,C,A	$v(B, C, A) - V(B, C) = 100$
C,A,B	$v(C, A) - V(C) = 100$
C,B,A	$v(C, B, A) - V(C, B) = 100$

- From them, we calculate the SV and obtain A's share (ϕ_A) as $\phi_A = \frac{1}{6} \times [0 + 0 + 100 + 100 + 100 + 100] = \frac{200}{3}$

Is the SV a good solution? - Definitions

Interchangeable players
Two players are interchangeable if each of them provides with the same worth to every sub-coalition

Dummy player
is a player whose contribution to any sub-coalition is the same value he/she can achieve alone.

Notes:

- We should of course define *good*
- We should do that in terms of desirable properties for a *good* rule
- Note that each sharing rule defines its own properties conveniently ;)
- If two players are *interchangeable* one desired property is that they receive the same share.
- Definition i, j are interchangeable if $v(S \setminus \{i\}) = v(S \setminus \{j\})$ $\forall S \cup N \setminus \{i, j\}$
- Definition, $i \in N$ is a dummy player if $v(S) = v(S \setminus \{i\}) = v(\{i\})$ $\forall S \subseteq N$

SV - Axiomatic characterization

Symmetry
A sharing rule verifies the Symmetry axiom if it gives the same shares to interchangeable players.

Additivity
Given three coalitional games, all with same set of players and with worth functions v_1, v_2 and $v_3 = v_1+v_2$, respectively. A sharing rule verifies the Additivity axiom if the share obtained for the third game is equal to the share in game 1 plus the share in game 2.

Notes:

- Symmetry Axiom: $\phi_i(N \setminus j, v) = \phi_j(N \setminus i, v)$ $\forall i, j \in N$
- Additivity Axiom: $\phi_i(N, v_1) + \phi_i(N, v_2) = \phi_i(N, v_1 + v_2)$, where the game $(N, v_1 + v_2)$ where $v_1 + v_2$ is defined as $(v_1 + v_2)(S) = v_1(S) + v_2(S)$ $\forall S \subseteq N$

SV - Axiomatic characterization

Dummy player
A rule verifying the Dummy player action gives to a dummy player a share equal to the worth that player achieves alone.

Notes:

- Dummy player axiom: if i is a dummy player then $\phi_i(N, v) = v(\{i\})$
- Remark: Is the dummy player a desirable property? Depends on the context, at least no in a social-aware context.

SV, axiomatic characterization

Theorem
There is only one sharing rule that verifies all the three previously stated axioms, and that rule is the SV.

Notes:

- So SV is only one possible sharing rule, we will see later on others
- Lets first address the question of which coalition will be formed.
- For that we are going to discuss about *Stability*
- The idea is that the grand coalition will remain stable if there are no incentives to form other sub coalitions.
- For that we will introduce the definition of the *core*.

Important solution concept: The Core

The Core

Is a set of payoff vectors such that:

. They are pre-imputations
. For **all** subcoalition, the sum of the shares is no less than the worth of that subcoalition

Notes:

- A payoff vector x is in the core of a coalitional game (N, v) if and only if $\sum_{i:i\in S} x_i \geq v(S) \forall S \subseteq N$

$$\sum_{i\in S} x_i \geq V(S) \forall S \subseteq N \quad (1)$$

$$\sum_{i\in N} x_i = V(N) \quad (2)$$

- A sharing rule lying in the Core is said to provide Stability
- Remark: similar concept to Nash equilibrium, when the payoff vector is in the core, there are no incentives to unilaterally deviate
- It is however stronger, since unilateral deviations are not possible not only for individual players but for all possible subcoalition

Is the core always non-empty?

Exercise: Three-player majority game. Three players dispute one unit of a divisible good. Any majority keeps the good. Define a coalitional game for this situation and show that the core of such game is empty.

Notes:

- Three-player majority game, we define the game as follows
- We define a coalitional game as follows
- $N = \{1, 2, 3\}$
- $v(\{1\}) = v(\{2\}) = v(\{3\}) = 0, v(\{1, 2, 3\}) = v(\{1, 3\}) = v(\{1, 2\}) = 1$
- Specify the constraints defining the core, and conclude

The core in convex games

Theorem

The core of a convex coalitional game is always non empty.

Notes:

Does the core determines a unique sharing vector?

Example: Two people produce together one unit, which they may share in any way they wish. If they are alone each produces zero units. Each person cares only about the amount of output she/he receives, preferring more to less.

Notes:

- Define a coalitional game
- Determine the core

Does the Shapley value lie in the Core?

Theorem
For every convex coalitional game, the SV lies in the core.

Notes:

Other sharing rules

- τ-value
- Nucleolus
- Aumann-Shapley
- Friedman-Moulin
- ...

Notes:

- The SV is not, of course, the only sharing rule
- For instance, we have already introduced the proportional one, which could be defied as dividing the whole value proportional to the contribution of each player, or proportional to the demand.
- back to the initial examples of come back to the contested garment and the bankrupt situations, and the solutions proposed in the Talmud. Aumman [1] has shown that the solutions proposed in the Talmud are the nucleolus of the bankrupt problem defined as a coalitional game, and that it is also garment-consistent for any two players.
- The nucleolus is a core refinement and it is unique, regardless of the game.

Example: Multicast Tree

- A group of customers must be connected to a service provided by some central facility
- a customer must either be directly connected to the facility or be connected to some other connected customer.
- We can model the customers and the facility as nodes on a graph, and the possible connections as edges with associated costs.
- Problem can be modeled as a coalitional game (N, v) .
- N is the set of customers, and $v(S)$ is the cost of connecting all customers in S directly to the facility minus the cost of the minimum spanning tree that spans both the customers in S and the facility.

Notes:

- Remark: the definition of the game implies that different solutions can verify different properties

The Nash Bargaining Solution

Define:

- a compact and convex set of all possible outcomes
- model each member's bargaining power (weight for negotiating)
- a disagreement point (outcome when there is no agreement)
- each member's utility function, i.e. their preferences over the set of possible outcomes.
- Assume there exits within the set of possible outcomes an outcome "suitable" for every member

Notes:

- X set of possible outcomes
- β_n will note the bargaining power of $n \in N$
- Let us assume utility functions are linear $u_n = x_n$ for all $n \in N$ where $x = \{x_n\}_N \in X$
- d is the disagreement point $d \in D \subset \mathbb{R}^{|N|}$
- we assume X is such that $\exists\, x \in X$ such that $x_n > d_n$, for all $n \in N$, where d_n is n's disagreement point.
- examples of bargaining powers: contribution to the coalition
- example of disagreement point: the stand alone revenue of each player, i.e. $d_n = v(\{n\})$, for all $n \in N$.
- The total amount to share is given by $v(N)$, and bounds the set of possible outcomes.

NBS

NBS
Is the outcome optimizing the product of differences of each member's outcome with his/her disagreement value, exponent to the his/her bargaining power, with feasibility and individually rational constraints.

Notes:

- The nash bargaining solution is the result of the following optimization problem

$$\max \prod_{n \in N} (x_n - d_n)^{\beta_n}$$
$$\text{s.t.} \sum_{n \in N} x_n = v(N)$$
$$x_n \geq d_n, \forall n \in N$$

which can be proven to be given by:

$$x_n^{nbs} = \frac{\beta_n}{\sum_{j \in N} \beta_j} \left(v(N) - \sum_{j \in N} d_n \right) + d_n. \quad (3)$$

Is the NBS a "good" solution?-Definitions

Symmetry
If the outcomes set is symmetric then if players are indistinguishable, they get the same outcome

Independence of Irrelevant alternatives
If the NBS is in a subset of the outcomes set, then the problem restricted to that subsets renders the same share.

Notes:

- *Symmetry:* if X is symmetric, and if players are indistinguishable, then players get the same outcome.
- *Independence of irrelevant alternatives:* let X and X' be such that $X' \subseteq X$. If $\phi^{nbs}(X, D) \in X'$ then $\phi^{nbs}(X', D) = \phi^{nbs}(X, D)$

Is the NBS a "good" solution?-Definitions

Pareto Efficiency
There's no outcome making every member indifferent with at least one happier.

Invariance to equivalent utility
a transformation of the utility functions that maintains the same ordering over preferences renders the same outcome

Notes:

- *Pareto efficiency: phi*$^{nbs}(X, D)$ is Pareto efficient if there is not $x \in X$ such that $x \geq \phi^{nbs}(x, D)$ and $x_n > \phi_n^{nbs}(x, D)$ for some $n \in N$.
- *Invariance to equivalent utility representations:* e.g. of transformation of the utility functions that maintains the same ordering over preferences: a linear transformation

Is the NBS a "good" solution?

- Nash proved that for 2-person bargaining games four previously introduced axioms characterize NBS [6], theory that was then extended to multiple players (see e.g. [2])

Notes:

Agenda

Notes:

Further Types of Games

- Non-atomic games
  - Large number of players
  - The individual effect of a player in the outcome is negligible but not the one of a portion of players
- Repeated games
  - Game is played several times, history is known
- Potential games
  - Utilities can be expressed through a common function
- ...

Notes:

To Sum up

- Game theory provides tools for analysing situations where multiple decisions makers interact
- Non-cooperative game theory
  - Study choices of rational selfish players
  - Nash Equilibrium
    - helps predict the possible rational outcomes of a game
- Cooperative game theory (with TU)
  - How to split costs/revenue of a coalition
  - The core set, stability
- A correct modeling is very important for coherent results
- Beyond Game Theory: evolutionary game theory

Notes:

¡Gracias!

Agenda

Notes:

Acknowledgments

Material for these slides have been taken from the following books

- An introduction to game theory [8]
- Multiagent Systems [11]
- Auction Theory [4]
- Pricing in communication networks [3]
- Telecommunication Networks Economics [5]

and from content from the following courses:

- Coursera online course: Game Theory
  `https://www.coursera.org/learn/game-theory-1/`
- Felix-Munoz Garcia: Strategy and Game Theory, Washington State University
- Bruno Tuffin and Patrick Maillé: Game Theory and Applications, IMT, INRIA, France

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Notes:

Quizz Lecture 5

The questions proposed here are taken from: the MOOC Game Theory on Coursera platform, created by Matthew Jackson, Kevin Leyton-Brown and Yoav Shoham.

Exercise 1 *Suppose* $N = 3$ *and* $v(1) = v(2) = v(3) = 1$.
Which of the following payoff functions is superadditive?

a. *v(1,2)=3, v(1,3)=4, v(2,3)=5, v(1,2,3)=5;*

b. *v(1,2)=3, v(1,3)=4, v(2,3)=5, v(1,2,3)=7;*

c. *v(1,2)=0, v(1,3)=4, v(2,3)=5, v(1,2,3)=7;*

d. *None of the above.*

Exercise 2 *Suppose N=2 and v(1)=0, v(2)=2, v(1,2)=2.*
What is the Shapley Value of both players?

a. $\phi_1(N, v) = 1$, $\phi_2(N, v) = 0$

b. $\phi_1(N, v) = 1/2$, $\phi_2(N, v) = 1/2$

c. $\phi_1(N, v) = 1/3$, $\phi_2(N, v) = 2/3$

d. $\phi_1(N, v) = 0$, $\phi_2(N, v) = 2$

Exercise 3 *Suppose N=3 and v(1)=v(2)=v(3)=0, v(1,2)=v(2,3)=v(3,1)=2/3, v(1,2,3)=1.*
Which allocation is in the core of this coalitional game?

a. *(0,0,0);*

b. *(1/3, 1/3, 0);*

c. *(1/3, 1/3, 1/3);*

d. *non of the above*