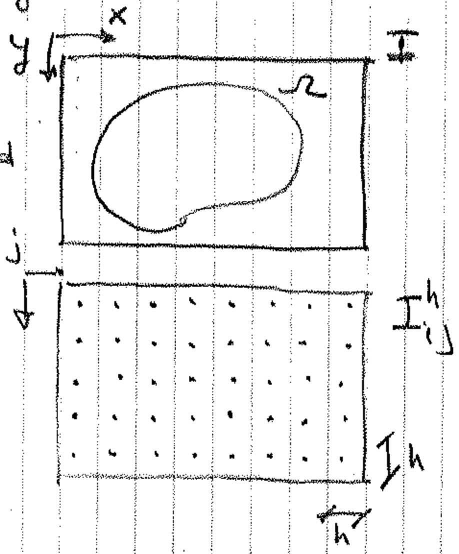
CLASE 1: Introduction to signal processing. 2017, 3D Vision

An image is a function $I(x,y): \Omega \subset \mathbb{R}^2 \to \mathbb{R}$, in practical aplications this image is represented by a sampled (Discrete) version of it:



$$I^h_{ij} \overset{DEF}{=} \int_{-\frac{h}{2}}^{\frac{h}{2}}\int_{-\frac{h}{2}}^{\frac{h}{2}} I(u-j, v-i)\, du\, dv \overset{\text{comm (H)}}{\approx} I(jh, ih)$$

- USEFULL DEFINITIONS AND PROPERTIES ON IMAGES:

(i) x/y - Partial derivatives:

- Cont. $I_x \overset{def}{=} \lim_{\delta \to 0^+} \frac{I(x+\delta, y) - I(x,y)}{\delta}$ (analogously for y)

- Discrete approx: $I_x(x,y) \approx$
  - $(I^h_{i(j+1)} - I^h_{i(j-1)})\frac{1}{2}$ (centered-scheme)
  - $I^h_{i(j+1)} - I^h_{ij}$ (Forward-scheme)
  - $I^h_{ij} - I_{i(j-1)}$ (Backward-scheme)

⊖ BE CAREFUL AT BOUNDARIES ⊖

> EXERCISE:
>
> Demonstrate that the first approx has an error of order $O(h^3)$ and the other two-ones are order $O(h^2)$.

Solution: (TAYLOR) $I^h_{i(j+1)} = I(x+h, y) = \underbrace{I(x,y)}_{= I^h_{ij}} + I_x(x,y)\,h + O(h^2)$

$\longrightarrow I^h_{i(j+1)} - I_{ij} = h\, I_x(x,y) + O(h^2)$. – [the other are similar]

(ii) Laplacian:

- Cont: $\Delta I \overset{DEF}{=} I_{xx} + I_{yy}$

- Discret: 4-points stencil

	1	
1	-4	1
	1	

$$\Delta^h I^h_{ij} = -4 I_{ij} + I_{(i+1)j} + I_{(i-1)j} + I_{i(j+1)} + I_{i(j-1)}$$

DEM: $\Delta^h I^h_{ij} = \underbrace{\overbrace{(I_{(i+1)j} - I_{ij})}^{I_x(i+1)j} - \overbrace{(I_{ij} - I_{(i-1)j})}^{I_x(ij)}}_{I_{xx}^{ij}} +$ " ✓

(iii) 2D - Convolution:

- Cont: DEF: $f*g_{(x,y)} = \iint_{\Omega} f(u,v)\, g(x-u, y-v)\, du\, dv$
- Discret: $(f*g)_{ij} = \sum_{i',j'} f_{i'j'} \cdot g_{(i-i')(j-j')}$ ⊖ BE CAREFUL AT BOUNDARIES

(iv) 2D - FOURIER TRANSFORM: $I \in \mathcal{L}^1(\Omega)$, i.e. $\int_{\Omega} |I| d\vec{x} < \infty$

DEFINITION

- Cont: $\mathcal{F}[I]_{(f_x, f_y)} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} I(x,y)\, e^{-j 2\pi (f_x x + f_y y)}\, dx\, dy = \hat{I}$

inverse $\longrightarrow \mathcal{F}^{-1}[\,] = \int \cdots e^{+j(2\pi \cdots)} df_x\, df_y$

Prop:

- Spatial shift: $\mathcal{F}[I(x+\Delta x, y)]_{(f_x, f_y)} = \mathcal{F}[I(x,y)]_{(f_x, f_y)} \cdot e^{j 2\pi f_x \Delta x}$

Dem: EXERCISE!

CV $\Rightarrow x' = x+\Delta x$

$\mathcal{F}[I(x+\Delta x, \cdot)] = \int I(x+\Delta x, y) e^{-j()} dx\, dy = \int I(x', \cdot) e^{-j(\,)}$

$= \mathcal{F}[I(x,y)]_{(f_x, f_y)}\, e^{j 2\pi f_x \Delta x}$

- DERIVATIVES: $\mathcal{F}[I_x(x,y)] = (+j 2\pi f_x)\, \mathcal{F}[I(x,y)]$ (the same y.

Dem: $\int I_x e^{-j()} dx\, dy = -\int I \, e^{-j 2\pi (f_x x + f_y y)} \cdot (-(j 2\pi f_x))\, dx\, dy + \left. I e^{-j 2\pi (f_x x + f_y y)} \right|_{-\infty}^{+\infty}$ (parts; last term $= 0$)

- Linearity: $\mathcal{F}[I_1 + \mu I_2] = \mathcal{F}[I_1] + \mu\, \mathcal{F}[I_2]$
- Convolution: $\mathcal{F}[f*g] = \mathcal{F}[f] \cdot \mathcal{F}[g]$

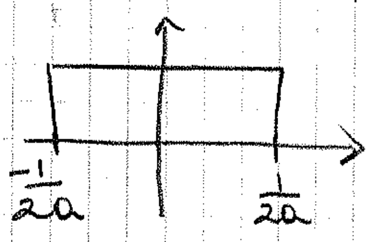
Dem: $\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \left(f(x-u)\, g(u)\, du \right) e^{-j 2\pi \nu x} dx = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x-u)\, g(u)\, e^{-j 2\pi \nu x}\, du\, dx$

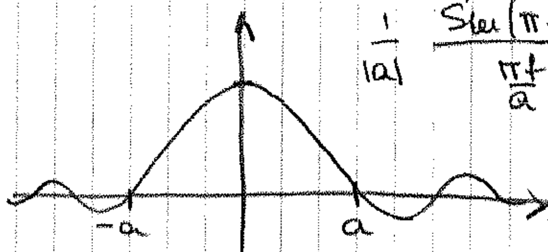
CV: $v = x - u$, $u = u$ $\Rightarrow \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(v)\, g(u)\, e^{-j 2\pi \nu (v+u)}\, dv\, du = \mathcal{F}[f] \cdot \mathcal{F}[g]$

JACOBIANO $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \rightarrow \det(J) = 1.$

Comon Fourier transforms:

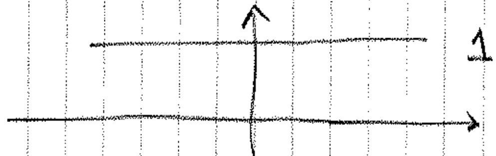
- $\Pi(ax)$



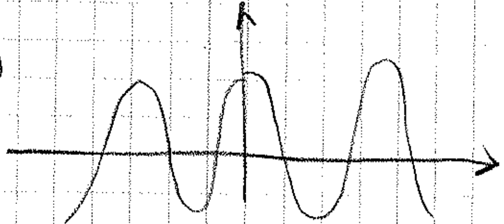


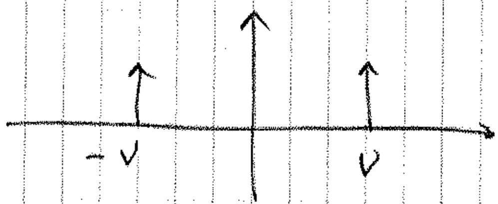
$\frac{1}{|a|} \frac{\sin(\pi f/a)}{\frac{\pi f}{a}}$

- $\delta(x)$

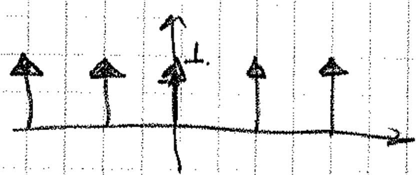


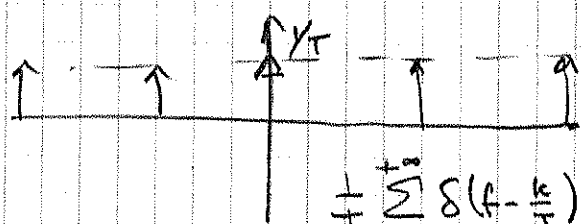
- $\cos(2\pi \nu x)$





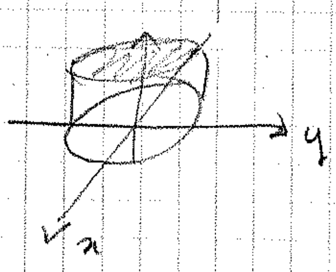
- $\sum_{-\infty}^{+\infty} \delta(x - nT)$

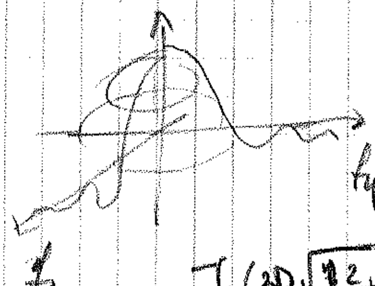




$\frac{1}{T} \sum_{k=-\infty}^{+\infty} \delta(f - \frac{k}{T})$

- $\mathrm{circ}(\sqrt{x^2+y^2})$





$\frac{J_1(2\pi\sqrt{f_x^2+f_y^2})}{\sqrt{f_x^2+f_y^2}}$

*

- Hankel transform: more in Iizuka Engineering Optics pg

$\mathcal{F}[g]_{(f_x, f_y)} = \iint g(x,y)\, e^{-j2\pi(f_x x + f_y y)}\, dx\,dy$ in cilindrical coords.

$dx\,dy \to r\,dr\,d\theta$
$f_x = \rho \cos\phi$
$f_y = \rho \sin\phi$

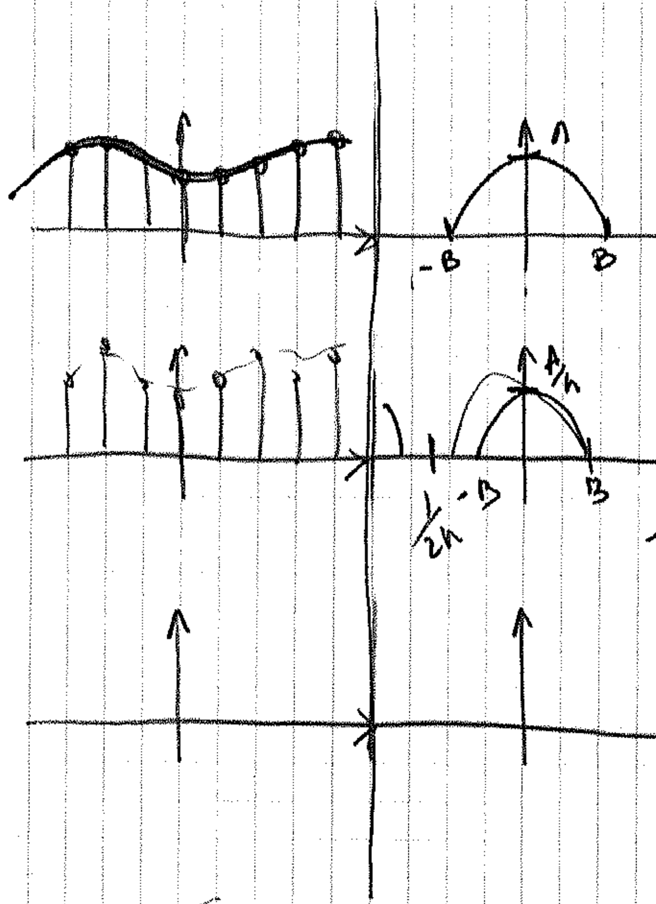
$\Rightarrow \mathcal{F}[g]_{(\rho,\phi)} = \int_{r=0}^{+\infty}\int_{\theta=0}^{2\pi} g(r,\theta)\, e^{-j2\pi(r\cos\theta\rho\cos\phi + r\sin\theta\rho\sin\phi)}$ — $r\rho(\cos\theta-\phi)$

$= \iint g(r,\theta)\, e^{-j2\pi r\rho\cos(\theta-\phi)}\, r\,dr\,d\theta$

(IV) Fourier in a Discrete Domain:

$$I^h = I(x,y) \cdot \sum_{k=-\infty}^{+\infty} \delta(x - hk)$$

$$\mathcal{F}[I^h] = \mathcal{F}[I] * \underbrace{\mathcal{F}[\textstyle\sum \ldots]}_{\frac{1}{h}\sum_{k=-\infty}^{+\infty} \delta\left(t - k\frac{1}{h}\right)}$$



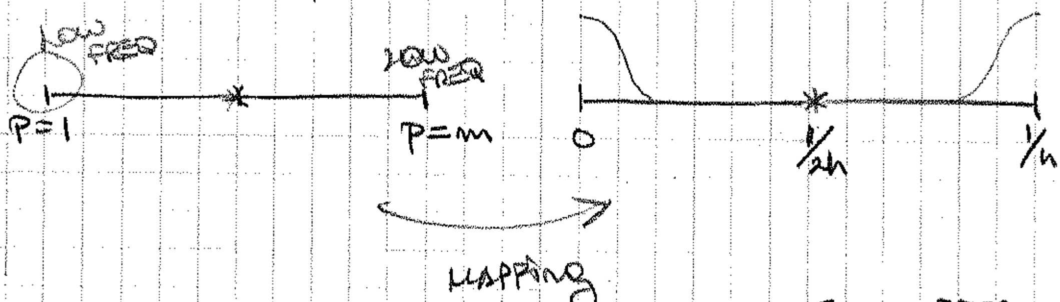
DFT

série de fourier $c_n = \frac{1}{T}\int_0^T f\, e^{-j2\pi n x/T}\, dx$

$$\tilde{I}^h_{pq} = \sum_{u=1}^{m}\sum_{v=1}^{n} I_{uv}\, e^{-j2\pi\left(\frac{(u-1)(p-1)}{m} + \frac{(v-1)(q-1)}{n}\right)}$$

DFT

freq cont.



SEE: FFT2, FFTSHIFT (MATLAB)