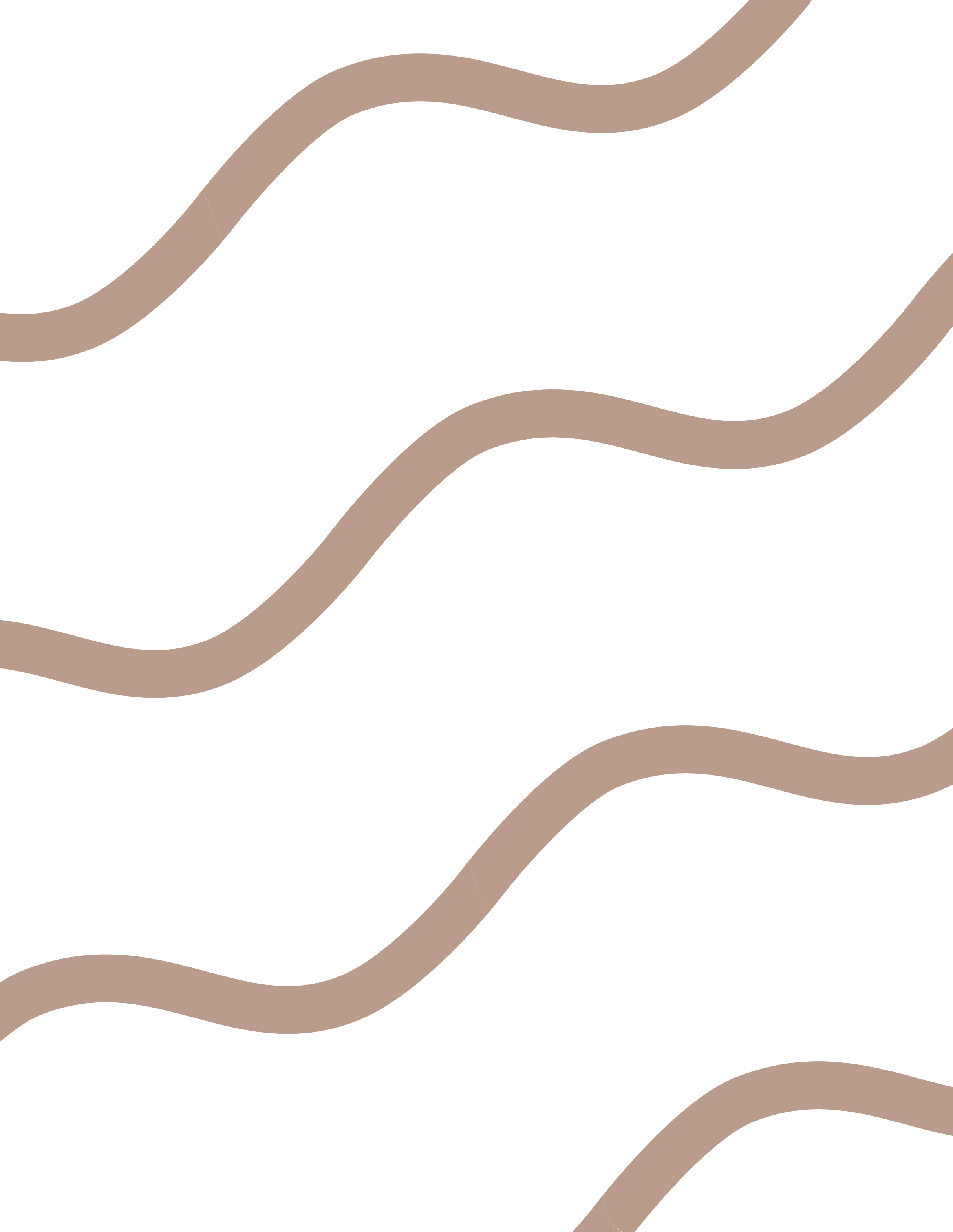

Aproximación usando polinomio de Taylor

Sea $f:(\alpha,\beta)\longrightarrow\mathbb{R}$ una función C^{∞}.

El desarrollo de Taylor de f en un $a\in(\alpha,\beta)$ es

$$f(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\cdots+\frac{f^{(k)}(a)}{k!}(x-a)^k+r_k(x)$$

en donde $r_k(x)$ es un infinitésimo de orden k

es decir $\lim\limits_{x\to a}\dfrac{r_k(x)}{(x-a)^k}=0$.

El resto se puede expresar $r_k(x)=\dfrac{f^{(k+1)}(c)\,(x-a)^{k+1}}{(k+1)!}$

Si queremos aproximar $\sqrt{7}$ por un racional podemos considerar el polinomio de Taylor de $f(x)=\sqrt{x}$ en un a tal que $\sqrt{a}$ sea racional. Por ejemplo en $a=4$

$f(x) = \sqrt{x}$ $\qquad a = 4$

$f'(x) = \frac{1}{2} \cdot x^{1/2 - 1} = \frac{1}{2} \cdot x^{-1/2}$ $\qquad f'(4) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

$f''(x) = \frac{1}{2} \cdot \frac{-1}{2} x^{-1-1/2} = -\frac{1}{4} x^{-3/2}$ $\qquad f''(4) = \frac{-1}{4} \cdot \frac{1}{8} = \frac{-1}{32}$

$f'''(x) = \frac{-1}{4} \cdot \frac{-3}{2} \cdot x^{-5/2} = \frac{3}{8} x^{-5/2}$ $\qquad f'''(4) = \frac{3}{8 \cdot 2^5} = \frac{3}{2^8}$

$f^{(4)}(x) = \frac{3}{8} \cdot \frac{-5}{2} x^{-7/2} = \frac{-15}{16} x^{-7/2}$ $\qquad f^{(4)}(4) = \frac{-15}{2^4 \cdot 2^7} = \frac{-15}{2^{11}}$

$f^{(5)}(x) = -\frac{15}{16} \cdot \left(\frac{-7}{2}\right) x^{-9/2} = \frac{135}{32} x^{-9/2}$

$\vdots$

$f^{(k)}(x) = \frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right) \cdots \left(\frac{1}{2}-k+1\right) x^{1/2 - k}$

$f(x) = \sqrt{4} + \frac{1}{4}(x-4) - \frac{1}{32 \cdot 2}(x-4)^2 + \frac{3}{2^8 \cdot 3!}(x-4)^3 - \frac{15}{2^{11} \cdot 4!} \cdot (x-4)^4$

$+ r_4(x)$

Si $\quad x = 7$

$\underset{\sqrt{7}}{\overset{}{f(7)}} = 2 + \frac{3}{4} - \frac{3^2}{2^6} + \frac{3}{2^8 \cdot 3!} \cdot 3^3 - \frac{15}{2^{11} \cdot 4!} 3^4 + r_4(7)$

$= 2 + \frac{3}{4} - \frac{3^2}{2^6} + \frac{3^3}{2^9} - \frac{15 \cdot 3^3}{2^{14}} + r_4(7)$

¿Cuál es el error $r_4(x)$?

Por la expresión del resto de Lagrange

$$r_4(7) = \frac{f^{(5)}(c)\, 3^5}{5!}$$ para algún $c \in [4,7]$

$$f^{(5)}(c) = \frac{105}{32} \cdot c^{-9/2}$$

$$4 \leq c \leq 7 \Rightarrow 2 \leq c^{1/2} \leq 7^{1/2}$$

$$2^9 \leq c^{9/2} \leq 7^{9/2}$$

$$\Rightarrow c^{-9/2} \leq 2^{-9}$$

$$\Rightarrow |r_4(7)| = \left| \frac{f^{(5)}(c)\, 3^5}{5!} \right| \leq \frac{105}{32} \cdot \frac{3^5}{5!} \cdot 2^{-9}$$

$$= \frac{7 \cdot \cancel{3} \cdot \cancel{5}}{2^5} \cdot \frac{3^5}{\cancel{5} \cdot 4 \cdot \cancel{3} \cdot 2} \cdot 2^{-9}$$

$$= \frac{7 \cdot 3^5}{2^{17}} = \frac{1701}{131072}$$

$$= 0{,}0129776$$

$$a = \frac{64}{9}$$

$$\sqrt{x} = \frac{8}{3} + f'\left(\frac{64}{9}\right)\left(x - \frac{64}{9}\right) + \frac{f''\left(\frac{64}{9}\right)}{2} \cdot \left(x - \frac{64}{9}\right)^2 + r_2(x)$$

$$r_2(x) = \frac{f^{(3)}(c)\,\left(x - \frac{64}{9}\right)^3}{3!} \qquad c \in \left[x, \frac{64}{9}\right]$$

$$\frac{49}{9} \leq 7 \leq c \leq \frac{64}{9} \qquad f^{(3)}(c) = \frac{3}{8\sqrt{c^5}} = \frac{3}{8} \cdot c^{-5/2}$$

$$\frac{7}{3} \leq c^{1/2} \leq \frac{8}{3}$$

$$\frac{3}{8} \leq c^{-1/2} \leq \frac{3}{7} \quad \Rightarrow \quad c^{-5/2} \leq \frac{3^5}{7^5}$$

$$\Rightarrow \quad f^{(3)}(c) \leq \frac{3}{8} \cdot \frac{3^5}{7^5}$$

$$|r_2(7)| \leq \frac{\left(\frac{64}{9} - 7\right)^3 \cdot \frac{3^6}{8 \cdot 7^5}}{3!} = \frac{2.28 \times 10^{-4} \cdot 3^6}{8 \cdot 7^5}$$

$$\sim 1.23 \times 10^{-6}$$

$$\sqrt{x} = \frac{8}{3} + f'\left(\frac{64}{9}\right)\left(x-\frac{64}{9}\right) + \frac{f''\left(\frac{64}{9}\right)}{2}\cdot\left(x-\frac{64}{9}\right)^2 + r_2(x)$$

Si $x = 7 \Rightarrow |r_2(7)| < 10^{-5}$

$$\boxed{\sqrt{7} = \frac{8}{3} + \frac{3}{16}\left(7-\frac{64}{9}\right) - \frac{3^3}{2^{12}}\cdot\left(7-\frac{64}{9}\right)^2 + r_2(x)}$$

$$f'\left(\frac{64}{9}\right) = \frac{1}{2\,{}^{8}\!/_{3}} = \frac{3}{16}$$

$$f''\left(\frac{64}{9}\right) = \frac{-1}{4\left(8/3\right)^3} = \frac{-3^3}{2^{11}}$$

$4 = 2^2 \qquad 8^3 = (2^3)^3 = 2^9$

esta aproximación racional cumple con lo pedido.