JHU vision lab

Low Rank Subspace Clustering (LRSC)

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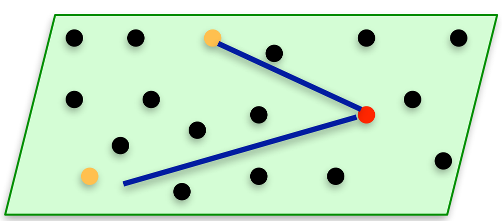
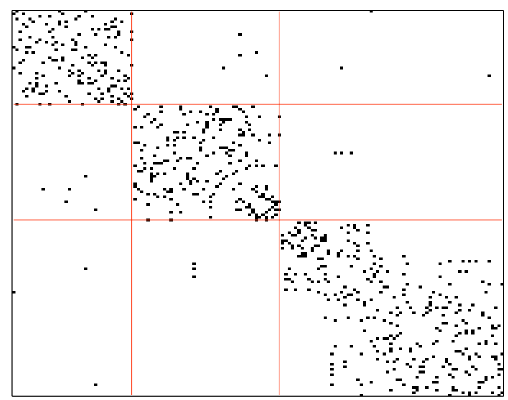
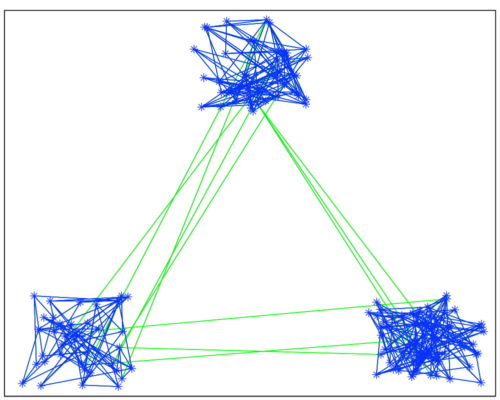
Sparse Subspace Clustering: Spectral Clustering

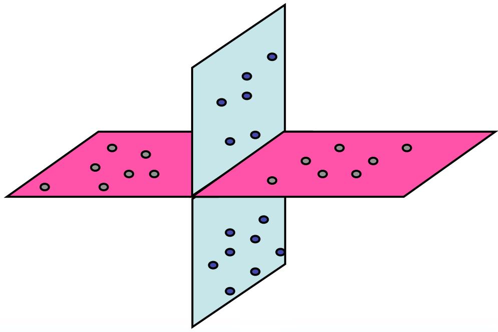
- Spectral clustering
  - Represent data points as nodes in graph G
  - Connect nodes i and j with weight c_{ij}
  - Infer clusters from Laplacian of G







- How to define a good affinity matrix C for subspaces?
  - points in the same subspace: $c_{ij} \neq 0$
  - points in different subspaces: $c_{ij} = 0$

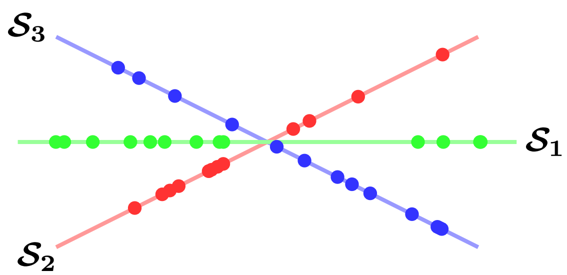


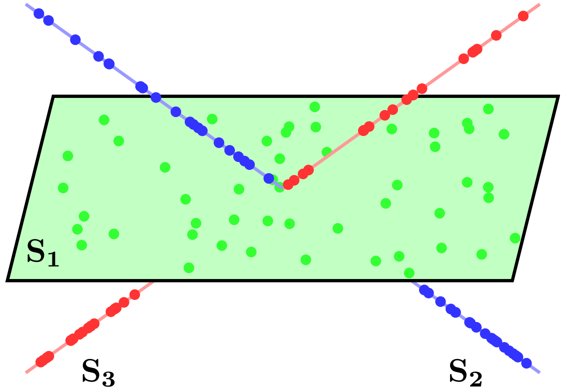
Sparse Subspace Clustering: Intuition

- Data in a union of subspaces are self-expressive

$$\boldsymbol{x}_j = \sum_{i=1}^{N} c_{ij}\boldsymbol{x}_i \implies \boldsymbol{x}_j = X\boldsymbol{c}_j \implies X = XC$$

- Union of subspaces admits subspace-sparse representation





- Sparse Subspace Clustering

$$\min_{C} \|C\|_1 \quad \text{s.t.} \quad X = XC, \quad \operatorname{diag}(C) = 0$$

E. Elhamifar and R. Vidal. Sparse Subspace Clustering. CVPR 2009.
E. Elhamifar and R. Vidal. Clustering Disjoint Subspaces via Sparse Representation. ICASSP 2010.
E. Elhamifar and R. Vidal. Sparse Subspace Clustering: Algorithm, Theory and Applications. TPAMI 2013.

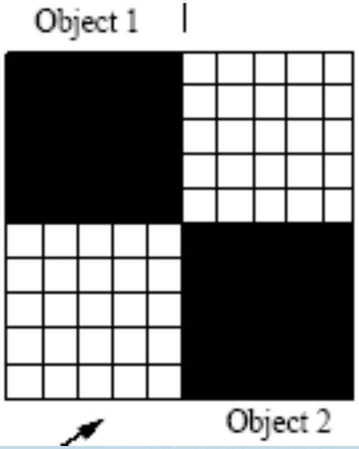
Subspace Clustering by Matrix Factorization

- Data from i-th subspace can be factorized as $Y_i = U_i V_i^\top$

$$Y\Gamma = [Y_1, Y_2, \ldots, Y_n] = [U_1, U_2, \ldots, U_n] \begin{bmatrix} V_1^\top & & & \\ & V_2^\top & & \\ & & \ddots & \\ & & & V_n^\top \end{bmatrix}$$

- Segmentation of the data can be obtained from
  - Leading singular vector of $Y = \mathcal{U}\Sigma\mathcal{V}^\top$ (Boult and Brown '91)
  - Shape interaction matrix $C = \mathcal{V}\mathcal{V}^\top$ (Costeira & Kanade '95, Gear '94)

- $C_{ij} = 0$ if points i and j lie in two independent subspaces (Kanatani et al. '01, Vidal et al. '08)



T. Boult, L. Brown. Factorization-based segmentation of motions. Workshop on Motion Understanding, pages 179–186, 1991.
J. Costeira, T. Kanade. A multibody factorization method for independently moving objects. IJCV, 29(3):159–179, 1998.
K. Kanatani. Motion segmentation by subspace separation and model selection. ICCV, volume 2, pages 586–591, 2011.
R. Vidal, R. Tron, R. Hartley. Multiframe motion segmentation with missing data using PowerFactorization and GPCA. IJCV,2008.

Low Rank Subspace Clustering

- Data in a union of subspaces are self-expressive

$$\boldsymbol{x}_j = \sum_{i=1}^{N} c_{ij}\boldsymbol{x}_i \implies \boldsymbol{x}_j = X\boldsymbol{c}_j \implies X = XC$$

  - C is sparse
  - C is low-rank

- Low Rank Subspace Clustering (noiseless case)

$$\min_C \|C\|_* \quad \mathrm{s.\,t.} \quad X = XC \quad \implies \quad \begin{cases} X = \mathcal{U}\Sigma\mathcal{V}^\top \\ C = \mathcal{V}\mathcal{V}^\top \end{cases}$$

- Low Rank Subspace Clustering (noisy case)

$$\min_C \|C\|_* + \frac{\tau}{2}\|X - XC\|_F^2 \quad \implies \quad C = \mathcal{V}(I - \frac{1}{\tau}\Sigma^{-2})\mathcal{V}^\top$$

G. Liu, Z. Lin, and Y. Yu. Robust subspace segmentation by low-rank representation. ICML 2010.
G. Liu, Z. Lin, S. Yan, J. Sun, and Y. Ma. Robust recovery of subspace structures by low-rank representation. TPAMI, 2013.
P. Favaro, R. Vidal and A. Ravichandran. A closed form solution to robust subspace estimation and clustering. CVPR 2011.
R. Vidal and P. Favaro. Low rank subspace clustering (LRSC). Pattern Recognition Letters, 43:47–61, 2014.

SSC versus LRSC

Sparse Subspace Clustering	Low-Rank Subspace Clustering
Convex Optimization	Closed Form Solution
Arbitrary Subspaces	Independent Subspaces
Provably Robust to Noise, Outliers	

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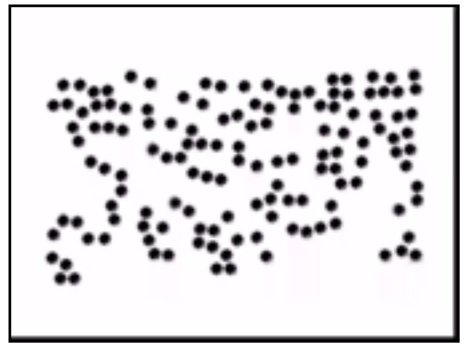
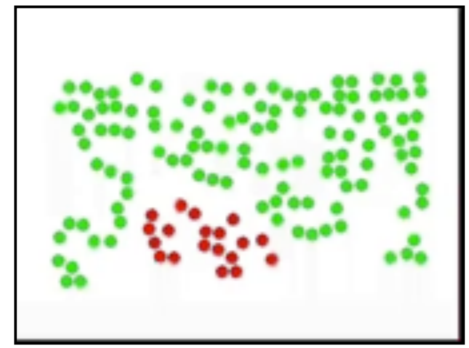
Applications in Computer Vision

Experiments on 3D Motion Segmentation

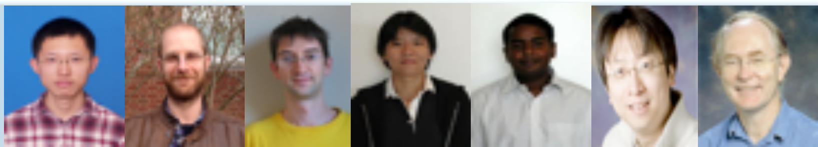
- Motion segmentation problem
  - Input: multiple images of a scene with multiple rigid-body motions
  - Output: number of motions, motion model parameters, segmentation

- Motion of a rigid-body: 4D subspace (Boult and Brown '91, Tomasi and Kanade '92)
  - P = #points
  - F = #frames

$$\underbrace{\begin{bmatrix} \boldsymbol{x}_{11} & \cdots & \boldsymbol{x}_{1P} \\ \vdots & \ddots & \vdots \\ \boldsymbol{x}_{F1} & \cdots & \boldsymbol{x}_{FP} \end{bmatrix}}_{2F \times P} = \underbrace{\begin{bmatrix} \boldsymbol{A}_1 \\ \vdots \\ \boldsymbol{A}_F \end{bmatrix}}_{2F \times 4} \underbrace{\begin{bmatrix} \boldsymbol{X}_1 & \cdots & \boldsymbol{X}_P \end{bmatrix}}_{4 \times P}$$

Vidal et al., ECCV02, IJCV06; Vidal, Ma and Sastry CVPR03, PAMI05; Vidal and Sastry CVPR03; Vidal and Ma ECCV04, JMIV06; Vidal and Hartley, CVPR04; Tron and Vidal, CVPR07; Li et al. CVPR07; Goh and Vidal CVPR07; Vidal and Hartley, PAMI08; Vidal, Tron and Hartley IJCV08; Rao et al. CVPR 08, PAMI 09; Elhamifar and Vidal, CVPR 09, TPAMI 13; Vidal SPM11; Tsakiris '15



Experiments on 3D Motion Segmentation

- 2 motions, 120 sequences, 266 points, 30 frames

	GPCA	LLMC	LSA	RANSAC	MSL	SCC	ALC	SSC
Checkerboard	6.09	3.96	2.57	6.52	4.46	1.30	1.55	**1.12**
Traffic	1.41	3.53	5.43	2.55	2.23	1.07	1.59	**0.02**
Articulated	2.88	6.48	4.10	7.25	7.23	3.68	10.70	**0.62**
All	4.59	4.08	3.45	5.56	4.14	1.46	2.40	**0.82**

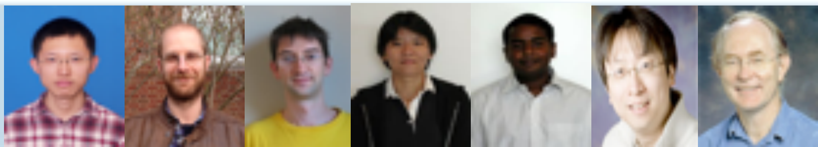
- 3 motions, 35 sequences, 398 points, 29 frames

	GPCA	LLMC	LSA	RANSAC	MSL	SCC	ALC	SSC
Checkerboard	31.95	8.48	5.80	25.78	10.38	5.68	5.20	**2.97**
Traffic	19.83	6.04	25.07	12.83	1.80	2.35	7.75	**0.58**
Articulated	16.85	9.38	7.25	21.38	2.71	10.94	21.08	**1.42**
All	28.66	8.04	9.73	22.94	8.23	5.31	6.69	**2.45**

- All

	GPCA	LLMC	LSA	RANSAC	MSL	SCC	ALC	LRR	LRSC	SSC
All	10.34	4.97	4.94	9.76	5.03	2.33	3.37	3.16	3.28	**1.24**

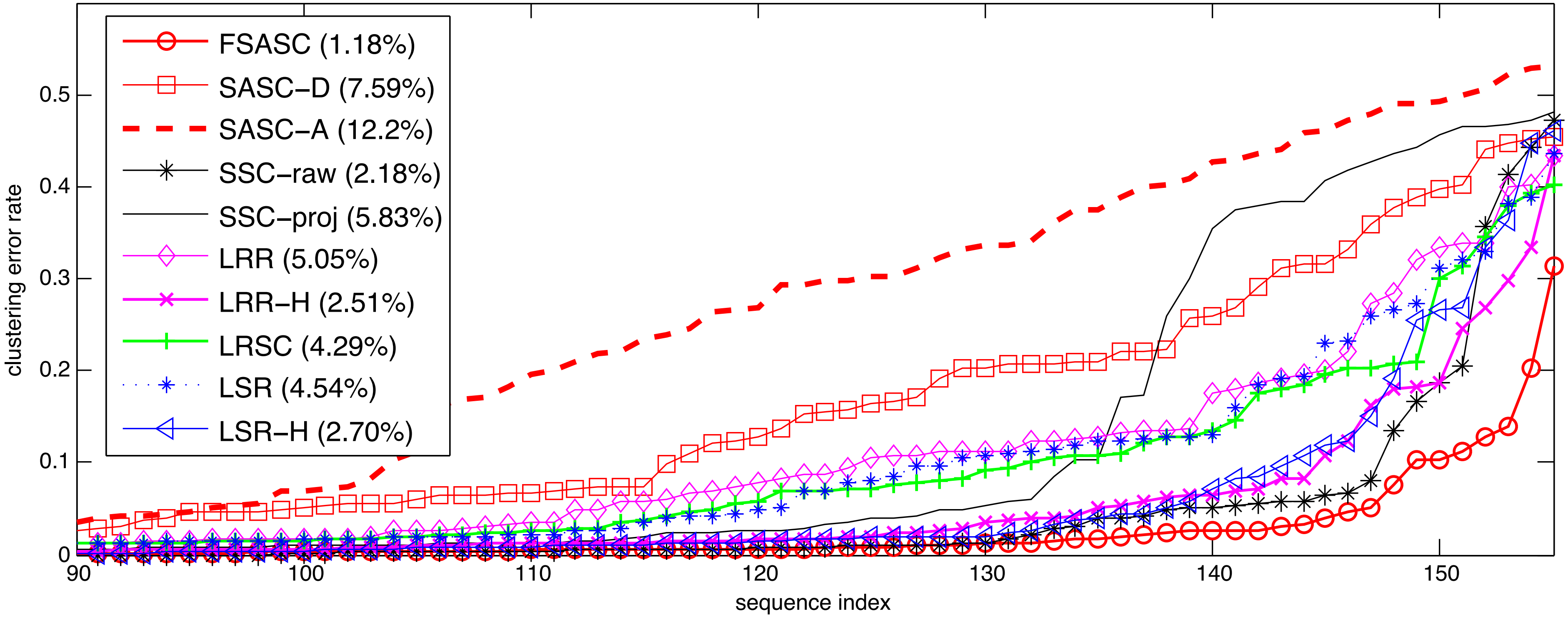
Vidal et al., ECCV02, IJCV06; Vidal, Ma and Sastry CVPR03, PAMI05; Vidal and Sastry CVPR03; Vidal and Ma ECCV04, JMIV06; Vidal and Hartley, CVPR04; Tron and Vidal, CVPR07; Li et al. CVPR07; Goh and Vidal CVPR07; Vidal and Hartley, PAMI08; Vidal, Tron and Hartley IJCV08; Rao et al. CVPR 08, PAMI 09; Elhamifar and Vidal, CVPR 09, TPAMI 13; Vidal SPM11; Tsakiris '15



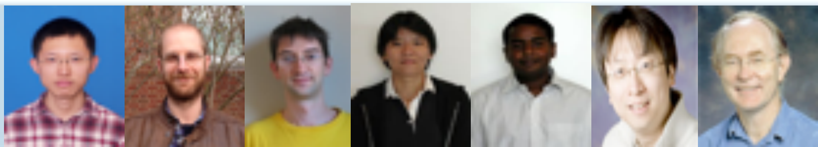
Experiments on 3D Motion Segmentation

- Misclassification rates on Hopkins 155 database

R. Tron and R. Vidal. A Benchmark for the Comparison of 3-D Motion Segmentation Algorithms. CVPR 2007.

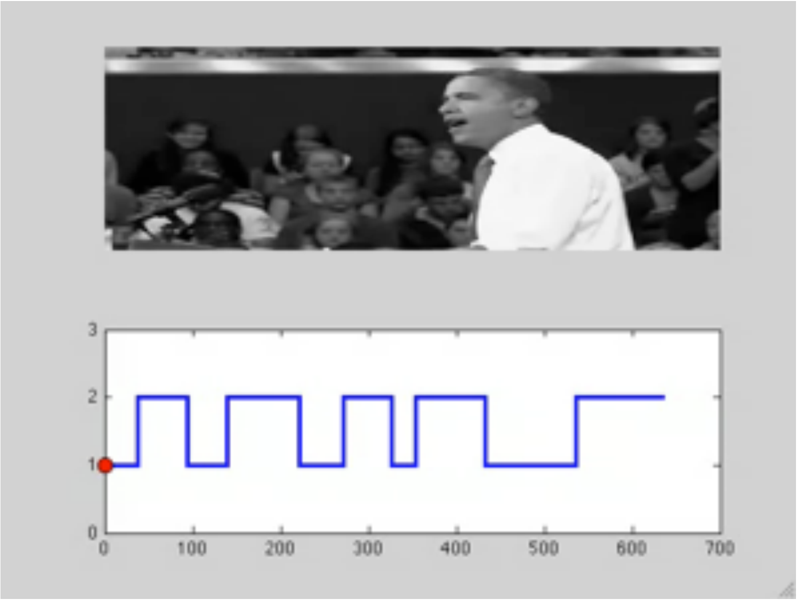


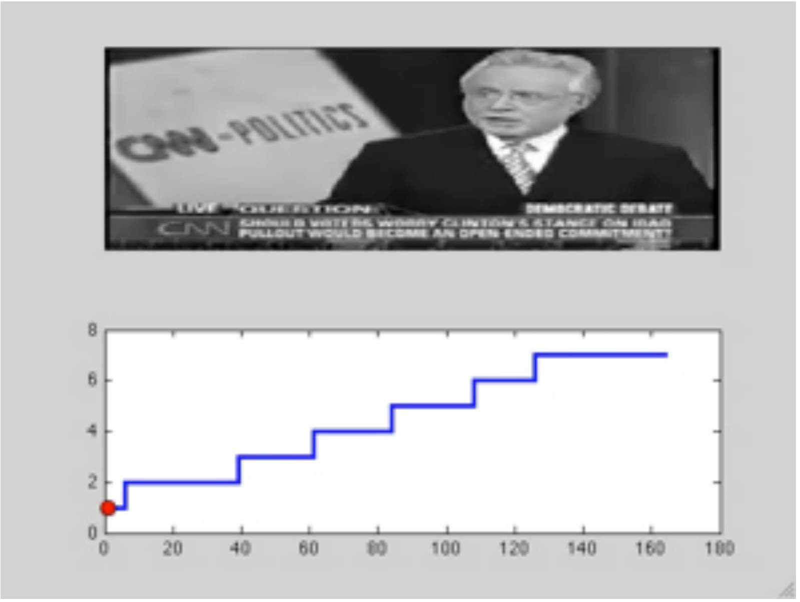
Vidal et al., ECCV02, IJCV06; Vidal, Ma and Sastry CVPR03, PAMI05; Vidal and Sastry CVPR03; Vidal and Ma ECCV04, JMIV06; Vidal and Hartley, CVPR04; Tron and Vidal, CVPR07; Li et al. CVPR07; Goh and Vidal CVPR07; Vidal and Hartley, PAMI08; Vidal, Tron and Hartley IJCV08; Rao et al. CVPR 08, PAMI 09; Elhamifar and Vidal, CVPR 09, TPAMI 13; Vidal SPM11; Tsakiris '15



Experiments on Video Segmentation

- Model each video segment as a low-dimensional subspace
- Cluster video frames into multiple segments

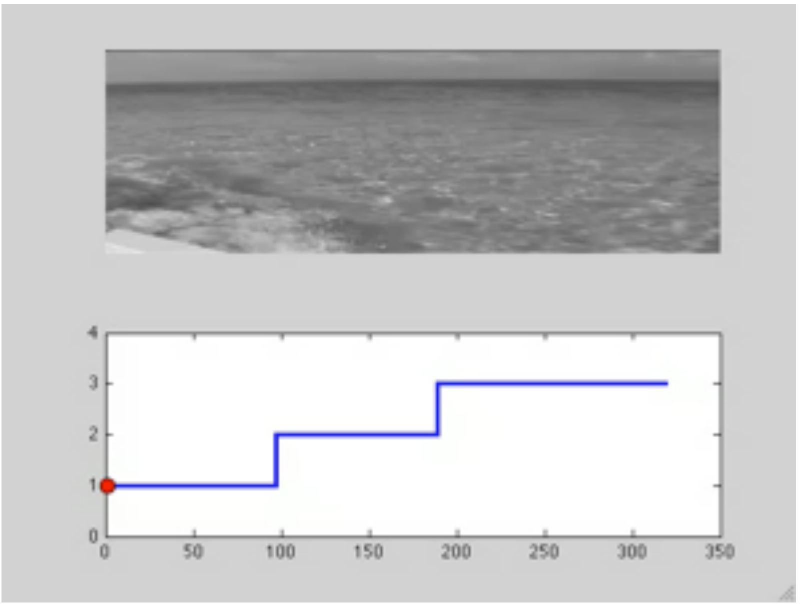


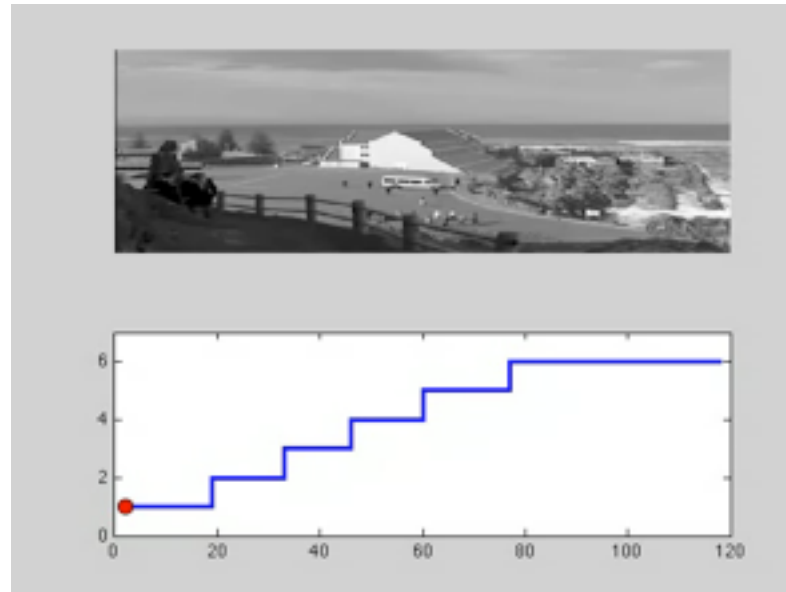


- Advantages
  - SSC easily detects sharp transitions in the video
  - SSC can handle camera motion and scene variations

Experiments on Video Segmentation

- Model each video segment as a low-dimensional subspace
- Cluster video frames into multiple segments

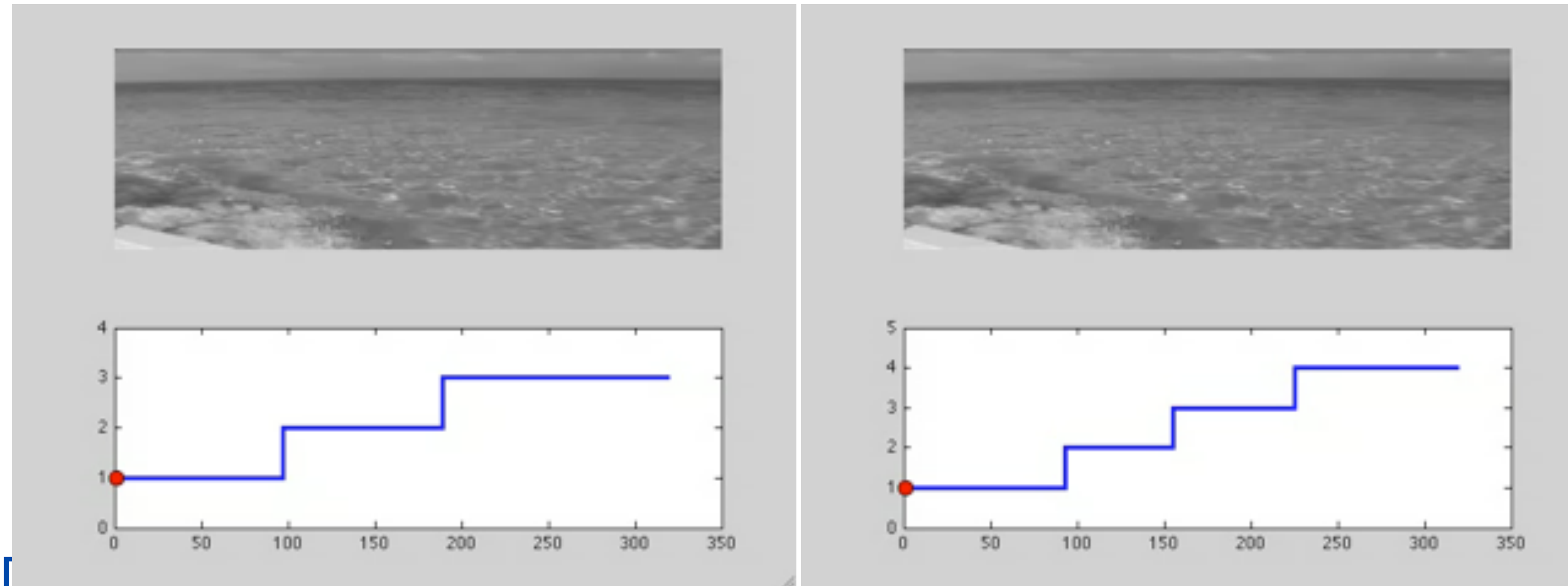




- Advantages
  - SSC easily detects sharp transitions in the video
  - SSC can handle camera motion and scene variations

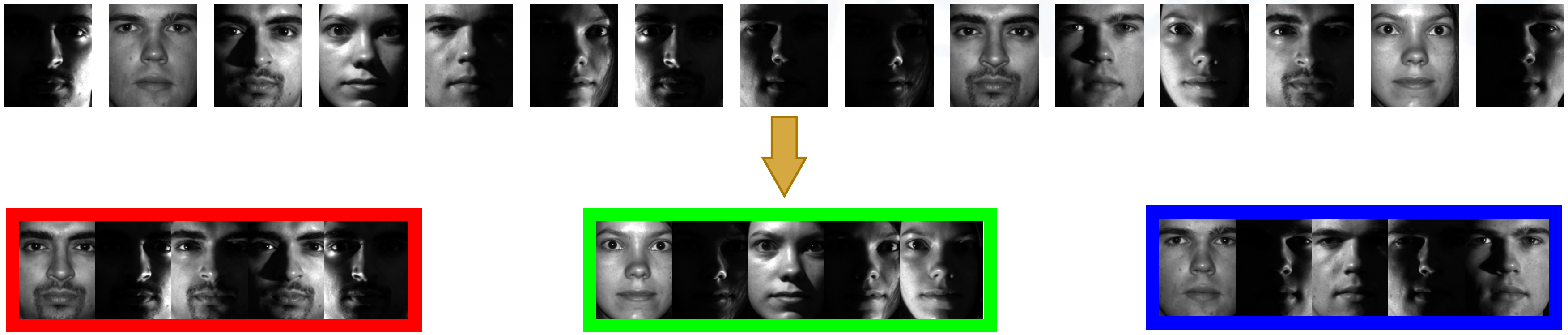
Experiments on Video Segmentation

- Model each video segment as a low-dimensional subspace
- Segment the video into multiple segments

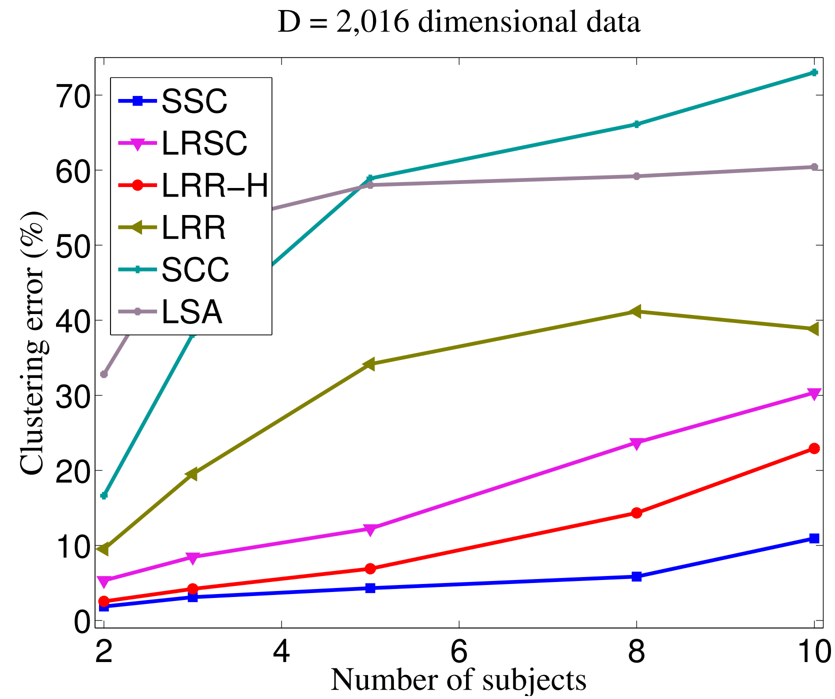


- Disadvantages
  - The segmentation depends on the number of subspaces
  - Continuous camera motion is not well handled

Experiments on Face Clustering

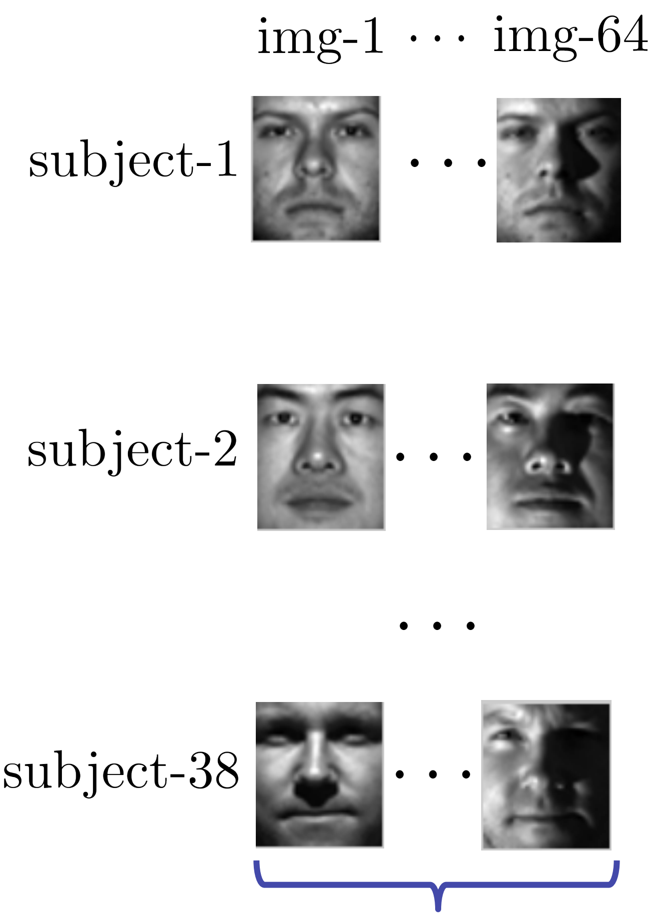


- Faces under varying illumination
  - 9D subspace
- Extended Yale B dataset
  - 38 subjects
  - 64 images per subject
- Clustering error
  - SSC < 2.0% error for 2 subjects
  - SSC < 11.0% error for 10 subjects



E. Elhamifar and R. Vidal, Sparse Subspace Clustering: Algorithm, Theory, and Applications, TPAMI13.

Experiment on extended Yale B



No. subjects	2	10	20	30	38
a%: average clustering accuracy					
SSC-OMP	99.21	88.43	**81.71**	**79.27**	**80.45**
SSC-BP	**99.45**	**91.85**	79.80	76.10	68.97
LSR	96.77	62.89	67.17	67.79	63.96
LRSC	94.32	66.98	66.34	67.49	66.78
SCC	78.91	NA	NA	14.15	12.80
t(sec.): running time					
SSC-OMP	0.3	1.7	4.7	9.4	**14.5**
SSC-BP	49.1	228.2	554.6	1240	1851
LSR	**0.1**	**0.8**	**3.1**	**8.3**	15.9
LRSC	1.1	1.9	6.3	14.8	26.5
SCC	50.0	NA	NA	520.3	750.7

$>$ 100 times faster

Experiment on MNIST

 . . . 

No. points	500	2,000	6,000	20,000	60,000
a%: average clustering accuracy					
SSC-OMP	**85.17**	**88.99**	**90.56**	**94.21**	**94.68**
SSC-BP	83.01	85.58	85.60	-	-
LSR	75.84	78.09	79.91	-	-
LRSC	75.02	79.44	79.88	-	-
SCC	53.45	66.43	70.60	-	-
t(sec.): running time					
SSC-OMP	**1.3**	**11.7**	**71.7**	**427**	**3219**
SSC-BP	20.1	635.2	13605	-	-
LSR	1.7	42.4	327.6	-	-
LRSC	1.9	43.0	312.9	-	-
SCC	31.2	101.3	366.8	-	-

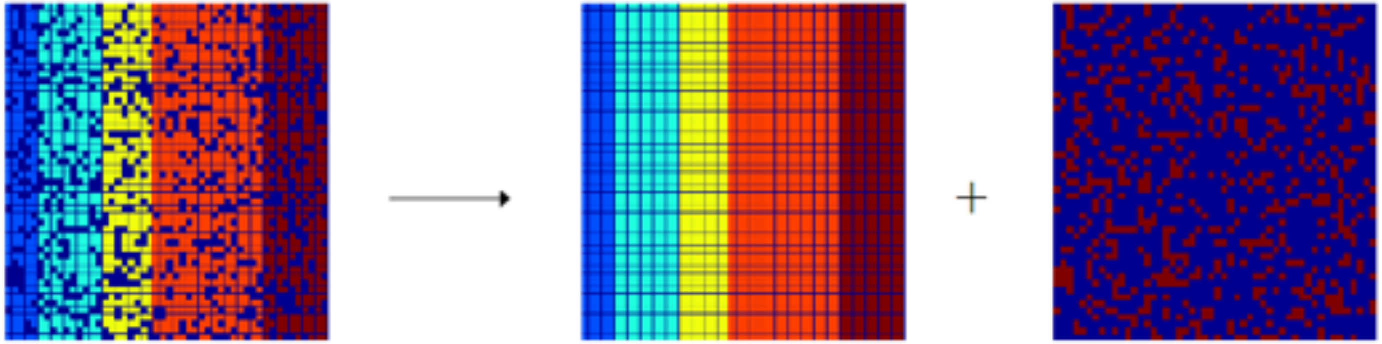
What's Next

- Big Data (Peng '13, Dyer '13, You '15)

	GPCA	SSC	OMP	?
Dimension of the data	10	10,000	10,000	1M
Number of data points	1000	10,000	100,000	1M

Chong You

- Missing Data: (Grubber '04, Eriksson '12, Balzano '12, Pimentel '14, Candes '14, Yang'15)



Matrix of corrupted observations *Underlying low-rank matrix* *Sparse error matrix*

Congyuan Yang

Conclusions

- Many problems in computer vision can be posed as subspace clustering and classification problems
  - Spatial and temporal video segmentation
  - Face clustering under varying illumination
  - Face classification
- These problems can be solved using
  - Generalized Principal Component Analysis (GPCA)
  - Sparse Subspace Clustering (SSC)
  - Low Rank Subspace Clustering (LRSC)
- This algorithms is provably correct when
  - Subspaces are sufficiently separated
  - Data are well distributed within each subspace

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Thank You!