Evaluación Final 23/07/20

Pregunta 1

1.1) $\boxed{r = a(1+\cos\theta)}$: $\dot r = -a\,\text{sen}\theta\,\dot\theta$ ($\Rightarrow \dot r(0) = 0$) : $\ddot r = -a\cos\theta\,\dot\theta^2 - a\,\text{sen}\theta\,\ddot\theta$ (i)

$\ell = m r^2\dot\theta = m\,2a v_0$: $\boxed{r^2\dot\theta = 2a v_0}$ (ii); $\dot\ell = 0 : \ddot\theta = -2\frac{\dot r}{r}\dot\theta = \frac{2a\,\text{sen}\theta}{r}\dot\theta^2$ (iii)

($\dot r(0) = 0$, velocidad inicial $\perp \vec r$)

$$F = m(\ddot r - r\dot\theta^2) \overset{(i)}{=} m(-a\cos\theta\,\dot\theta^2 - a\,\text{sen}\theta\,\ddot\theta - r\dot\theta^2) \overset{(iii)}{=} m\Big(-a\cos\theta\,\dot\theta^2 - \frac{2(a\,\text{sen}\theta)^2}{\underset{=a(1+\cos\theta)}{\boxed{r}}}\dot\theta^2 - \boxed{r}\dot\theta^2\Big)$$

$$F = -m\underbrace{\dot\theta^2}_{\overset{(ii)}{\left(\frac{2av_0}{r^2}\right)^2}} a\left[\cos\theta + \frac{2\,\text{sen}^2\theta}{1+\cos\theta} - (1+\cos\theta)\right]$$

$$F = -\frac{4ma^3v_0^2}{r^4}\Bigg[\underbrace{\cancel{\cos\theta} + \frac{2\overbrace{(1-\cos^2\theta)}^{(1+\cos\theta)(1-\cancel{\cos\theta})}}{\cancel{(1+\cos\theta)}} + (1+\cancel{\cos\theta})}_{3}\Bigg] = \boxed{-\frac{\overbrace{12ma^3v_0^2}^{K}}{r^4} = -\frac{K}{r^4}}$$

1.2, versión 1) tiempo de 0 a π:

$\frac{\ell}{m} = r^2\dot\theta = r^2\frac{d\theta}{dt}$: $dt = \frac{r^2 d\theta}{(\ell/m)} = \frac{r^2 d\theta}{2av_0}$;

$r(\pi) = 0$ (para $\theta = \pi$ alcanza el origen) $\Rightarrow t^* = \int_0^\pi \frac{r^2 d\theta}{2av_0} = \frac{a}{2v_0}\underbrace{\int_0^\pi (1+\cos\theta)^2 d\theta}_{\frac{3\pi}{2}}$

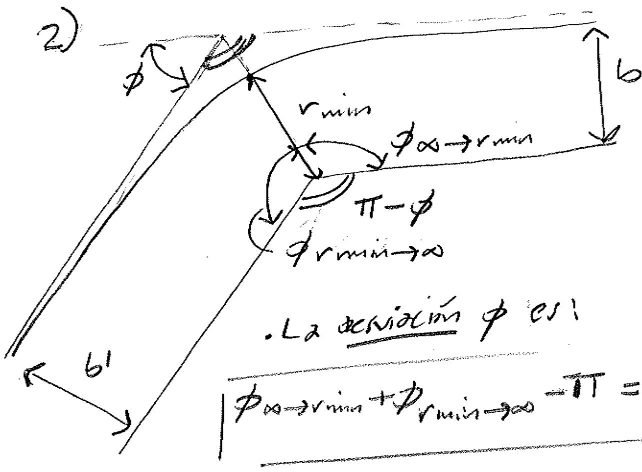
$$\Rightarrow \boxed{t^* = \frac{3\pi a}{4v_0}}$$

1.2 versión 2) movimiento circular: $\underbrace{F(2a)}_{\frac{-K}{(2a)^4}} = -m(2a)\underset{\left(\frac{v_0}{2a}\right)^2}{\boxed{\dot\theta^2}}$: $\boxed{v_0^2 = \frac{K}{8ma^3}}$

$\forall t\ r = 2a : \dot r = 0, \ddot r = 0$

$$T = 2\pi/\dot\theta = 2\pi\frac{2a}{v_0} = 2\pi\sqrt{\frac{8ma^3}{K}(2a)^2} = 2\pi\sqrt{\frac{32ma^5}{K}}$$

2)



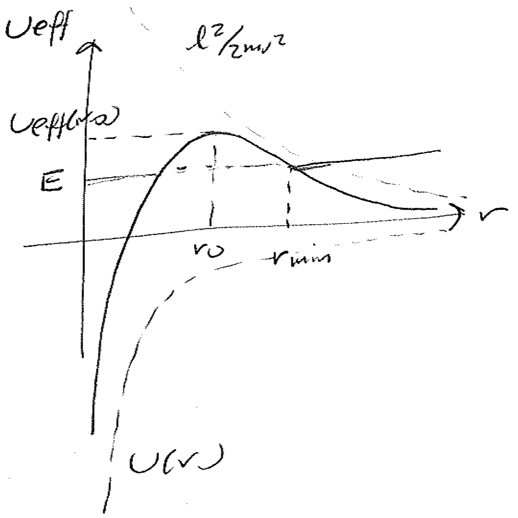
. La deviación ϕ es:

$$\boxed{\phi_{\infty\to r_{min}} + \phi_{r_{min}\to\infty} - \pi = \phi}$$

$\ell = mbv_0$, $E = \frac{1}{2}mv_\infty^2$

$$E = \frac{1}{2}m\dot r^2 + \frac{\ell^2}{2mr^2} + \underbrace{U(r)}_{= -\frac{K}{3r^3}} \quad \left(\vec F = -\frac{K}{r^4}\hat{e_r}\right)$$

$$\boxed{E = \frac{1}{2}m\dot r^2 + \underbrace{\frac{\ell^2}{2mr^2} - \frac{K}{3r^3}}_{U_{eff}(r)}} \ \text{(iii)}$$



Para que la partícula no alcance el origen (viniendo desde el infinito) se debe cumplir $E \le U_{eff}(r_0)$, en cuyo caso se aproxima hasta $r = r_{min}$, siendo r_{min} la raíz más grande de $E - U_{eff}(r) = 0$:

$$\boxed{\frac{1}{2} m v_\infty^2 - \frac{\ell^2}{2mr^2} + \frac{K}{3r^3} = 0}$$

Para cumplir $E \le U_{eff}(r_0)$: $r_0 \,/\, \frac{dU_{eff}}{dr} = 0$: $\frac{-\ell^2}{mr^3} + \frac{K}{r^4} = 0$; $\boxed{r_0 = \frac{mK}{\ell^2}}$

$$\Rightarrow U_{eff}(r_0) = \frac{\ell^2}{2m\left(\frac{mK}{\ell^2}\right)^2} - \frac{K}{3\left(\frac{mK}{\ell^2}\right)^3} = \frac{\ell^6}{6m^3K^2}$$

$$\Rightarrow \frac{1}{2} m v_\infty^2 \le \ell^6 / 6m^3K^2 = (mbv_\infty)^6 / 6m^3K^2 \; ; \; \boxed{v_\infty \ge \left(\frac{3K^2}{m^2 b^6}\right)^{1/4}}$$

- Recorrido angular:

$$\frac{d\theta}{dr} = \frac{d\theta}{dt}\frac{dt}{dr} = \dot\theta / \dot r = \frac{(\ell/mr^2)}{\dot r} \overset{(iii)}{=} \frac{(\ell/mr^2)}{\pm\sqrt{2/m\,(E - U_{eff})}} :$$

↳ consideremos un tramo con signo definido, por ejemplo de ∞ a r_{min} : $\dot r < 0$

$$d\theta = \frac{(\ell/mr^2)\,dr}{-\sqrt{2/m\left(1/2\, m v_\infty^2 - \ell^2/2mr^2 + K/3r^3\right)}} :$$

$$\phi_{\infty \to r_{min}} = \int_0^{\phi_{\infty\to r_{min}}} d\theta = -\int_\infty^{r_{min}} \frac{\left(\overset{mbv_\infty}{\boxed{\ell}}/\cancel{m}r^2\right) dr}{\sqrt{2/m\,(E - U_{eff}(r))}} = \sqrt{\frac{m}{2}}\, b v_\infty \int_{r_{min}}^\infty (\ldots)$$

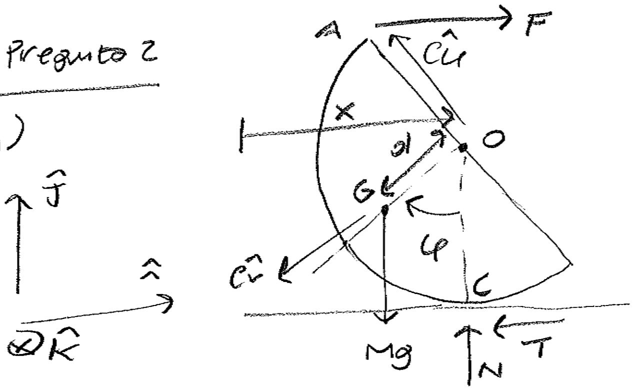
La expresión para $\phi_{r_{min}\to\infty}$ es la misma (tomo la raíz + de $\dot r$ e integro entre r_{min} e ∞)

$$\Rightarrow \phi_{r_{min}\to\infty} = \phi_{\infty\to r_{min}} = \sqrt{\frac{m}{2}}\, b v_\infty \int_{r_{min}}^\infty \frac{dr}{r^2\sqrt{(E - U_{eff})}}$$

$$\Rightarrow \boxed{\phi = 2\sqrt{\frac{m}{2}}\, b v_\infty \int_{r_{min}}^\infty \frac{dr}{r^2\sqrt{E - U_{eff}}}}$$

Pregunta 2

1)



Según la versión de la pregunta tenemos un semi-aro o un semi-disco, para los cuales: semi-aro: $\begin{cases} d = 2R/\pi \\ I_O = MR^2 \end{cases}$

semi-disco: $\begin{cases} d = 4R/3\pi \\ I_O = MR^2/2 \end{cases}$

(Vamos a considerar una solución genérica y sustituir al final de cada parte.)

Rodadura sin deslizamiento: $\vec{v}_C = 0$, $\vec{v}_O = \dot{x}\hat{\imath}$ $\Big| \Rightarrow \dot{x}\hat{\imath} = \dot{\varphi}\hat{k}\times(O-C) = R\dot{\varphi}\hat{\imath} : \boxed{\dot{x} = R\dot{\varphi}}$

(vale mientras: $|T| \leq f|N|$)

vínculo entre x y φ, $\rightarrow \boxed{x = R\varphi} + (x_0 - R\varphi_0)$

el rígido tiene 1 grado de libertad mientras r.s.d.

T, N: potencia nula, pero conservativa, vamos a ver que F es conservativa para probar que la energía se conserva y hallar la ecuación de movimiento:

$$\mathcal{P}_F = \vec{F}\cdot\vec{v}_A = \vec{F}\cdot(\underbrace{\vec{v}_O}_{\dot{x}\hat{\imath}} + \dot{\varphi}\hat{k}\times\underbrace{(A-O)}_{R\hat{e}_\varphi}) = \underset{F\hat{\imath}}{\vec{F}}\cdot(\dot{x}\hat{\imath} - R\dot{\varphi}\hat{e}_r) = F(\dot{x} + R\dot{\varphi}\,\mathrm{sen}\varphi)$$

$$\mathcal{P}_F = FR(\dot{x} + \dot{\varphi}\,\mathrm{sen}\varphi) : \quad W_F = F(x - R\cos\varphi)(+cte.)$$

: sólo depende de las coordenadas x, φ (integro en el tiempo) $\Rightarrow$ F cons.: $U_F = -W_F$

$$U_F = -F(x - R\cos\varphi) \underset{x = R\varphi}{=} -FR(\varphi - \cos\varphi)$$

$$\Rightarrow E = T + U_g + U_F = cte \;;\; U_g = -Mgd\cos\varphi \;,\; T = \frac{1}{2}I_C\dot{\varphi}^2$$

$I_C \overset{(Steiner)}{=} I_G + M(G-C)^2$

$I_G = I_O - Md^2$ (Steiner)

$$T = \frac{1}{2}[I_O - Md^3 + M\underbrace{(G-C)}_{(G-O)+(O-C) = d\hat{e}_r + R\hat{j}}{}^2]\dot{\varphi}^2$$

$$T = \frac{1}{2}[I_O - \cancel{Md^2} + M(\cancel{d^2} + R^2 - 2dR\cos\varphi)]\dot{\varphi}^2$$

$$T = \frac{1}{2}[I_O + MR^2 - 2MdR\cos\varphi]\dot{\varphi}^2$$

$$\boxed{E = \frac{1}{2}[I_O + MR^2 - 2MdR\cos\varphi]\dot{\varphi}^2 - Mgd\cos\varphi - FR(\varphi - \cos\varphi)} = cte.$$

derivando en el tiempo:

$$\boxed{\dot{\varphi}[I_O + MR^2 - 2MdR\cos\varphi] + MdR\,\mathrm{sen}\varphi\,\dot{\varphi}^2 + Mgd\,\mathrm{sen}\varphi - FR(1 + \mathrm{sen}\varphi) = 0} \quad (I)$$

Vale mientras $|T| \leq f|N|$;

T, N : $\vec{a}_G$: $\vec{r}_G = x\hat{\imath} + d\hat{e}_r$: $\dot{\vec{r}}_G = \dot{x}\hat{\imath} + d\dot{\varphi}\hat{e}_\varphi$: $\ddot{\vec{r}}_G = \ddot{x}\hat{\imath} + d\ddot{\varphi}\hat{e}_\varphi - d\dot{\varphi}^2\hat{e}_r$

$\Rightarrow \vec{a}_G = \textcircled{$\ddot{x}$}\,\hat{\imath} + d\ddot{\varphi}\hat{e}_\varphi - d\dot{\varphi}^2\hat{e}_r$; $\boxed{\vec{a}_G(0) = (R-d)\ddot{\varphi}\,\hat{\imath}}$

$R\ddot{\varphi}$ (mientras r/d) $\qquad (\dot{\varphi}(0) = 0,\ \hat{e}_\varphi(0) = -\hat{\imath})$

Primera Cardinal en $t \approx 0$: $\left[\begin{array}{lll} \hat{\imath}) & M(R-d)\ddot{\varphi}(0) = F - T & \text{(i)} \\ \hat{\jmath}) & 0 = N - Mg : N \approx Mg & \text{(ii)} \end{array}\right.$

Evaluando (I) en $t \approx 0$ $(\varphi \approx 0, \dot{\varphi} \approx 0)$: $\ddot{\varphi}(0)\left[I_O + MR^2 - 2MdR\right] \approx FR$

$$\overset{\text{(i)}}{\Rightarrow} \quad M(R-d)\frac{FR}{(I_O + MR^2 - 2MdR)} = F - T : T = \left(\frac{I_O - MdR}{I_O + MR^2 - 2MdR}\right)F$$

$$\Rightarrow \quad |T| \leq f|N| = fMg \longleftrightarrow \left|\frac{I_O - MdR}{I_O + MR^2 - 2MdR}\right| F \leq fMg :$$

$$f \geq \left(\frac{F}{Mg}\right)\left|\frac{I_O - MdR}{I_O + MR^2 - 2MdR}\right|$$

2) El rígido se desliza con respecto al piso, $P_T \neq 0$: la energía no se conserva :

debo trabajar por cardinales ; $\vec{a}_G(0) \approx [\ddot{x}(0) - d\ddot{\varphi}(0)]\hat{\imath}$

Primera Cardinal : $\left\{\begin{array}{ll} \hat{\imath}) & \boxed{M(\ddot{x}(0) - d\ddot{\varphi}(0)) = F - T} \text{ (iii)} \\ \hat{\jmath}) & N \approx Mg \end{array}\right.$

(podemos verificar luego que $\vec{v}_c(t \approx 0)\cdot\hat{\imath} > 0$, lo que confirma que el sentido supuesto para T es correcto)

Segunda Cardinal desde O :

$$M\underbrace{(G-O)}_{d\hat{e}_r(0) = -d\hat{\jmath}} \times \underbrace{\textcircled{$\vec{a}_O$}}_{\ddot{x}(0)\hat{\imath}} \cdot \hat{k} + I_O\ddot{\varphi} = R\,\textcircled{T} = fMgR$$

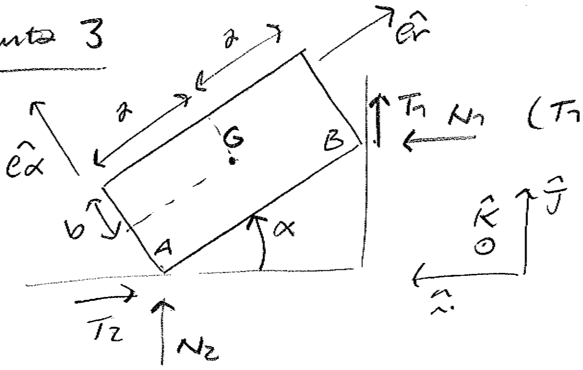
$$\textcircled{T} = fN = fMg$$

$-Md\ddot{x}(0) + I_O\ddot{\varphi}(0) = fMgR$ (iv) ; despejando $\ddot{\varphi}(0)$ entre (iii) y (iv) :

$$\boxed{\ddot{\varphi}(0) = \frac{f(R-d)Mg + Fd}{I_O - Md^2}}$$

Pregunta 3

1)



$(T_1 = fN_1)$

Para que la placa se mantenga en equilibrio (respecto al piso), se debe cumplir:

$$\begin{cases} N_1 \geq 0 & (I) \\ N_2 \geq 0 & (II) \\ T_2 \leq fN_2 & (III) \end{cases}$$

Primera Cardinal :
$$\begin{cases} \hat{i}) \quad T_2 = N_1 & (i) \\ \hat{j}) \quad T_1 + N_2 = Mg & (ii) \end{cases}$$

Segunda Cardinal desde A: $0 = 2a\cos\alpha \underset{fN_1}{T_1} + 2a\,\text{sen}\alpha\, N_1 + b\,\text{sen}\alpha\, mg - a\cos\alpha\, mg$ (iii)

$\overset{(iii)}{\Rightarrow} N_1 = \frac{mg}{2}\,\frac{1 - b/a\, tg\alpha}{f + tg\alpha}$: se cumple (I) si $\boxed{tg\alpha \leq a/b}$ (Condición de no desprendimiento de la cinta)

Luego $N_2 \overset{(iii)}{=} mg - fN_1 = mg\left(1 - \frac{f}{2}\left(\frac{1 - b/a\, tg\alpha}{f + tg\alpha}\right)\right) = mg\left[\frac{f + (2 + f b/a)\,tg\alpha}{2(f + tg\alpha)}\right] > 0$

(II) ✓

(III) $\underset{\underset{(i)}{= N_0}}{T_2} \leq fN_2$: $\cancel{\frac{mg}{2}}\left(\frac{1 - b/a\, tg\alpha}{\cancel{f + tg\alpha}}\right) \leq f\,\cancel{\frac{mg}{2}}\left[\frac{f + (2 + f\, b/a)\,tg\alpha}{\cancel{(f + tg\alpha)}}\right]$:

$$\boxed{tg\alpha \geq \frac{1 - f^2}{2f + (1 + f^2)\,b/a}}$$

(Condición de no deslizamiento con respecto al piso)

2) $b = a$: el equilibrio se mantendría si $\frac{\overset{(\cancel{1+f})(1-f)}{1 - f^2}}{\underset{(1+f)^{\cancel{2}}}{2f + 1 + f^2}} \leq tg\alpha \leq 1$: $\frac{1-f}{1+f} \leq tg\alpha \leq 1$

Para $\alpha_0 = \pi/6$, $f = \overset{\text{(versión 1)}}{1/3\sqrt{3}}$ o $\overset{\text{(versión 2)}}{1/5}$, no se cumple la condición de no deslizamiento : la placa parte del reposo manteniéndose en contacto con la cinta y deslizándose hacia la izq. (se verifica con el sg de $\ddot{\alpha}$)

Debemos ahora hallar la aceleración de G en términos de $\ddot{\alpha}$ para poder plantear la Primera Cardinal a la placa cuadrada:

$$\vec{a}_G(0) = \underbrace{\vec{a}_A(0)}_{} + \underset{\ddot{\alpha}\hat{k}}{\dot{\vec{\omega}}} \times \underbrace{(G - A)}_{a(\hat{e}_r + \hat{e}_\alpha)} \qquad (\vec{\omega}(0) = 0)$$

$\hookrightarrow \ddot{x}_A(0)\,\hat{i}$; $x_A = 2a\cos\alpha$: $\dot{x}_A = -2a\,\text{sen}\alpha\,\dot{\alpha}$: $\ddot{x}_A = -2a\cos\alpha\,\dot{\alpha}^2 - 2a\,\text{sen}\alpha\,\ddot{\alpha}$

$$\Rightarrow \boxed{\ddot{x}_A(0) = -2a\,\text{sen}\alpha_0\,\ddot{\alpha}(0)}$$

$$\Rightarrow \vec{a}_G(0) = -2a\,\text{sen}\alpha_0\,\ddot\alpha(0)\,\hat\imath + \ddot\alpha(0)\,a\,(\hat{e}_\alpha - \hat{e}_r) \qquad \left(\begin{array}{l}\hat{e}_r = -\cos\alpha\,\hat\imath + \text{sen}\alpha\,\hat\jmath \\ \hat{e}_\alpha = \text{sen}\alpha\,\hat\imath + \cos\alpha\,\hat\jmath\end{array}\right)$$

$$\boxed{\vec{a}_G(0) = \ddot\alpha(0)\,a\,(\cos\alpha_0 - \text{sen}\alpha_0)\,\hat\imath + \ddot\alpha(0)\,a\,(\cos\alpha_0 - \text{sen}\alpha_0)\,\hat\jmath}$$

- Primera Cardinal :

$$\begin{cases} \hat\imath)\ N_1 - T_2 = m a\,\ddot\alpha(0)(\cos\alpha_0 - \text{sen}\alpha_0)\ (iv), & T_2 = fN_2 \\ \hat\jmath)\ T_1 + N_2 - mg = m a\,\ddot\alpha(0)(\cos\alpha_0 - \text{sen}\alpha_0)\ (v), & T_1 = fN_1 \end{cases}$$

- Segunda Cardinal desde G :

$$\underset{\frac{2}{3}ma^2}{I_G}\,\ddot\alpha(0) = N_1 a(\text{sen}\alpha_0 - \cos\alpha_0) + T_1 a(\text{sen}\alpha_0 - \cos\alpha_0)$$
$$+ T_2 a(\text{sen}\alpha_0 - \cos\alpha_0) - N_2 a(\cos\alpha_0 - \text{sen}\alpha_0)$$

$$\frac{2}{3} m a\,\ddot\alpha(0) = (N_1 + N_2)\Big[(1+f)\,\text{sen}\alpha_0 - (1-f)\cos\alpha_0\Big]\ (vi)$$

$(v) - f(iv)$: $(1+f^2)N_2 - mg = (1-f)ma\,\ddot\alpha(0)(\cos\alpha_0 - \text{sen}\alpha_0)$:

$$\boxed{N_2 = \frac{ma(1-f)(\cos\alpha_0 - \text{sen}\alpha_0)\,\ddot\alpha(0) + mg}{(1+f^2)}}$$

$(iv) + f(v)$: $N_1(1+f^2) - fmg = (1+f)ma(\cos\alpha_0 - \text{sen}\alpha_0)\,\ddot\alpha(0)$:

$$\boxed{N_1 = \frac{ma(1+f)(\cos\alpha_0 - \text{sen}\alpha_0)\,\ddot\alpha(0) + fmg}{(1+f^2)}}$$

Sustituyendo en (vi) y despejando:

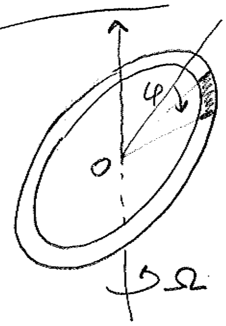
$$\boxed{\ddot\alpha(0) = \frac{(g/a)(1+f)\big[(1+f)\,\text{sen}\alpha - (1-f)\cos\alpha\big]}{\frac{2}{3}(1+f^2) - 2(\cos\alpha - \text{sen}\alpha)\big[(1+f)\,\text{sen}\alpha - (1-f)\cos\alpha\big]}}$$

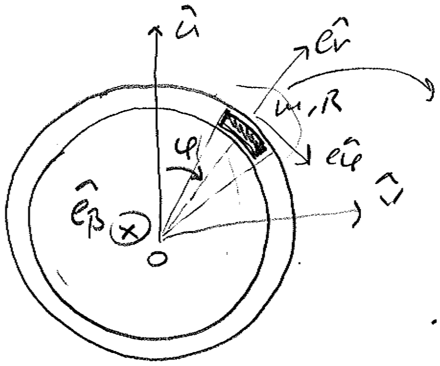
$$= \begin{cases} \approx -0.160\ g/a & f = 1/3\sqrt{3} \\ \approx -0.146\ g/a & f = 1/5 \end{cases}$$

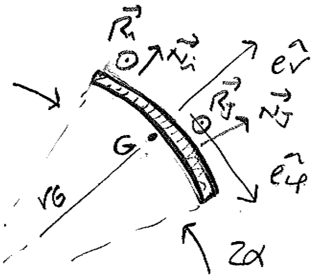
(y se verifica $N_2, N_1 > 0$ ✓)

Pregunta 4

1)







· Como el contacto entre el arco y el tubo es liso, las fuerzas que se ejercen mutuamente son en dirección radial o ⊥ plano del tubo :

$$\mathcal{F}: \left\{ P_i,\ \vec F_i(\varphi+\theta_i) = N_i(\varphi+\theta_i)\hat e_r(\varphi+\theta_i) + R_i(\varphi+\theta_i)\hat e_\beta \quad \theta_i \in [-\alpha,\alpha] \right\}$$

Ec. de movimiento :

A partir de la Segunda Cardinal al arco desde O :

$$\dot{\vec L}_O = \vec M_O^{(ext)} \quad (\dot O = 0)$$

proyectada según $\hat e_\beta$:

$$\dot{\vec L}_O \cdot \hat e_\beta = \vec M_O^{(ext)} \cdot \hat e_\beta = \left(\vec M_O^{(tubo)} + \vec M_O^{(peso)}\right)\cdot \hat e_\beta = \vec M_O^{(peso)} \cdot \hat e_\beta$$

$$\vec M_O^{(tubo)} \cdot \hat e_\beta = \sum_i R\hat e_r(\varphi+\theta_i) \times \left(N_i(\varphi+\theta_i)\hat e_r(\varphi+\theta_i) + R_i(\varphi+\theta_i)\hat e_\beta\right)\cdot \hat e_\beta = 0$$

$$\Rightarrow \boxed{\dot{\vec L}_O \cdot \hat e_\beta = \vec M_O^{(peso)} \cdot \hat e_\beta}$$

$$\bullet\ \vec M_O^{(peso)} = r_G \hat e_r \times (-mg\hat k) = -mgR\frac{\operatorname{sen}\alpha}{\alpha}\underbrace{(\hat e_r \times \hat k)\cdot \hat e_\beta}_{\underbrace{(\hat e_\beta \times \hat e_r)}_{\hat e_\varphi = -\operatorname{sen}\varphi\,\hat u + \cos\varphi\,\hat v}\cdot \hat k = -\operatorname{sen}\varphi\cos\beta} = mgR\frac{\operatorname{sen}\alpha}{\alpha}\cos\beta\operatorname{sen}\varphi$$

$$\bullet\ \vec L_O = \mathbb{I}_O \vec\omega \quad (\vec v_O = 0); \quad \{\hat e_r, \hat e_\varphi, \hat e_\beta\} \text{ es base principal para } \mathbb{I}_O$$

$$\vec\omega = \Omega\hat k + \dot\varphi\hat e_\beta = \Omega(\cos\beta\underbrace{\hat u}_{\cos\varphi\hat e_r - \operatorname{sen}\varphi\hat e_\varphi} - \operatorname{sen}\beta\,\hat e_\beta) + \dot\varphi\hat e_\beta$$

$$= \Omega\cos\beta\cos\varphi\,\hat e_r - \Omega\cos\beta\operatorname{sen}\varphi\,\hat e_\varphi + (\dot\varphi - \Omega\operatorname{sen}\beta)\hat e_\beta$$

$$\Rightarrow \mathbb{I}_O\vec\omega = \underbrace{I_{O,\hat e_r}}_{\frac{mR^2}{2}\left(1-\frac{\operatorname{sen}2\alpha}{2\alpha}\right)}\Omega\cos\beta\cos\varphi\,\hat e_r - \underbrace{I_{O,\hat e_\varphi}}_{\substack{I_{O,\hat e_\beta} - I_{O,\hat e_r} \\ = \frac{mR^2}{2}\left(1+\frac{\operatorname{sen}2\alpha}{2\alpha}\right)}}\Omega\cos\beta\operatorname{sen}\varphi + \underbrace{I_{O,\hat e_\beta}}_{mR^2}(\dot\varphi - \Omega\operatorname{sen}\beta)\hat e_\beta$$

$$\dot{\vec{\omega}} = -\Omega\cos\beta \,\mathrm{sen}\varphi\,\dot{\varphi}\,\hat{e_r} - \Omega\cos\beta\cos\varphi\,\dot{\varphi}\,\hat{e_\varphi} + \ddot{\varphi}\,\hat{e_\beta}$$

$$\Rightarrow \mathbb{I}_O\dot{\vec{\omega}}\cdot\hat{e_\beta} = mR^2\ddot{\varphi}$$

$$\vec{\omega}\times(\mathbb{I}_O\vec{\omega})\cdot\hat{e_\beta} = -\frac{mR^2}{2}\left(1+\frac{\mathrm{sen}\,2\alpha}{2\alpha}\right)(\Omega\cos\beta)^2\,\mathrm{sen}\varphi\cos\varphi + \frac{mR^2}{2}\left(1-\frac{\mathrm{sen}\,2\alpha}{2\alpha}\right)(\Omega\cos\beta)^2\cdot \mathrm{sen}\varphi\cos\varphi$$

$$= -mR^2\,\frac{\mathrm{sen}\,2\alpha}{2\alpha}(\Omega\cos\beta)^2\,\mathrm{sen}\varphi\cos\varphi$$

Luego, $\dot{\vec{L_O}} = \frac{d}{dt}(\mathbb{I}_O\vec{\omega}) = \mathbb{I}_O\dot{\vec{\omega}} + \vec{\omega}\times\mathbb{I}_O\vec{\omega} \Rightarrow$

$$\Rightarrow \dot{\vec{L_O}}\hat{e_\beta} = mR^2\ddot{\varphi} - mR^2\underbrace{\frac{\mathrm{sen}\,2\alpha}{2\alpha}}_{\frac{\cancel{2}\,\mathrm{sen}\alpha\cos\alpha}{\cancel{2}\alpha}}(\Omega\cos\beta)^2\,\mathrm{sen}\varphi\cos\varphi = mgR\,\frac{\mathrm{sen}\,\alpha}{\alpha}\cos\beta\,\mathrm{sen}\varphi$$

$$\Rightarrow \boxed{\ddot{\varphi} - \left[\Omega^2\cos\alpha\cos\beta\cos\varphi + g/R\right]\frac{\mathrm{sen}\alpha}{\alpha}\cos\beta\,\mathrm{sen}\varphi = 0}$$

2) equilibrio y estabilidad: (versión 1)

eq: $\frac{dU_{eff}}{d\varphi} = -\left[\Omega^2\cos\alpha\cos\beta\cos\varphi + g/R\right]\frac{\mathrm{sen}\alpha}{\alpha}\cos\beta\,\mathrm{sen}\varphi = 0 = \begin{cases} \varphi = 0 \\ \varphi = \pi \\ \cos\varphi_{eq} = \dfrac{-g/R}{\Omega^2\cos\alpha\cos\beta} \end{cases}$

$\left(\exists \text{ si } \Omega^2 \geq \dfrac{g/R}{\cos\alpha\cos\beta}\right)$

estabilidad:

$\frac{d^2U_{eff}}{d\varphi^2}$: $\begin{cases} <0 \quad \varphi = 0 \quad (\text{inestable}) \\ (-\Omega^2\cos\alpha\cos\beta + g/R)\frac{\mathrm{sen}\alpha}{\alpha}\cos\beta > 0 \text{ si } \underbrace{\Omega^2 < \dfrac{g/R}{\cos\alpha\cos\beta}}_{\varphi=\pi \text{ estable si se cumple esta condición}} \\ \Omega^2\cos\alpha\cos^2\beta\left(\frac{\mathrm{sen}\alpha}{\alpha}\right)\mathrm{sen}^2\varphi_{eq} > 0 : \varphi_{eq} \text{ es estable mientras exista} \end{cases}$

2) preintegro la ec. de movimiento: (versión 2)

$$\frac{1}{2}\dot{\varphi}^2 - \left[\Omega^2\cos\alpha\cos\beta\,\frac{1}{2}\mathrm{sen}^2\varphi - g/R\cos\varphi\right]\frac{\mathrm{sen}\alpha}{\alpha}\cos\beta = \frac{1}{2}\dot{\varphi}_0^2 - g/R\,\frac{\mathrm{sen}\alpha}{\alpha}\cos\beta$$

$\Omega^2 = g/(R\cos\alpha\cos\beta)$:

$$\frac{1}{\left(\frac{g}{R}\frac{\mathrm{sen}\alpha}{\alpha}\cos\beta\right)}\left(\frac{1}{2}\dot{\varphi}^2\right) = \frac{1}{2}\mathrm{sen}^2\varphi - (1+\cos\varphi) + \frac{1}{2}\dot{\varphi}_0^2\,\frac{1}{\left(\frac{g}{R}\frac{\mathrm{sen}\alpha}{\alpha}\cos\beta\right)} > 0 \quad \forall\varphi\in[0,2\pi)$$

para dar una vuelta completa

$$\Rightarrow \frac{1}{2}\underbrace{\dot{\varphi}_0^2}_{(v_0/R)^2}\Big/\left(g/R\,\frac{\mathrm{sen}\alpha}{\alpha}\cos\beta\right) > \text{máx}\left(1+\cos\varphi - \frac{1}{2}\mathrm{sen}^2\varphi\right) = 2 :$$

$$\boxed{v_0^2 > 4gR\,\frac{\mathrm{sen}\alpha}{\alpha}\cos\beta}$$