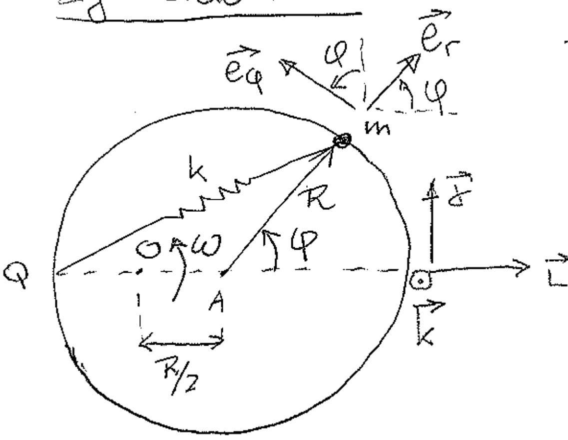
Ejercicio Nº 1

Parcial Mecánica Newtoniana (1122) 5/5/2018

1/4



parte a: $m\vec{a} = \vec{F} = -k(P-Q)$

punto de masa m

$$m\vec{a}\cdot\vec{e}_\varphi = -k(P-Q)\cdot\vec{e}_\varphi$$

$$P-Q = \underbrace{P-A}_{R\vec{e}_r} + \underbrace{A-Q}_{R\vec{i}}$$

$$(P-Q)\cdot\vec{e}_\varphi = R\underbrace{\vec{i}\cdot\vec{e}_\varphi}_{-\operatorname{sen}\varphi} = -R\operatorname{sen}\varphi$$

$$\vec{a} = \vec{a}_R + \vec{a}_T + \vec{a}_C$$

$$\vec{a}_R = R\ddot{\varphi}\,\vec{e}_\varphi - R\dot{\varphi}^2\vec{e}_r \Rightarrow \vec{a}_R\cdot\vec{e}_\varphi = R\ddot{\varphi}$$

$$\vec{a}_T = \underbrace{\vec{a}_O}_{=0} + \underbrace{\frac{d\vec{\omega}}{dt}}_{=0}\wedge(P-O) + \vec{\omega}\wedge[\vec{\omega}\wedge(P-O)]$$

$$P-O = P-A+A-O = R\vec{e}_r + \frac{R}{2}\vec{i}$$

$$\vec{\omega}\wedge(P-O) = \omega\vec{k}\wedge\left(R\vec{e}_r + \frac{R}{2}\vec{i}\right) = R\omega\left(\vec{e}_\varphi + \frac{\vec{j}}{2}\right)$$

$$\vec{a}_T = \omega\vec{k}\wedge R\omega\left(\vec{e}_\varphi + \frac{\vec{j}}{2}\right) = R\omega^2\left(-\vec{e}_r - \frac{\vec{i}}{2}\right)$$

$$\vec{a}_T\cdot\vec{e}_\varphi = -\frac{R\omega^2}{2}\underbrace{\vec{i}\cdot\vec{e}_\varphi}_{-\operatorname{sen}\varphi} = \frac{R\omega^2}{2}\operatorname{sen}\varphi$$

$$\vec{a}_C = 2\vec{\omega}\wedge\vec{v}_R = 2\omega\vec{k}\wedge R\dot{\varphi}\,\vec{e}_\varphi = -2\omega R\dot{\varphi}\,\vec{e}_r \Rightarrow \vec{a}_C\cdot\vec{e}_\varphi = 0$$

$$m\cancel{R}\ddot{\varphi} + \frac{m\cancel{R}\omega^2}{2}\operatorname{sen}\varphi = k\cancel{R}\operatorname{sen}\varphi$$

$$\Rightarrow \boxed{\ddot{\varphi} = \left(\frac{k}{m} - \frac{\omega^2}{2}\right)\operatorname{sen}\varphi}$$

parte b: $R\dot{\varphi} = cte \Rightarrow \ddot{\varphi} = 0\ \forall\varphi \Rightarrow \frac{k}{m} = \frac{\omega_0^2}{2} \Rightarrow \boxed{\omega_0 = \sqrt{\frac{2k}{m}}}$

parte c: $\omega = \sqrt{2}\,\omega_0 \Rightarrow \omega = 2\sqrt{\frac{k}{m}} \Rightarrow \omega^2 = \frac{4k}{m}$

$$\ddot{\varphi} = \left(\frac{k}{m} - \frac{2k}{m}\right)\operatorname{sen}\varphi = -\frac{k}{m}\operatorname{sen}\varphi$$

Eq $\Rightarrow \ddot{\varphi} = 0$ y $\operatorname{sen}\varphi = 0 \Rightarrow \boxed{\varphi = 0, \pi}$

$\varphi = 0 \Rightarrow \varphi \gtrsim 0 \Rightarrow \operatorname{sen}\varphi > 0 \Rightarrow \ddot{\varphi} < 0$
$\varphi \lesssim 0 \Rightarrow \operatorname{sen}\varphi < 0 \Rightarrow \ddot{\varphi} > 0$
} la partícula vuelve a la posición de equilibrio $\Rightarrow$ $\varphi = 0$ es estable

$\varphi = \pi \Rightarrow \varphi \gtrsim \pi \Rightarrow \operatorname{sen}\varphi < 0 \Rightarrow \ddot{\varphi} > 0$
$\varphi \lesssim \pi \Rightarrow \operatorname{sen}\varphi > 0 \Rightarrow \ddot{\varphi} < 0$
} la partícula se aleja de la posición de equilibrio $\Rightarrow$ $\varphi = \pi$ es inestable

Otra forma: preintegro

$$\ddot{\varphi}\dot{\varphi} = -\frac{k}{m}\operatorname{sen}\varphi\,\dot{\varphi} \Rightarrow \frac{\dot{\varphi}^2}{2} = +\frac{k}{m}\cos\varphi + C$$

$$\frac{m\dot{\varphi}^2}{2} - k\cos\varphi = C'$$

$$\underbrace{\frac{mR^2\dot{\varphi}^2}{2}}_{T_{rel.}} - \underbrace{kR^2\cos\varphi}_{U_{eff}} = C''$$

$$U_{eff} = -kR^2\cos\varphi$$

$$\frac{\partial U_{eff}}{\partial\varphi} = +kR^2\operatorname{sen}\varphi = 0 \Rightarrow \operatorname{sen}\varphi = 0 \Rightarrow \boxed{\varphi = 0, \pi}$$

$$\frac{\partial^2 U_{eff}}{\partial\varphi^2} = +kR^2\cos\varphi \qquad \varphi = 0 \Rightarrow \frac{\partial^2 U_{eff}}{\partial\varphi^2} = kR^2 > 0 \Rightarrow \text{estable}$$

$$\varphi = \pi \Rightarrow \frac{\partial^2 U_{eff}}{\partial\varphi^2} = -kR^2 < 0 \Rightarrow \text{inestable}$$

parte d: $\varphi(0) = 0$

$$R\dot{\varphi}(0) = v_R \Rightarrow \dot{\varphi}(0) = \frac{v_R}{R}$$

$$\frac{mR^2\dot{\varphi}^2}{2} - kR^2\cos\varphi = \frac{mv_R^2}{2} - kR^2$$

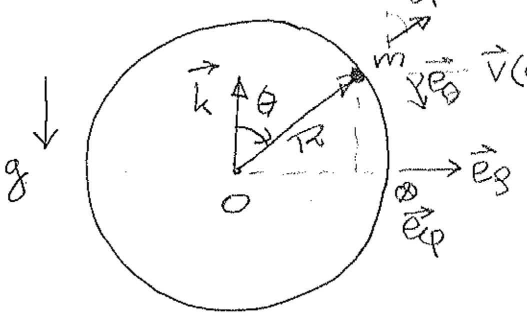
$$\dot{\varphi}^2 \geq 0 \text{ en } \varphi = \pi \Rightarrow kR^2\underbrace{\cos\pi}_{-1} + \frac{mv_R^2}{2} - kR^2 \geq 0$$

$$\frac{mv_R^2}{2} \geq 2kR^2$$

$$\boxed{v_R \geq 2R\sqrt{\frac{k}{m}}}$$

Ejercicio N° 2

$\theta(0) = \frac{\pi}{2}$



$\vec{v}(0) = v_0 \vec{e}_\varphi$

parte a: $\vec{L}_O = m\vec{r}\wedge\vec{v}$

$$\mathring{\vec{L}}_O = m\underbrace{\mathring{\vec{r}}\wedge\vec{v}}_{=0} + m\vec{r}\wedge\mathring{\vec{v}} = \vec{r}\wedge m\vec{a}$$

$$m\vec{a} = \vec{F} = -mg\vec{k} + N\vec{e}_r$$

$$\mathring{\vec{L}}_O = r\vec{e}_r\wedge\left(-mg\vec{k} + N\vec{e}_r\right) = -mgr\underbrace{\vec{e}_r\wedge\vec{k}}_{-\operatorname{sen}\theta\,\vec{e}_\varphi}$$

$$\mathring{\vec{L}}_O\cdot\vec{k} = 0$$

$$\left(\vec{L}_O\cdot\vec{k}\right)^{\circ} = 0 \Rightarrow \vec{L}_O\cdot\vec{k} = cte$$

parte b: $T = \frac{m\vec{v}^2}{2}$ $\vec{v} = \mathring{\vec{r}}$

$$\vec{r} = R\vec{e}_r$$

$$\mathring{\vec{r}} = R\mathring{\vec{e}}_r$$

$$\vec{e}_r = \operatorname{sen}\theta\,\vec{e}_\rho + \cos\theta\,\vec{k}$$

$$\mathring{\vec{e}}_r = \cos\theta\,\mathring{\theta}\,\vec{e}_\rho + \operatorname{sen}\theta\underbrace{\mathring{\vec{e}}_\rho}_{=\mathring{\varphi}\vec{e}_\varphi} - \operatorname{sen}\theta\,\mathring{\theta}\,\vec{k} = \operatorname{sen}\theta\,\mathring{\varphi}\,\vec{e}_\varphi + \mathring{\theta}\,\vec{e}_\theta$$

$$\vec{e}_\theta = \cos\theta\,\vec{e}_\rho - \operatorname{sen}\theta\,\vec{k}$$

$$\Rightarrow \vec{v} = R\operatorname{sen}\theta\,\mathring{\varphi}\,\vec{e}_\varphi + R\mathring{\theta}\,\vec{e}_\theta$$

$$T = \frac{m\vec{v}^2}{2} = \boxed{\frac{mR^2\operatorname{sen}^2\theta\,\mathring{\varphi}^2}{2} + \frac{mR^2\mathring{\theta}^2}{2} = T}$$

parte c: $T + U = E$

$$U = U_g = mgz = mgR\cos\theta$$

$$\vec{L} = mR\vec{e}_r\wedge\left(R\operatorname{sen}\theta\,\mathring{\varphi}\,\vec{e}_\varphi + R\mathring{\theta}\,\vec{e}_\theta\right) = mR^2\operatorname{sen}\theta\,\mathring{\varphi}\left(-\vec{e}_\theta\right) + mR^2\mathring{\theta}\,\vec{e}_\varphi$$

$$\vec{L}\cdot\vec{k} = -mR^2\operatorname{sen}\theta\,\mathring{\varphi}\underbrace{\vec{e}_\theta\cdot\vec{k}}_{-\operatorname{sen}\theta} = mR^2\operatorname{sen}^2\theta\,\mathring{\varphi}$$

$$\Rightarrow \operatorname{sen}^2\theta\,\mathring{\varphi} = cte$$

$$\vec{v}_0 = v_0\vec{e}_\varphi = R\mathring{\varphi}(0)\vec{e}_\varphi \Rightarrow \mathring{\theta}(0) = 0$$

$$\theta(0) = \frac{\pi}{2}$$

$$\mathring{\varphi}(0) = \frac{v_0}{R}$$

$$\text{sen}^2\theta\,\dot{\varphi} = \frac{v_0}{R} \Rightarrow \dot{\varphi} = \frac{v_0}{R\,\text{sen}^2\theta}$$

$$T + U = E$$

$$\frac{m\cancel{R^2}\,\text{sen}^2\theta}{2}\,\frac{v_0^2}{\cancel{R^2}\,\text{sen}^{\cancel{4}}\theta} + \frac{mR^2\dot{\theta}^2}{2} + mgR\cos\theta = E$$

$$\frac{mR^2}{2}\dot{\theta}^2 + \frac{mv_0^2}{2\,\text{sen}^2\theta} + mgR\cos\theta = \frac{mv_0^2}{2}$$

$$\dot{\theta}(0) = 0$$
$$\theta(0) = \frac{\pi}{2}$$

$$\dot{\theta}^2 + \underbrace{\frac{v_0^2}{R^2\,\text{sen}^2\theta} + \frac{2g}{R}\cos\theta - \frac{v_0^2}{R^2}}_{\boxed{f(\theta) = \frac{v_0^2}{R^2}\left(\frac{1}{\text{sen}^2\theta} - 1\right) + \frac{2g}{R}\cos\theta}} = 0$$

parte d: $\dot{\theta} = 0$ en $\theta = \frac{5\pi}{6} = 150°$

$$\cos\left(\frac{5\pi}{6}\right) = \cos\left(\frac{3\pi}{6} + \frac{2\pi}{6}\right) = \cos\left(\frac{\pi}{2} + \frac{2\pi}{6}\right) = \text{sen}\left(-\frac{2\pi}{6}\right) = -\text{sen}\,\frac{\pi}{3} = -\text{sen}\,60° = -\frac{\sqrt{3}}{2}$$

$$\text{sen}\left(\frac{5\pi}{6}\right) = \text{sen}\left(\frac{\pi}{2} + \frac{2\pi}{6}\right) = \cos\frac{\pi}{3} = \cos 60° = \frac{1}{2}$$

$$f\left(\frac{5\pi}{6}\right) = \frac{v_0^2}{R^{\cancel{2}}}(4-1) - \frac{g}{\cancel{R}}\sqrt{3} = 0$$

$$\frac{3v_0^2}{R} = g\sqrt{3} \Rightarrow \boxed{v_0 = \sqrt{\frac{Rg}{\sqrt{3}}}}$$