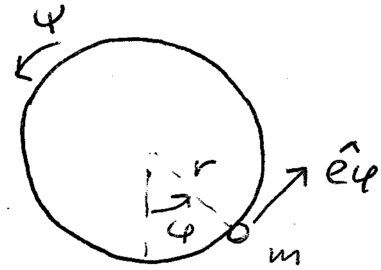
Mecánica Newtoniana agosto 09

Ej ① 1)



$\vec{M}_O^{(ext)} \cdot \hat{z} = 0 \quad \forall O \in$ eje

$\Rightarrow \vec{L}_O \cdot \hat{z} = L_z = cte.$

$$L_z = \frac{1}{2} m r^2 \dot{\psi} + m r \, \vec{v}_m \cdot \hat{e}_\varphi$$

$$\vec{v}_m = \vec{v}_m' + r\dot{\psi}\,\hat{e}_\varphi = r(\dot{\varphi}+\dot{\psi})\,\hat{e}_\varphi + \dot{z}\,\hat{z}$$

$$\Rightarrow L_z = \frac{1}{2} m r^2 \dot{\psi} + m r^2 (\dot{\varphi}+\dot{\psi})$$

$$\dot{L}_z = 0: \quad \boxed{\ddot{\varphi} + \frac{3}{2}\ddot{\psi} = 0} \quad (i)$$

Se conserva también la energía:

$$E = mgz + \frac{1}{2}\left(\frac{1}{2} m r^2\right)\dot{\psi}^2 + \frac{1}{2} m \left[r^2(\dot{\varphi}+\dot{\psi})^2 + \dot{z}^2 \right]$$

$$= mgr\varphi + \frac{1}{2}\left(\frac{1}{2} m r^2\right)\dot{\psi}^2 + \frac{1}{2} m r^2 \left[(\dot{\varphi}+\dot{\psi})^2 + \dot{\varphi}^2 \right]$$

$$\dot{E} = 0: \quad \boxed{\frac{g}{r}\dot{\varphi} + \frac{1}{2}\dot{\psi}\ddot{\psi} + \left[(\dot{\varphi}+\dot{\psi})(\ddot{\varphi}+\ddot{\psi}) + \dot{\varphi}\ddot{\varphi} \right] = 0} \quad (ii)$$

Combinando (i) y (ii):

$$\boxed{\ddot{\varphi} = -\frac{3}{4} g/r} \quad (1)$$

$$\boxed{\ddot{\psi} = \frac{1}{2} g/r} \quad (2)$$

2) preintegro (1): $\frac{1}{2}(\dot{\varphi}^2 - \dot{\varphi}_0^2) = -\frac{3}{4} g/r\, \varphi \Leftrightarrow \dot{\varphi}^2 = \dot{\varphi}_0^2 - \frac{3}{2} g/r\, \varphi$

¿llega arriba? $z = 2r \Rightarrow \varphi = 2$ si $\dot{\varphi}^2 \geq 0 \quad \forall \varphi \in [0,2]$

$$\Updownarrow$$

$$\boxed{\dot{\varphi}_0^2 \geq 3 g/r}$$

3) Integro (1) dos veces en el tiempo:

$$\varphi(t) = \dot\varphi_0 t - \frac{3}{8} g/r\, t^2$$

$$\varphi(t^*) = 2 \; : \; t^* = \frac{4}{3}\frac{r}{g}\left(\dot\varphi_0 \pm \sqrt{\dot\varphi_0^2 - 3g/r}\right)$$

tomo el que me da el menor $t^* > 0$

$$\boxed{t^* = \frac{4}{3}\frac{r}{g}\left(\dot\varphi_0 - \sqrt{\dot\varphi_0^2 - 3g/r}\right)}$$

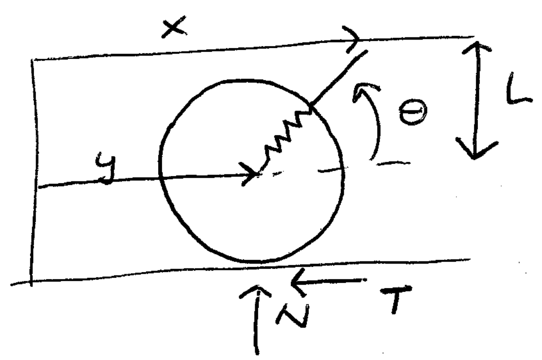
4) $\vec v \cdot \hat e_\varphi = 0$: $\dot\varphi(t^*) + \dot\psi(t^*) = 0$

integrando una vez en el tiempo (1) y (2) : $\dot\varphi = \dot\varphi_0 - 3/4\, g/r\, t$

$$\dot\psi = \dot\psi_0 + 1/2\, g/r\, t$$

$$\Rightarrow \dot\varphi_0 + \dot\psi_0 - \frac{1}{4} g/r\, t^* = 0 \; : \; \boxed{\dot\psi_0 = -\frac{1}{3}\left(2\dot\varphi_0 + \sqrt{\dot\varphi_0^2 - 3g/r}\right)}$$

Ej 2 1)



1era cardinal al disco: $M\ddot y = -T + K(x-y)$ (1)

$$0 = N + KL - Mg \quad (2)$$

2da " " : $\frac{1}{2} M R^2 \dot\omega = RT$ donde $\dot\omega = \ddot y / R$ (r.s.d.)

$$\frac{1}{2} M \ddot y = T \quad (3)$$

de (2) : $N = Mg - KL \geq 0$ para que el disco no despegue

sustituyendo (3) en (1) : $2T = -T + K(x-y)$

usando que $tg\theta = L/(x-y)$

$$T = \frac{KL}{3}\,\mathrm{ctg}\,\theta$$

para que ruede sin deslizar: $|T| \leq fN \Leftrightarrow$

$$-f(Mg - KL) \leq \frac{KL}{3}\,\mathrm{ctg}\,\theta \leq f(Mg - KL)$$

2) a) sust. (3) en (1): $M\ddot{y} = -\frac{1}{2}M\ddot{y} + K(x-y)$

$$\boxed{\frac{3}{2}M\ddot{y} + Ky = Kx(t)}$$

b) $x(t) = x_0 + vt$; la solución gral. a la ec. de movimiento es:

$$y(t) = A\,\mathrm{sen}\,\omega t + B\cos\omega t + x_0 + vt \quad , \quad \omega^2 = \frac{2}{3}\frac{K}{M}$$

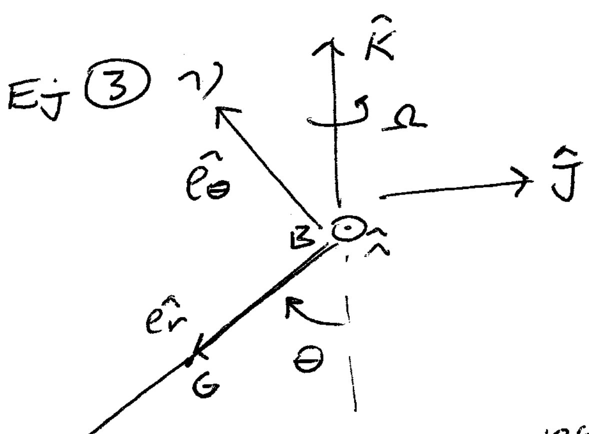
usando que $y(0) = R$

$x_0 = R$

$\dot{y}(0) = 0$

$$\boxed{y(t) = R + v\left(t - \frac{1}{\omega}\,\mathrm{sen}\,\omega t\right)}$$

Ej (3) 1)



$$\left(\hat{n} = \frac{B-A}{|B-A|}\right)$$

2da cardinal a la barra desde B proyectada según $\hat{n}$:

$$\boxed{\left[M(G-B)\times\vec{a}_B + \frac{d}{dt}(\mathbb{I}_B\vec{\omega})\right]\cdot\hat{n} = \vec{M}_B^{(ext)}\cdot\hat{n} = Mg\frac{L}{2}\,\mathrm{sen}\,\theta} \quad (1)$$

Consideremos el sistema $\{\hat{n}, \hat{e_\theta}, \hat{e_r}\}$ solidario al rígido y principal

$$\frac{d}{dt}(\mathbb{I}_B\vec{\omega}) = \mathbb{I}_B\dot{\vec{\omega}} + \vec{\omega}\times(\mathbb{I}_B\vec{\omega})$$

siendo $\vec{\omega} = -\dot{\theta}\hat{n} + \Omega\hat{K} = -\dot{\theta}\hat{n} + \Omega\,\mathrm{sen}\,\theta\,\hat{e_\theta} - \Omega\cos\theta\,\hat{e_r}$

$$\vec{\omega} = -\dot{\theta}\hat{\imath} + (\dot{\Omega}\,\text{sen}\theta + \Omega\cos\theta\,\dot{\theta})\hat{e_\theta} - (\dot{\Omega}\cos\theta - \Omega\,\text{sen}\theta\dot{\theta})\hat{e_r}$$

$$I_B^{\{\hat{\imath},\hat{e_\theta},\hat{e_r}\}} = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad , \quad I = \int_0^L \left(\frac{M}{L}\right) x^2 dx = \frac{1}{3} M L^2$$

$$\mathbb{I}_B \dot{\vec{\omega}} \cdot \hat{\imath} = -I\ddot{\theta}$$

$$\mathbb{I}_B \vec{\omega} = I(-\dot{\theta}\hat{\imath} + \Omega\,\text{sen}\theta\,\hat{e_\theta}) \; ; \; \vec{\omega} \times \mathbb{I}_B \vec{\omega} \cdot \hat{\imath} = I\Omega^2\,\text{sen}\theta\cos\theta$$

$$\vec{a_B} = L\dot{\Omega}\hat{\jmath} - L\Omega^2\hat{\imath} = L\dot{\Omega}(-\cos\theta\,\hat{e_\theta} - \text{sen}\theta\,\hat{e_r}) - L\Omega^2\hat{\imath} \quad \Bigg| \Rightarrow$$

$$G - B = \frac{L}{2}\hat{e_r}$$

$$\Rightarrow M(G-B) \times \vec{a_B} \cdot \hat{\imath} = \frac{ML^2}{2}\dot{\Omega}\cos\theta$$

y la ec. de movimiento (1) es:

$$\boxed{\ddot{\theta} + \text{sen}\theta\left(\frac{3}{2}g/L - \Omega^2\cos\theta\right) - \frac{3}{2}\dot{\Omega}\cos\theta = 0}$$

2) $\Omega = \Omega_0$: $\ddot{\theta} + \text{sen}\theta\left(\frac{3}{2}g/L - \Omega_0^2\cos\theta\right) = 0$

equilibrio ($\ddot{\theta} = 0$) : $\text{sen}\theta = 0 \quad \theta = 0, \pi$

$$\cos\theta = \frac{3}{2}g/L\,\Omega_0^{-2} \qquad \exists \text{ si } \Omega_0^2 \geq \frac{3}{2}g/L$$

Los ptos. de eq. estable son los mínimos de U' / $\frac{dU'}{d\theta} = \text{sen}\theta\left(\frac{3}{2}g/L - \Omega_0^2\cos\theta\right)$

$$\frac{d^2U'}{d\theta^2} = \cos\theta\left(\frac{3}{2}g/L - \Omega_0^2\cos\theta\right) + \Omega_0^2\,\text{sen}^2\theta$$

$$\left.\frac{d^2U'}{d\theta^2}\right|_\pi = -\left(\frac{3}{2}g/L + \Omega_0^2\right) < 0 \Rightarrow \theta = \pi \text{ inestable}$$

$$\left.\frac{d^2U'}{d\theta^2}\right|_0 = \frac{3}{2}g/L - \Omega_0^2 > 0 \text{ si } \Omega_0^2 < \frac{3}{2}g/L$$

$$\left.\frac{d^2U'}{d\theta^2}\right|_{\cos\theta = \frac{3}{2} g/L\, \Omega_0^{-2}} = \Omega_0^2\left(1-\left(\frac{3}{2} g/L\, \Omega_0^{-2}\right)^2\right) > 0 \text{ si}$$

$$\Omega_0^2 > \frac{3}{2} g/L$$

Para $\Omega_0^2 = \frac{3}{2} g/L$: $\frac{dU'}{d\theta} = \Omega_0^2 \operatorname{sen}\theta\,(1-\cos\theta)$ $\quad \theta \to 0^+ \ \frac{dU'}{d\theta} > 0$

$\theta \to 0^- \ \frac{dU'}{d\theta} < 0$

$\Rightarrow \frac{dU'}{d\theta}$ creciente alrededor de $\theta = 0$: $\theta = 0$ es un mínimo

resumiendo:

$$\boxed{\begin{array}{l} \theta = 0 \text{ estable si } \Omega_0^2 \leq \frac{3}{2} g/L \\ \theta = \pi \text{ inestable} \\ \cos\theta = \frac{3}{2} g/L\, \Omega_0^{-2} \ \exists \text{ y es estable para } \Omega_0^2 \geq \frac{3}{2} g/L \end{array}}$$