Variance reduction in Monte Carlo evaluation of residual connectedness network reliability

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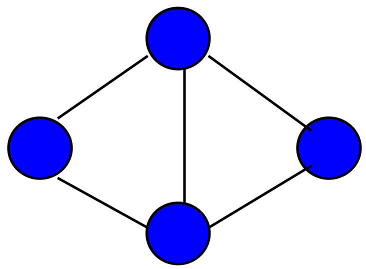
A Model

- Communication network:

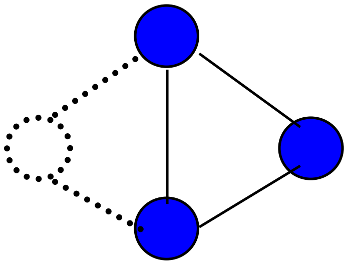
  undirected graph $\boldsymbol{G = (V,E)}$

  - **perfect lines (without failures)**
  - **nodes subject to independent failures**
  - **static model**
  - **operation probability of**

    node $u \in V$ is r_u



Network residual connectedness reliability



- Reliability measure $R(G)$:
  - $\tilde{G}$: subgraph of G induced by operational nodes
  - $Y(G)$: random variable "state of network G"

$$Y(G)=\begin{cases}1 & \text{if } \tilde{G} \text{ connected}\\ 0 & \text{otherwise}\end{cases}$$

  - Network residual connectedness reliability:

$$R(G)=E\{Y(G)\}=P\left\{\tilde{G}\text{ is connected}\right\}$$

Standard Monte Carlo Method (MCM)

- Unbiased estimator of $R(G)$

$$\hat{Y}(G) = \frac{1}{N}\sum_{i=1}^{N} Y^{(i)} \quad ; \quad Y^{(i)} \text{ is r.v. i.i.d. } Y(G)$$

- Variance

$$Var\{\hat{Y}(G)\} = \frac{Var\{Y(G)\}}{N} = \frac{R(G)\left(1-R(G)\right)}{N}$$

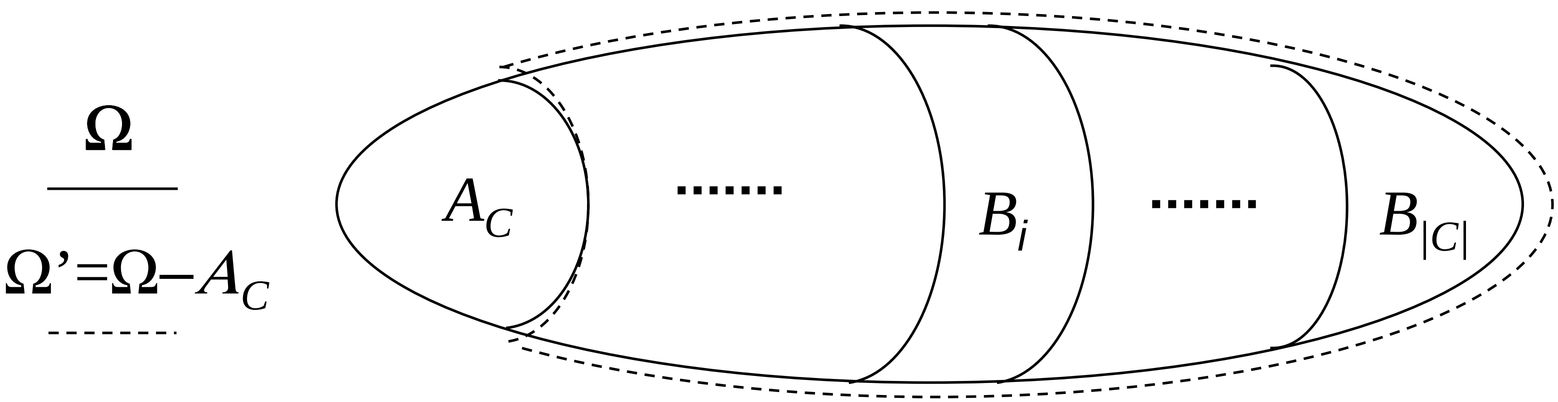
- Level ε confidence interval (applying CLT, with N big enough)

$$\hat{Y} \pm \xi\sqrt{\frac{Var\{Y(G)\}}{N}} \text{ with } \xi = G^{-1}\left(1-\frac{\varepsilon}{2}\right), \; G(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}dt$$

Recursive Variance Reduction (RVR) - preliminary definitions

- $C = \{u \in \mathrm{N} \,/\, r_u < 1\} = \{u_1, \ldots, u_{|C|}\}$ **: set of non-perfect nodes**
- A_C **: event "all nodes of C are operational"**
- B_i**: event "nodes** $u_1, \ldots, u_{i-1}$ **are operational, node** u_i **fails"**

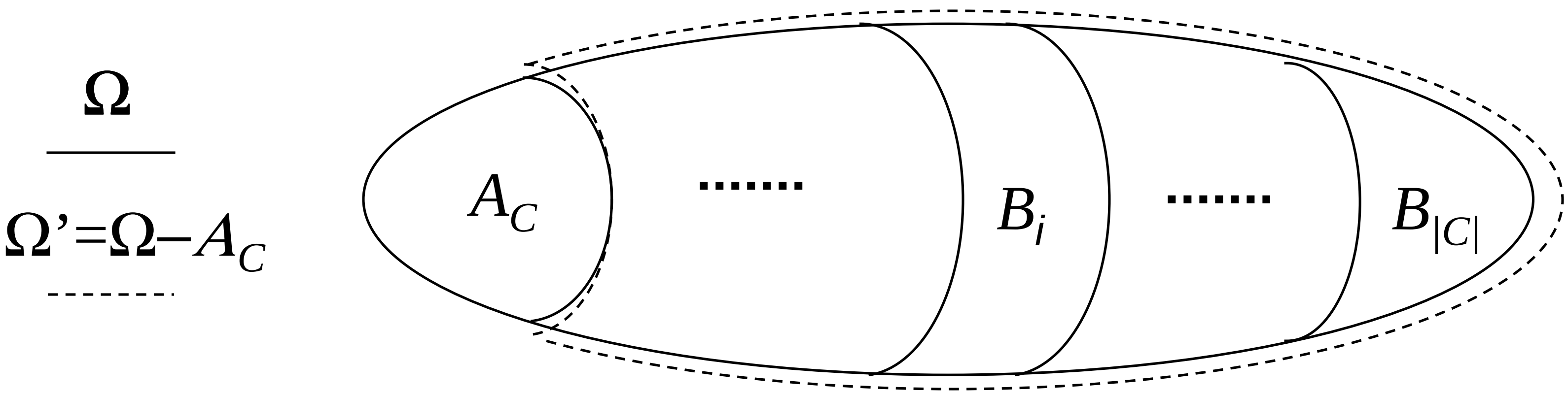
  $1 <= i <= |C|$



Recursive Variance Reduction (RVR) - preliminary definitions

- B_i: event "nodes $u_1,\ldots,u_{i-1}$ are operational, node u_i fails"
- G_i : network $(G/u_1\ldots/u_{i-1}) - u_i$, $1 \le i \le |C|$
- W: r.v. with distribution:

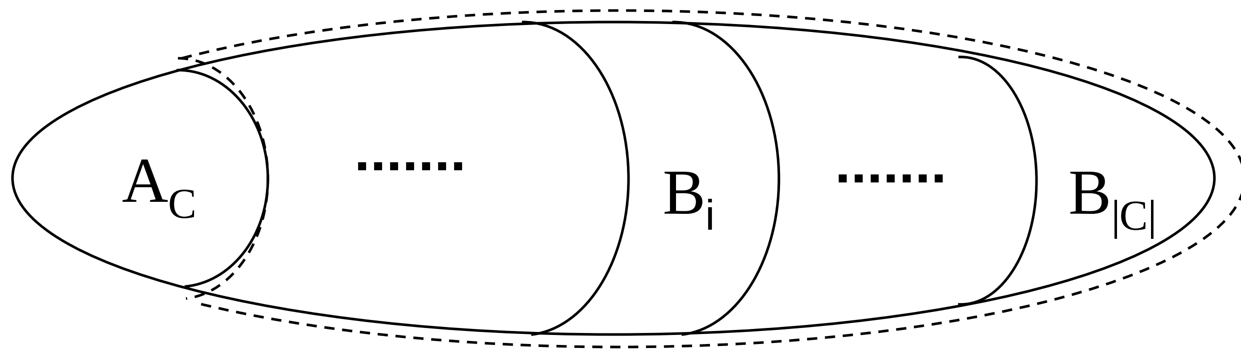
$$P\{W=i\} = \frac{P\{B_i\}}{(1-P\{A_C\})} = \frac{P\{B_i\}}{P\{\Omega'\}}$$



RVR estimator

Ω ———

$\Omega' = \Omega - A_C$ - - - - - - -



- Exact reliability

$$R(G) = \begin{cases} 1 \times P(A_C) + \sum_i R(G_i) P(B_i) & \textbf{if } G \textbf{ is connected} \\ 0 \times P(A_C) + \sum_i R(G_i) P(B_i) & \textbf{if } G \textbf{ is not connected} \end{cases}$$

- RVR estimator

$$Z(G) = \begin{cases} P(A_C) + P(\Omega') \sum_i Z(G_i)\, IND_{(W=i)} & \textbf{if } G \textbf{ is connected} \\ P(\Omega') \sum_i Z(G_i)\, IND_{(W=i)} & \textbf{if } G \textbf{ is not connected} \end{cases}$$

W r.v. with distribution $P\{W = i\} = \dfrac{P\{B_i\}}{(1 - P\{A_C\})} = \dfrac{P\{B_i\}}{P\{\Omega'\}}$

Recursive Variance Reduction RVR- Properties

- Property:
  - Z(G) is an unbiased estimator of R(G):

$$E(Z(G)) = R(G)$$

- Property:
  - Z(G) is more accurate than standard Monte Carlo

$$Var(Z(G)) \leq R(G)(1 - R(G)) = Var(Y(G))$$

- Relative accuracy index (equiv. efficiency)

$$S = Var(Y(G))/Var(Z(G))$$

RVR method

⇒ Input : network G

⇒ Output: a value of r.v. $Z(G)$

⇒ Pseudocode:

- End recursion conditions:
  - If all nodes are perfect and G connected : Return (1)
  - If all nodes are perfect and G not connected: Return (0)
- Preliminary computations:
  - Find C=set of non-perfect nodes
  - Generate a trial of W with distribution $P\{W = i\}=P\{Bi\}/P\{\Omega'\}$
  - Build network $G_i =(G /u_1.../u_{i-1})-u_i$
- Recursive call:
  - If G connected: Return $(P\{A_C\}+ P\{\Omega'\}.\mathrm{RVR}(G_i))$
  - Si G not connected Return ($P\{\Omega'\}.\mathrm{RVR}(G_i)$)

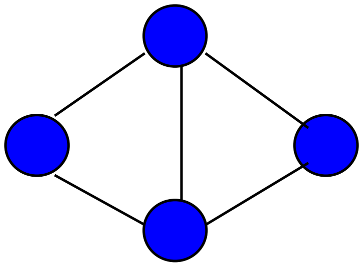
RVR method
Computational complexity

- Property:
  - complexity of a single RVR call: O(max(|V|,|*E*|))
  - maximum recursion depth: |V|
- Property:
  - if |V|=*n*, and all nodes have reliability *r*, the average recursion depth is bounded by

$$Depth(n) \le \sum_{k=1}^{n} \left(1 \Big/ \sum_{i=0}^{k-1} r^i \right)$$

  - Unreliable networks: if r→0, Depth(n) →n
  - Very reliable networks: if r→1, Depth(n) →H_n~ log(n)

Numerical illustration

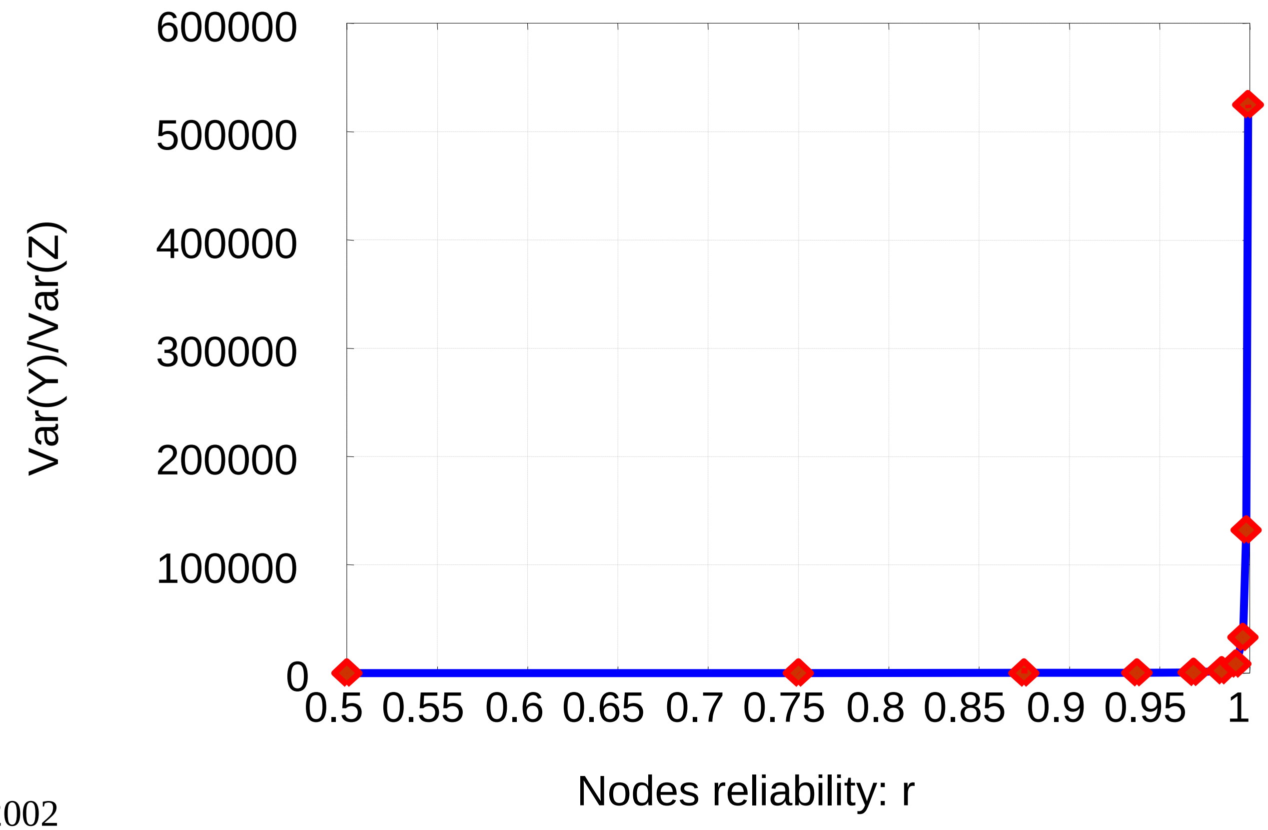


- “Bridge” network
- Node reliability $r=1-2^{-k}$, k=1,2,...,10
- Results:

k	r	R(G)	S=Var(Y)/Var(Z)
1	0.5	0.6875000	4.4
2	0.75	0.9023438	12.3
3	0.875	0.9724121	40.3
4	0.9375	0.9926605	144
5	0.96875	0.9981070	544
6	0.984375	0.9995193	2110
7	0.9921875	0.9998789	8320
8	0.99609375	0.9999696	33000
9	0.998046875	0.9999924	132000
10	0.9990234375	0.9999981	525000

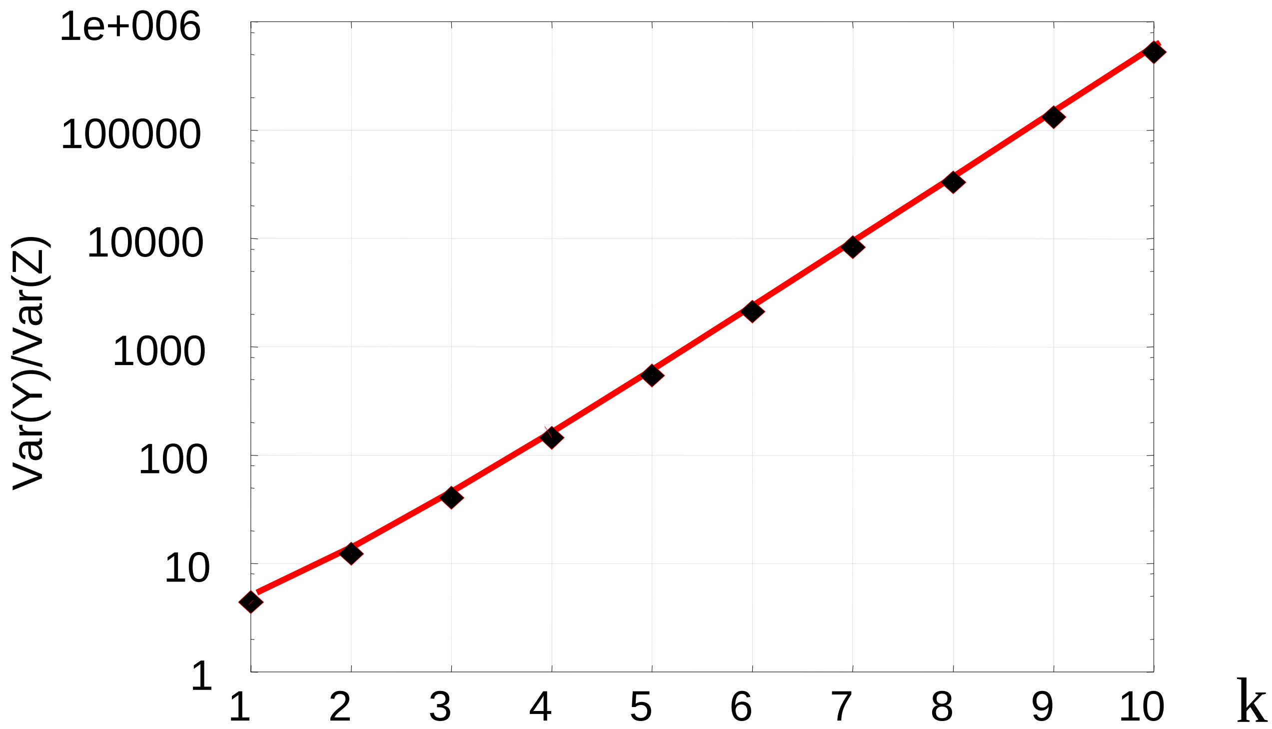
Relative efficiency index or speed-up behavior

- Relative precision S=Var(Y)/Var(Z) as function of $r=1-2^{-k}$ (usual scales)



Relative efficiency index behavior

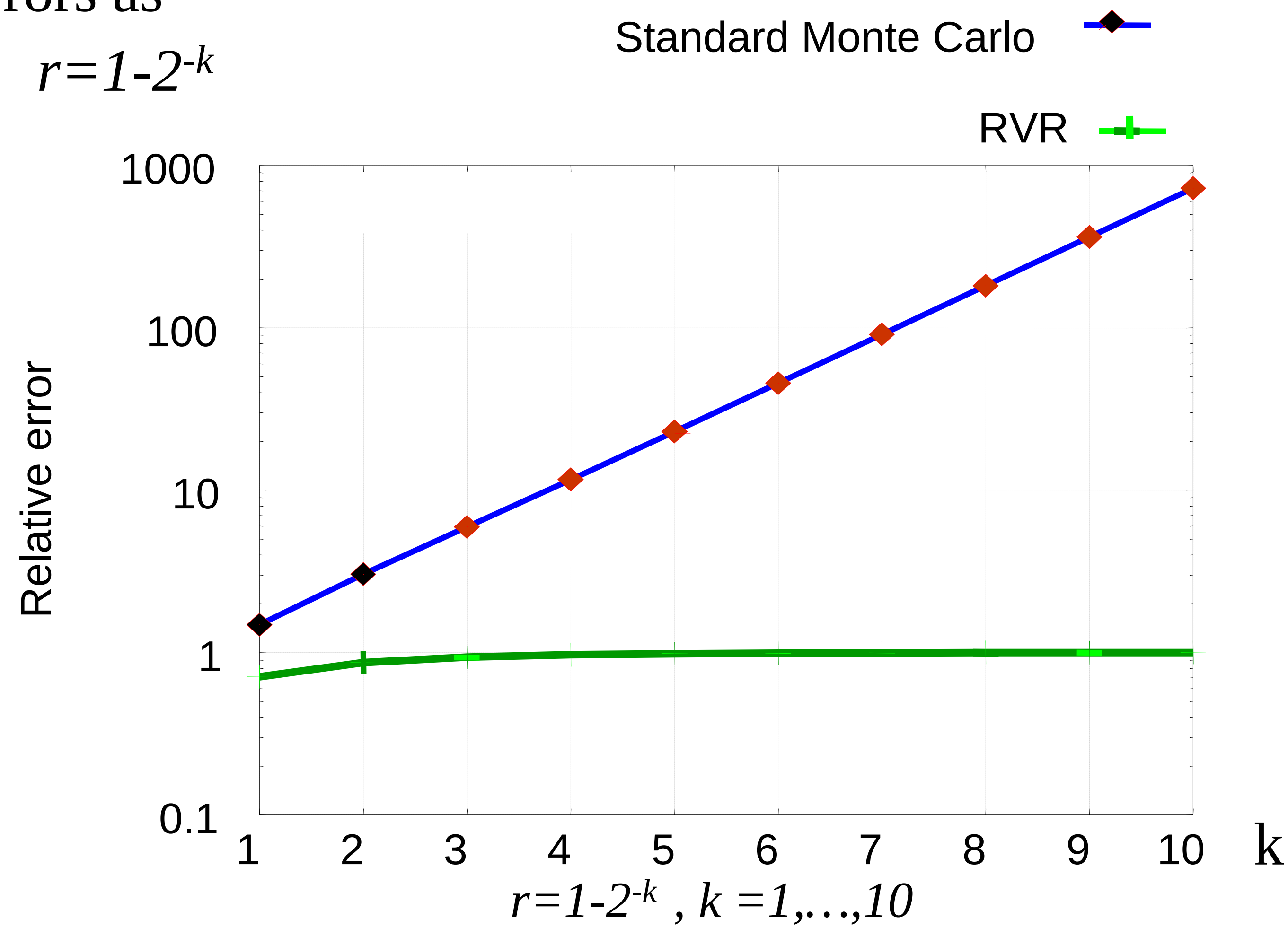
- Relative precision S=Var(Y)/Var(Z) as function of $r=1-2^{-k}$ (logarithmic scales on both axes)



$r=1-2^{-k}$, $k=1,\ldots,10$

Relative error comparison

Relative errors as function of $r=1-2^{-k}$



Conclusions

- RVR method is useful for reliability evaluation (in particular, residual connectivity).
- Specially adapted for very reliable systems:
  - Smaller computational complexity.
  - Greater precision.
- Numerical example suggests stable relative error.
- Future work:
  - Theoretical analysis of relative error behavior
  - Study of recursion depth for non-identical nodes.

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