Introduction to Game Theory

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October 2017

Notes:

Agenda

Notes:

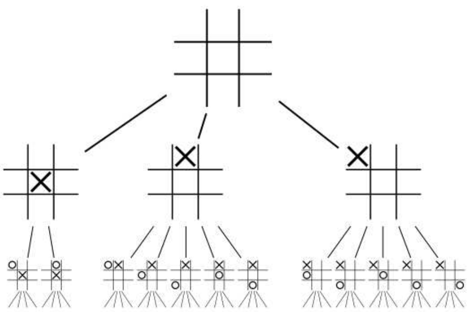
Extensive Form Games

Notes:

- In many situations, time is important, in other words player's moves order are important
- We can model sequential moves through extensive form games
- Perfect information refers to the fact that each player moves knowing all previous moves of all players
- extensive games models with imperfect information also exist

Tree Representation of an extensive form game

First two play of a game tree for tic-tac-toe



Notes:

- image source: wikipedia.org
- We can model both cases through trees
- We will formalize this, for the moment observe that indeed we can decompose the game in sequential states each node being a state, and we go from one node to the other with an given action, performed by a given player
- We don't see it in this picture but at the end of the game we will have the payoffs (in this case either (1,0),(0,1) or (0,0)
- From an extensive game we can define a normal form game (converse not always true)
- We will focus in perfect information and finite games

Definition

Formalization

- A set of players
- A set of actions
- The tree, with choice nodes, end nodes, branches and labels (actions)
- Players affectation to nodes
- Payoffs at end nodes

Notes:

- Mathematically it is defined by the tuple (N, A,H,Z,χ,ρ,σ,u)
- N is the set of players and A the set of actions
- Defining the tree:
  - H is the set of choice nodes
  - Z is the set of terminal nodes
  - χ is a function that maps nodes into actions $\chi : H \to 2^A$
  - ρ is the player function, affects players to the nodes $\rho : H \to N$
  - σ is the successor function, maps nodes and actions into nodes (or terminal nodes) $\sigma : H \times A \to \{H \cup Z\}$ In order to obtain a tree, we need to have only one possible path towards one node. We impose that: if $\sigma(h_1, a_1) = \sigma(h_2, a_2)$ then $h_1 = h_2$ and $a_1 = a_2$.
- $u = \{u_i\}_{i \in N}$, where u_i is player's y utility functions that maps every terminal node into a real $u_i : Z \to \mathbb{R}$

Definitions

Pure Strategy

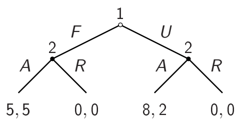
for player i is a complete specification of the action to take at each node belonging to i

- With this definitions of pure strategies, we can keep the same definitions for mixed strategies, best response, and nash equilibrium

Notes:

- Mathematically, a pure strategy in an extensive form game with perfect information is the obtained from the cross product of all possible actions, which we can define as $\prod_{h\ inH:\rho(h)=i} \chi(h)$
- Mixed strategy: a distribution probability over pure strategies
- Best response in ms: is the mixed strategy that maximizes the expected utility for given other agents strategies
- Nash equilibrium, a strategy profile such that every players strategy is a best response for the other agents strategies
- Theorem Any finite Extensive Form Game with perfect information has a pure strategy Nash equilibrium

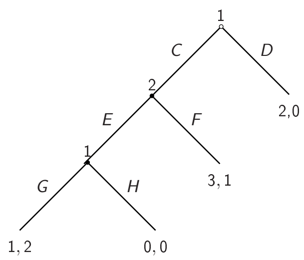
Example: the ultimatum game



Notes:

- Actions for player 1 offer unfair (U), or fair (F) share
- Actions for player 2 accept (A) or reject (R) the offer
- Compute all the available pure strategies for player 1, and for player 2

Finding NE



Notes:

- How to find the equilibria?
- For every game in extensive form we can define the normal form game, and fin the equilibrium through the search of BR in the table
- Find the NE of this game
- However, we can observe in this example, that not all NE are intuitively rational
- Informally, a subgame is every subtree with terminal nodes, without “breaking“ any node

Solution Concept - Subgame Perfect Equilibrium

Subgame
every subtree with terminal nodes and without "breaking" any node

Subgame Perfect Equilibruim (SPE)
A strategy s^* such that in no subgame can any player do better off by unilaterally deviating.

Notes:

- More formally: given a perfect-information extensive-form game G , the **subgame** of G rooted at node h is the restriction of G to the descendants of h.
- The SPE of a game G are all strategy profiles s such that for any subgame G' of G , the restriction of s to G' is a Nash equilibrium of G' .
- From previous example, find which of the NE are SPE
- A subgame perfect equilibrium induces a NE in every sub game of an extensive form game
- Every finite extensive game with perfect information has at least one SPE

Computing SPE - Backward Induction

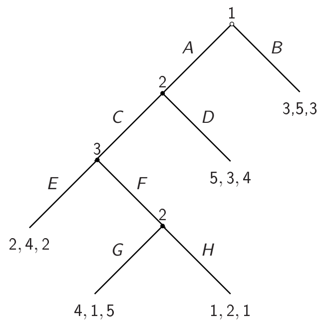
1. `Start from the smallest subgame`
2. `Find the best strategy for the player in that turn`
3. `Report the payoffs to the node reached backwardly with that strategy`
4. `back to 2 till the first node is reached`

- `Restart from the subsequent smallest subgame`
- `If several possible answers for 2, start one backward search for each of the possibilities`

Notes:

- An algorithmic way of finding the SPE is the so called backward induction method

Exercise

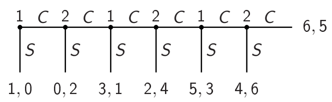


Notes:

- compute the SPE using backward induction

SPE is not always appealing

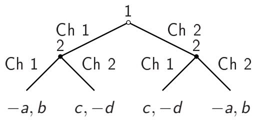
Example: the centipede game



Notes:

- Show that the unique SPE (which is found by backward induction) makes that the game ends at the first move
- Actually, in real experiments people were observed to end the game closest to the end; which shows the limitations of backward inductions

Example, extending the Jamming game [1]



Notes:

- Note that the game is not the same Jamming game written in extended form, but rather, is another game, similar to the Jamming game but where players play sequentially
- Note that every extensive form game with perfect information can be converted into a normal form one, but the converse is not true
- In particular, verify that in this example there are two NE in pure strategies, while in the original normal form there were no

Stackelberg Games

- A duopoly model
- One player moves after the other one
- Second player knows first player's move before playing
- Modeled by extensive form (tree) games
- Also called Leader-Follower games
- One round games

Notes:

- Typical setting proposed by Heinrich Freiherr von Stackelberg in 1934
- One player (leader) plays first, the other player plays afterwards (follower)
- Is a player better off-moving first or secondly?
- Solution method: backwards induction

Stackelberg Games-Pricing example

- Leader: service provider fixing its price p
- Followers: users, demand is a function of price say $D(p)$ equilibrium population accepting the service for a given price.
- We have assumed that there is an equilibrium population that accepts the service for that price (demand function)
- The leader sets price p to maximize its revenue, which is proportional to demand: $R(p) = pD(p)$.
- Assuming demand is known, p maximizing the revenue can be obtained by derivation

Notes:

Quizz Lecture 2

The questions proposed here are taken from: the MOOC Game Theory on Coursera platform, created by Matthew Jackson, Kevin Leyton-Brown and Yoav Shoham and from the Book An Introduction to Game Theory, by Martin J. Osborne.

Exercise 1 (Normal form games - Mixed Strategies) *Consider the following game in normal form.*

		Player 2	
		L	*R*
Player 1	*T*	*(2,2)*	*(0,2)*
	B	*(1,2)*	*(3,3)*

Find all pure-strategy and mixed-strategy Nash equilibria:

a. (T, L);

b. (B, R);

c. Player 1 plays T with prob q=1, player 2 plays L with prob p=3/4;

d. All of above.

Exercise 2 (Normal form games - Maxmin strategies) *Consider the following game in normal form:*

		Player 2	
		L	*R*
Player 1	*T*	*(3,0)*	*(1,2)*
	B	*(2,1)*	*(0,3)*

What is a maxmin strategy for player 1?

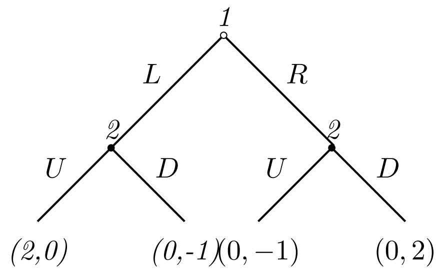
a. T;

b. B;

c. mixed, playing T with proba. 1/2 and B with proba. 1/2

d. mixed, playing T with proba. 1/3 and B with proba. 2/3.

Exercise 3 (Extensive form games - SPE) *Consider the following game in extensive form:*



How many subgames are in this game? Which is a subgame perfect equilibrium?

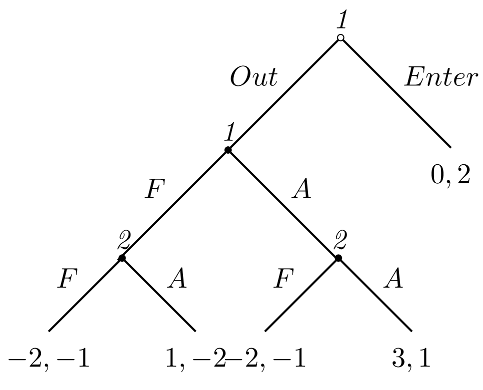
a. There are 1 subgames; (L), (U,D);

b. There are 1 subgames; (L), (U,U);

c. There are 3 subgames; (L), (U,D);

d. There are 3 subgames; (L), (U,U).

Exercise 4 (extensive form game - backward induction) *Consider the following extensive form game.*



Which is the backward induction solution of this game?

a. (Enter, Acc.), (Fight, Fight).

b. (Enter, Fight), (Acc., Acc.).

c. (Stay out, Acc.), (Fight, Acc.).

d. (Enter, Acc.), (Fight, Acc.).

Exercise 5 (extensive form game, backward induction) *Consider the following game:*

- *Player A makes an offer x in 0,1,...10 to player B;*
- *Player B can accept or reject;*
- *A gets 10-x and B gets x if accepted;*
- *If rejected, player A gets 0 and player B gets a punishment of -1.*

Which is a possible outcome (payoff to players A,B) from backward induction?

a. (9, 1).

b. (5, 5).

c. (0, -1).

d. (10, 0).

Exercise 6 (Stackelbergs duopoly game with quadratic costs) *Consider a market in which there are two firms, both producing the same good. Firm i's cost of producing q_i units of the good is $c_i(q_i)$; the price at which output is sold when the total output is Q is $p(Q)$*

Each firms strategic variable is output, the firms make their decisions sequentially, simultaneously: one firm chooses its output, then the other firm does so, knowing the output chosen by the first firm.

1. Find the sub-game perfect equilibrium of Stackelbergs duopoly game when $C_i(q_i) = q_i^2$ for $i = 1, 2$, and $p(Q) = aQ$ for all $Q \leq a$ (with $p(Q) = 0 for Q > a$).

In a Cournot duopoly, both firms chose their output simultaneously.

2. Compare the equilibrium outcome with the Nash equilibrium of Cournots game under the same assumptions

Agenda

Notes:

Pricing, what for?

- Revenue maximiser mechanism
- Signalling mechanism
  - Feedback to the seller
  - Incentive provider mechanism

Example: Mobile telephony tariffs

Offer 1) Week-end and night free calls
Offer 2) All day free calls

Notes:

- The more clients choosing offer 1 $\Rightarrow$ the more demand is going to be greater nights and week-ends
- Clients of Offer 1 are encouraged to make calls the night and week-ends

Pricing, what for?

- In data networks, pricing has also been claim as a mechanism for admission control/congestion control
- If prices increase, demand decreases and thus congestion decreases

Notes:

Some Terms and Definitions

- **Charge** Amount that is billed for a service
- **Price** Amount of money associated with a unit of service
- **Tariff** Part of the contract between two parts which specifies the way the charge will be computed for the services

Example: Taxi tariffs. Charge for a ride of X kms and T minutes

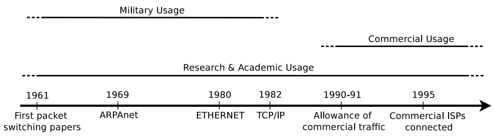
$$ride\_price(X,T) = \begin{cases} a + bX & speed > s* \\ a + cT & speed \leq s* \end{cases}$$ where:

a fixed price per ride, b price per km, c price per minute, $speed$ taxi's speeds, $s*$ threshold

Notes:

- What is the incentive to the taxi driver?
- and for customers?

Internet History is Very Related to its Economic Model

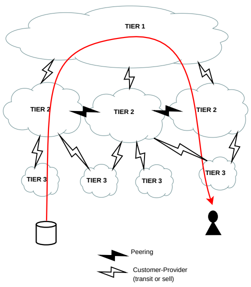


- Initially free. Objective: reduce costs by interconnecting
- Dependency on the telephony wired network for initial deployment

Notes:

Business Models in the Internet

- Network access pricing schemes
- Investments for network service providers (NSPs or ISPs)
- Economic relations between NSPs
- Economic relations between content/application and NSPs
- Economic model of content/application service providers



Current interconnection business model

Notes:

Some Facts of Nowadays Internet

- Commoditization of the Internet services
- Providers claim the need for product differentiation e.g. quality
- Internet is best effort, congestion might degrade quality
- How to offer different quality levels?
  - Upgrade capacity?
  - Access control mechanism?
- As in any good, if price raises demand decreases
- Problem at different levels: network, contents

Notes:

Particularity of Pricing Network Services

- Externalization: the more clients/connections the more value the network has
- Congestion
- Not centralized control in the Internet
- Statistical multiplexing

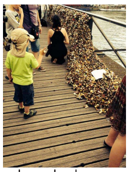
Notes:

Some Types of Access Pricing Schemes

Flat tariffs

- Fixed payment for unlimited consumption
  - Easy to apply
  - Predictable
  - Unfair fore some users
  - Wrong incentives
  - *The tragedy of the commons*

Waste of resources if no control
Source: http://www.maniacworld.com/



May lead to deplete common resources. Source: http://www.lemonde.fr

Notes:

Some Types of Access Pricing Schemes

Usage-based tariffs

- Users are charged according to consumption (e.g. for exchanged *Mbs*)
  - Less predictable
  - Needs accurate measurement
  - Problem in communications networks: difficult to accurately predict consumption

Dynamic pricing

- Price per unit time varies dynamically to reflect demand
  - Non predictable for users
  - May provide right incentives
  - Needs careful design
  - E.g. Prices that depend on congestion

Notes:

Proposed Pricing Schemes for the Internet

- Paris metro pricing (A. Odlyzko [2])
  - Proposed for differentiated services in a packet network (e.g. the Internet)
  - Inspired in the old Paris RER pricing scheme
    - Two types of wagons: expensive and normal price
    - Wagons are the same, but less people in the expensive one
  - Self regulated

Source: http://dozodomo.com

Source: http://www.rtl.fr

Notes:

And what about Net neutrality ?

- Started end 2005 by the CEO of AT&T claiming that content providers should pay ISPs to which they are not connected
- Investments are mainly made by ISPs but content providers also benefit from them
- Content providers receive much revenue from adds and content
- ISPs received revenue per transit, which is very low (1$ per Mbps /month in 2004)
- Reaction of some ISPs: block some traffic (e.g. P2P)
- Reaction of content providers and users associations: filtering traffic is against speech freedom and human rights
- Discussion at the legal level
- Pricing & QoS: can we treat users' traffic differently and remain neutral? And differentiate per application? Or per provider?

Notes: