Regeneration Theory

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Regeneration or feed-back is of considerable importance in many applications of vacuum tubes. The most obvious example is that of vacuum tube oscillators, where the feed-back is carried beyond the singing point. Another application is the 21-circuit test of balance, in which the current due to the unbalance between two impedances is fed back, the gain being increased until singing occurs. Still other applications are cases where portions of the output current of amplifiers are fed back to the input either unintentionally or by design. For the purpose of investigating the stability of such devices they may be looked on as amplifiers whose output is connected to the input through a transducer. This paper deals with the theory of stability of such systems.

Preliminary Discussion

WHEN the output of an amplifier is connected to the input through a transducer the resulting combination may be either stable or unstable. The circuit will be said to be stable when an impressed small disturbance, which itself dies out, results in a response which dies out. It will be said to be unstable when such a disturbance results in a response which goes on indefinitely, either staying at a relatively small value or increasing until it is limited by the non-linearity of the amplifier. When thus limited, the disturbance does not grow further. The net gain of the round trip circuit is then zero. Otherwise stated, the more the response increases the more does the non-linearity decrease the gain until at the point of operation the gain of the amplifier is just equal to the loss in the feed-back admittance. An oscillator under these conditions would ordinarily be called stable but it will simplify the present paper to use the definitions above and call it unstable. Now, this fact as to equality of gain and loss appears to be an accident connected with the non-linearity of the circuit and far from throwing light on the conditions for stability actually diverts attention from the essential facts. In the present discussion this difficulty will be avoided by the use of a strictly linear amplifier, which implies an amplifier of unlimited power carrying capacity. The attention will then be centered on whether an initial impulse dies out or results in a runaway condition. If a runaway condition takes place in such an amplifier, it follows that a non-linear amplifier having the same gain for small current and decreasing gain with increasing current will be unstable as well.

Steady-State Theories and Experience

First, a discussion will be made of certain steady-state theories; and reasons why they are unsatisfactory will be pointed out. The most obvious method may be referred to as the series treatment. Let the complex quantity $AJ(i\omega)$ represent the ratio by which the amplifier and feed-back circuit modify the current in one round trip, that is, let the magnitude of AJ represent the ratio numerically and let the angle of AJ represent the phase shift. It will be convenient to refer to AJ as an admittance, although it does not have the dimensions of the quantity usually so called. Let the current

$$I_0 = \cos \omega t = \text{real part of } e^{i\omega t} \tag{a}$$

be impressed on the circuit. The first round trip is then represented by

$$I_1 = \text{real part of } AJe^{i\omega t} \tag{b}$$

and the nth by

$$I_m = \text{real part of } A^nJ^ne^{i\omega t}. \tag{c}$$

The total current of the original impressed current and the first n round trips is

$$I_n = \text{real part of } (1 + AJ + A^2J^2 + \cdots A^nJ^n)e^{i\omega t}. \tag{d}$$

If the expression in parentheses converges as n increases indefinitely, the conclusion is that the total current equals the limit of (d) as n increases indefinitely. Now

$$1 + AJ + \cdots A^nJ^n = \frac{1 - A^{n+1}J^{n+1}}{1 - AJ}. \tag{e}$$

If $|AJ| < 1$ this converges to $1/(1 - AJ)$ which leads to an answer which accords with experiment. When $|AJ| > 1$ an examination of the numerator in (e) shows that the expression does not converge but can be made as great as desired by taking n sufficiently large. The most obvious conclusion is that when $|AJ| > 1$ for some frequency there is a runaway condition. This disagrees with experiment, for instance, in the case where AJ is a negative quantity numerically greater than one. The next suggestion is to assume that somehow the expression $1/(1 - AJ)$ may be used instead of the limit of (e). This, however, in addition to being arbitrary, disagrees with experimental results in the case where AJ is positive and greater than 1, where the expression $1/(1 - AJ)$ leads to a finite current but where experiment indicates an unstable condition.

The fundamental difficulty with this method can be made apparent by considering the nature of the current expressed by (*a*) above. Does the expression cos ωt indicate a current which has been going on for all time or was the current zero up to a certain time and cos ωt thereafter? In the former case we introduce infinities into our expressions and make the equations invalid; in the latter case there will be transients or building-up processes whose importance may increase as n increases but which are tacitly neglected in equations (*b*) — (*e*). Briefly then, the difficulty with this method is that it neglects the building-up processes.

Another method is as follows: Let the voltage (or current) at any point be made up of two components

$$V = V_1 + V_2, \tag{f}$$

where V is the total voltage, V_1 is the part due directly to the impressed voltage, that is to say, without the feed-back, and V_2 is the component due to feed-back alone. We have

$$V_2 = AJV. \tag{g}$$

Eliminating V_2 between (*f*) and (*g*)

$$V = V_1/(1 - AJ). \tag{h}$$

This result agrees with experiment when $|AJ| < 1$ but does not generally agree when AJ is positive and greater than unity. The difficulty with this method is that it does not investigate whether or not a steady state exists. It simply assumes tacitly that a steady state exists and if so it gives the correct value. When a steady state does not exist this method yields no information, nor does it give any information as to whether or not a steady state exists, which is the important point.

The experimental facts do not appear to have been formulated precisely but appear to be well known to those working with these circuits. They may be stated loosely as follows: There is an unstable condition whenever there is at least one frequency for which AJ is positive and greater than unity. On the other hand, when AJ is negative it may be very much greater than unity and the condition is nevertheless stable. There are instances of $|AJ|$ being about 100 without the conditions being unstable. This, as will appear, accords closely with the rule deduced below.

Notation and Restrictions

The following notation will be used in connection with integrals:

$$\int_I \phi(z)dz = \lim_{M\to\infty} \int_{-iM}^{+iM} \phi(z)dz, \tag{1}$$

the path of integration being along the imaginary axis (see equation 9), i.e., the straight line joining $-iM$ and $+iM$;

$$\int_{s+} \phi(z)dz = \lim_{M\to\infty} \int_{-iM}^{iM} \phi(z)dz, \tag{2}$$

the path of integration being along a semicircle [1] having the origin for center and passing through the points $-iM$, M, iM;

$$\int_C \phi(z)dz = \lim_{M\to\infty} \int_{-iM}^{-iM} \phi(z)dz, \tag{3}$$

the path of integration being first along the semicircle referred to and then along a straight line from iM to $-iM$. Referring to Fig. 1 it

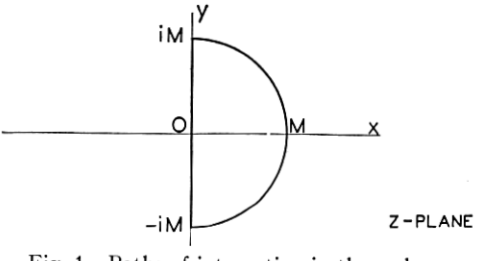

Fig. 1—Paths of integration in the z-plane.

will be seen that

$$\int_{s+} - \int_I = \int_C \cdot \tag{4}$$

The total feed-back circuit is made up of an amplifier in tandem with a network. The amplifier is characterized by the amplifying ratio A which is independent of frequency. The network is characterized by the ratio $J(i\omega)$ which is a function of frequency but does not depend on the gain. The total effect of the amplifier and the network is to multiply the wave by the ratio $AJ(i\omega)$. An alternative way of characterizing the amplifier and network is to say that the amplifier is

[1] For physical interpretation of paths of integration for which $x > 0$ reference is made to a paper by J. R. Carson, "Notes on the Heaviside Operational Calculus," *B. S. T. J.*, Jan. 1930. For purposes of the present discussion the semicircle is preferable to the path there discussed.

characterized by the amplifying factor A which is independent of time, and the network by the real function $G(t)$ which is the response caused by a unit impulse applied at time $t = 0$. The combined effect of the amplifier and network is to convert a unit impulse to the function $AG(t)$. Both these characterizations will be used.

The restrictions which are imposed on the functions in order that the subsequent reasoning may be valid will now be stated. There is no restriction on A other than that it should be real and independent of time and frequency. In stating the restrictions on the network it is convenient to begin with the expression G. They are

$$G(t) \text{ has bounded variation, } -\infty < t < \infty. \tag{AI}$$

$$G(t) = 0, \qquad -\infty < t < 0. \tag{AII}$$

$$\int_{-\infty}^{\infty} |G(t)|\,dt \text{ exists.} \tag{AIII}$$

It may be shown [2] that under these conditions $G(t)$ may be expressed by the equation

$$G(t) = \frac{1}{2\pi i}\int_I J(i\omega)e^{i\omega t}d(i\omega), \tag{5}$$

where

$$J(i\omega) = \int_{-\infty}^{\infty} G(t)e^{-i\omega t}dt. \tag{6}$$

These expressions may be taken to define J. The function may, however, be obtained directly from computations or measurements; in the latter case the function is not defined for negative values of ω. It must be defined as follows to be consistent with the definition in (6):

$$J(-i\omega) = \text{complex conjugate of } J(i\omega). \tag{7}$$

While the final results will be expressed in terms of $AJ(i\omega)$ it will be convenient for the purpose of the intervening mathematics to define an auxiliary and closely related function

$$w(z) = \frac{1}{2\pi i}\int_I \frac{AJ(i\omega)}{i\omega - z}d(i\omega), \qquad 0 < x < \infty, \tag{8}$$

where

$$z = x + iy \tag{9}$$

and where x and y are real. Further, we shall define

$$w(iy) = \lim_{x \to 0} w(z). \tag{10}$$

[2] See Appendix II for fuller discussion.

The function will not be defined for $x < 0$ nor for $|z| = \infty$. As defined it is analytic [3] for $0 < x < \infty$ and at least continuous for $x = 0$.

The following restrictions on the network may be deduced:

$$\lim_{y \to \infty} y|J(iy)| \text{ exists.} \tag{BI}$$

$$J(iy) \text{ is continuous.} \tag{BII}$$

$$w(iy) = AJ(iy). \tag{BIII}$$

Equation (5) may now be written

$$AG(t) = \frac{1}{2\pi i}\int_I w(z)e^{zt}dz = \frac{1}{2\pi i}\int_{s+} w(z)e^{zt}dz. \tag{11}$$

From a physical standpoint these restrictions are not of consequence. Any network made up of positive resistances, conductances, inductances, and capacitances meets them. Restriction (AII) says that the response must not precede the cause and is obviously fulfilled physically. Restriction (AIII) is fulfilled if the response dies out at least exponentially, which is also assured. Restriction (AI) says that the transmission must fall off with frequency. Physically there are always enough distributed constants present to insure this. This effect will be illustrated in example 8 below. Every physical network falls off in transmission sooner or later and it is ample for our purposes if it begins to fall off, say, at optical frequencies. We may say then that the reasoning applies to all linear networks which occur in nature. It also applies to other linear networks which are not physically producible but which may be specified mathematically. See example 7 below.

A temporary wave $f_0(t)$ is to be introduced into the system and an investigation will be made of whether the resultant disturbance in the system dies out. It has associated with it a function $F(z)$ defined by

$$f_0(t) = \frac{1}{2\pi i}\int_I F(z)e^{zt}dz = \frac{1}{2\pi i}\int_{s+} F(z)e^{zt}dz. \tag{12}$$

$F(z)$ and $f_0(t)$ are to be made subject to the same restrictions as $w(z)$ and $G(t)$ respectively.

Derivation of a Series for the Total Current

Let the amplifier be linear and of infinite power-carrying capacity. Let the output be connected to the input in such a way that the

[3] W. F. Osgood, "Lehrbuch der Funktionentheorie," 5th ed., Kap. 7, § 1, Hauptsatz. For definition of "analytic" see Kap. 6, § 5.

amplification ratio for one round trip is equal to the complex quantity AJ, where A is a function of the gain only and J is a function of ω only, being defined for all values of frequency from 0 to ∞.

Let the disturbing wave $f_0(t)$ be applied anywhere in the circuit. We have

$$f_0(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(i\omega)e^{t\omega t}d\omega \tag{13}$$

or

$$f_0(t) = \frac{1}{2\pi i}\int_{s+} F(z)e^{zt}dz. \tag{13'}$$

The wave traverses the circuit and on completing the first trip it becomes

$$f_1(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} w(i\omega)\,F(i\omega)e^{t\omega t}d\omega \tag{14}$$

$$= \frac{1}{2\pi i}\int_{s+} w(z)\,F(z)e^{zt}dz. \tag{14'}$$

After traversing the circuit a second time it becomes

$$f_2(t) = \frac{1}{2\pi i}\int_{s+} Fw^2e^{zt}dz, \tag{15}$$

and after traversing the circuit n times

$$f_n(t) = \frac{1}{2\pi i}\int_{s+} Fw^ne^{zt}dz. \tag{16}$$

Adding the voltage of the original impulse and the first n round trips we have a total of

$$s_n(t) = \sum_{k=0}^{n} f_k(t) = \frac{1}{2\pi i}\int_{s+} F(1 + w + \cdots w^n)e^{zt}dz. \tag{17}$$

The total voltage at the point in question at the time t is given by the limiting value which (17) approaches as n is increased indefinitely [4]

$$s(t) = \sum_{k=0}^{\infty} f_k(t) = \lim_{n\to\infty}\frac{1}{2\pi i}\int_{s+} S_n(z)e^{zt}dz, \tag{18}$$

where

$$S_n = F + Fw + Fw^2 + \cdots Fw^n = \frac{F(1 - w^{n+1})}{1 - w}. \tag{19}$$

[4] Mr. Carson has called my attention to the fact that this series can also be derived from Theorem IX, p. 49, of his Electric Circuit Theory. Whereas the present derivation is analogous to the theory expressed in equations (*a*)–(*e*) above, the alternative derivation would be analogous to that in equations (*f*)–(*h*).

Convergence of Series

We shall next prove that the limit $s(t)$ exists for all finite values of t. It may be stated as of incidental interest that the limit

$$\int_{s+} S_\infty(z)e^{izt}dz \tag{20}$$

does not necessarily exist although the limit $s(t)$ does. Choose M_0 and N such that

$$|f_0(\lambda)| \leq M_0. \qquad 0 \leq \lambda \leq t. \tag{21}$$

$$|G(t-\lambda)| \leq N. \quad 0 \leq \lambda \leq t. \tag{22}$$

We may write [5]

$$f_1(t) = \int_{-\infty}^{\infty} G(t-\lambda)f_0(\lambda)d\lambda. \tag{23}$$

$$|f_1(t)| \leq \int_0^t M_0 N d\lambda = M_0 N t. \tag{24}$$

$$f_2(t) = \int_{-\infty}^{\infty} G(t-\lambda)f_1(\lambda)d\lambda. \tag{25}$$

$$|f_2(t)| \leq \int_0^t M_0 N^2 t dt = M_0 N^2 t^2/2! \tag{26}$$

Similarly

$$|f_n(t)| \leq M_0 N^n t^n/n! \tag{27}$$

$$|s_n(t)| \leq M_0(1 + Nt + \cdots N^n t^n/n!). \tag{28}$$

It is shown in almost any text [6] dealing with the convergence of series that the series in parentheses converges to e^{Nt} as n increases indefinitely. Consequently, $s_n(t)$ converges absolutely as n increases indefinitely.

Relation Between $s(t)$ and w

Next consider what happens to $s(t)$ as t increases. As t increases indefinitely $s(t)$ may converge to zero, indicating a condition of stability, or it may go beyond any value however large, indicating a runaway condition. The question which presents itself is: *Referring to (18) and (19), what properties of $w(z)$ and further what properties of $AJ(i\omega)$ determine whether $s(t)$ converges to zero or diverges as t increases*

[5] G. A. Campbell, "Fourier Integral," *B. S. T. J.*, Oct. 1928, Pair 202.
[6] E.g., Whittaker and Watson, "Modern Analysis," 2d ed., p. 531.

indefinitely? From (18) and (19)

$$s(t) = \lim_{n\to\infty} \frac{1}{2\pi i}\int_{s+} F\left(\frac{1}{1-w} - \frac{w^{n+1}}{1-w}\right) e^{zt}dz. \tag{29}$$

We may write

$$s(t) = \frac{1}{2\pi i}\int_{s+} [F/(1-w)]e^{zt}dz - \lim_{n\to\infty}\frac{1}{2\pi i}\int_{s+}[Fw^{n+1}/(1-w)]e^{zt}dz \tag{30}$$

provided these functions exist. Let them be called $q_0(t)$ and $\lim_{n\to\infty} q_n(t)$ respectively. Then

$$q_n(t) = \int_{-\infty}^{\infty} q_0(t-\lambda)\phi(\lambda)d\lambda. \tag{31}$$

where

$$\phi(\lambda) = \frac{1}{2\pi i}\int_{s+} w^{n+1}e^{z\lambda}dz. \tag{32}$$

By the methods used under the discussion of convergence above it can then be shown that this expression exists and approaches zero as n increases indefinitely provided $q_0(t)$ exists and is equal to zero for $t < 0$. Equation (29) may therefore be written, subject to these conditions

$$s(t) = \frac{1}{2\pi i}\int_{s+} [F/(1-w)]e^{zt}dz. \tag{33}$$

In the first place the integral is zero for negative values of t because the integrand approaches zero faster than the path of integration increases. Moreover,

$$\int_I [F/(1-w)]e^{zt}dz \tag{34}$$

exists for all values of t and approaches zero for large values of t if $1 - w$ does not equal zero on the imaginary axis. Moreover, the integral

$$\int_C [F/(1-w)]e^{zt}dz \tag{35}$$

exists because

1. Since F and w are both analytic within the curve the integrand does not have any essential singularity there,
2. The poles, if any, lie within a finite distance of the origin because $w \to 0$ as $|z|$ increases, and
3. These two statements insure that the total number of poles is finite.

We shall next evaluate the integral for a very large value of t. It will suffice to take the C integral since the I integral approaches zero. Assume originally that $1 - w$ does not have a root on the imaginary axis and that $F(z)$ has the special value $w'(z)$. The integral may be written

$$\frac{1}{2\pi i}\int_C [w'/(1 - w)]e^{zt}dz. \tag{36}$$

Changing variables it becomes

$$\frac{1}{2\pi i}\int_D [1/(1 - w)]e^{zt}dw, \tag{37}$$

where z is a function of w and D is the curve in the w plane which corresponds to the curve C in the z plane. More specifically the imaginary axis becomes the locus $x = 0$ and the semicircle becomes a small curve which spirals around the origin. See Fig. 2. The function

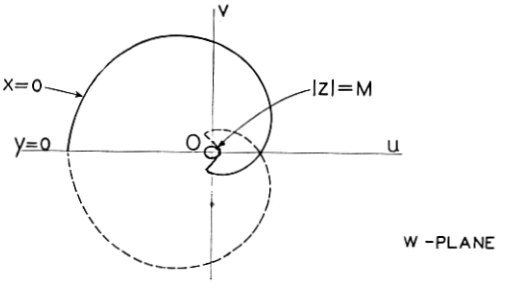

Fig. 2—Representative paths of integration in the w-plane corresponding to paths in Fig. 1.

z and, therefore, the integrand is, in general, multivalued and the curve of integration must be considered as carried out over the appropriate Riemann surface.[7]

Now let the path of integration shrink, taking care that it does not shrink across the pole at $w = 1$ and initially that it does not shrink across such branch points as interfere with its passage, if any. This shrinking does not alter the integral [8] because the integrand is analytic at all other points. At branch points which interfere with the passage of the path the branches stopped may be severed, transposed and connected in such a way that the shrinking may be continued past the branch point. This can be done without altering the value of the integral. Thus the curve can be shrunk until it becomes one or more very small circles surrounding the pole. The value of the total integral

[7] Osgood, loc. cit., Kap. 8.
[8] Osgood, loc. cit., Kap. 7, § 3, Satz 1.

(for very large values of t) is by the method of residues [9]

$$\sum_{j=1}^{n} r_j e^{z_j t}, \tag{38}$$

where z_j $(j = 1, 2 \cdots n)$ is a root of $1 - w = 0$ and r_j is its order. The real part of z_j is positive because the curve in Fig. 1 encloses points with $x > 0$ only. The system is therefore stable or unstable according to whether

$$\sum_{j=1}^{n} r_j$$

is equal to zero or not. But the latter expression is seen from the procedure just gone through to equal the number of times that the locus $x = 0$ encircles the point $w = 1$.

If F does not equal w' the calculation is somewhat longer but not essentially different. The integral then equals

$$\sum_{j=1}^{n} \frac{F(z_j)}{w(z_j)} e^{z_j t} \tag{39}$$

if all the roots of $1 - w = 0$ are distinct. If the roots are not distinct the expression becomes

$$\sum_{j=1}^{n} \sum_{k=1}^{r_j} A_{jk} t^{k-1} e^{z_j t}, \tag{40}$$

where A_{jr_j}, at least, is finite and different from zero for general values of F. It appears then that unless F is specially chosen the result is essentially the same as for $F = w'$. The circuit is stable if the point lies wholly outside the locus $x = 0$. It is unstable if the point is within the curve. It can also be shown that if the point is on the curve conditions are unstable. We may now enunciate the following

Rule: Plot plus and minus the imaginary part of $AJ(i\omega)$ against the real part for all frequencies from 0 to ∞. If the point $1 + i0$ lies completely outside this curve the system is stable; if not it is unstable.

In case of doubt as to whether a point is inside or outside the curve the following criterion may be used: Draw a line from the point $(u = 1, v = 0)$ to the point $z = -i\infty$. Keep one end of the line fixed at $(u = 1, v = 0)$ and let the other end describe the curve from $z = -i\infty$ to $z = i\infty$, these two points being the same in the w plane. If the net angle through which the line turns is zero the point $(u = 1, v = 0)$ is on the outside, otherwise it is on the inside.

If AJ be written $|AJ|(\cos \theta + i \sin \theta)$ and if the angle always

[9] Osgood, loc. cit., Kap. 7, § 11, Satz 1.

changes in the same direction with increasing ω, where ω is real, the rule can be stated as follows: The system is stable or unstable according to whether or not a real frequency exists for which the feed-back ratio is real and equal to or greater than unity.

In case $d\theta/d\omega$ changes sign we may have the case illustrated in Figs. 3 and 4. In these cases there are frequencies for which w is real and

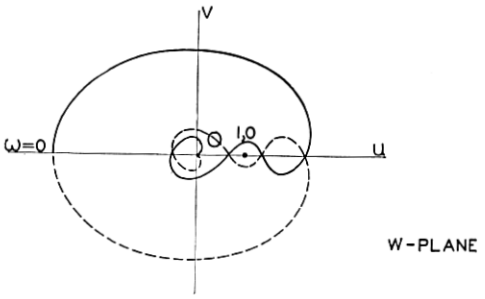

Fig. 3—Illustrating case where amplifying ratio is real and greater than unity for two frequencies, but where nevertheless the path of integration does not include the point 1, 0.

greater than 1. On the other hand, the point (1, 0) is outside of the locus $x = 0$ and, therefore, according to the rule there is a stable condition.

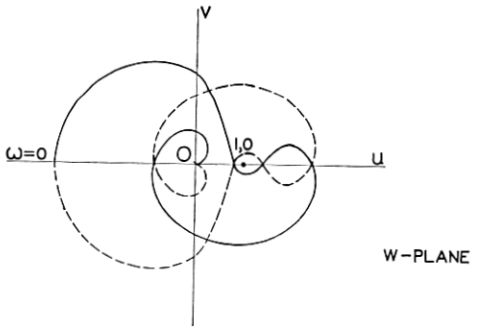

Fig. 4—Illustrating case where amplifying ratio is real and greater than unity for two frequencies, but where nevertheless the path of integration does not include the point 1, 0.

If networks of this type were used we should have the following interesting sequence of events: For low values of A the system is in a stable condition. Then as the gain is increased gradually, the system becomes unstable. Then as the gain is increased gradually still further, the system again becomes stable. As the gain is still further increased the system may again become unstable.

Examples

The following examples are intended to give a more detailed picture of certain rather simple special cases. They serve to illustrate the previous discussion. In all the cases F is taken equal to AJ so that f_0 is equal to AG. This simplifies the discussion but does not detract from the illustrative value.

1. Let the network be pure resistance except for the distortionless amplifier and a single bridged condenser, and let the amplifier be such that there is no reversal. We have

$$AJ(i\omega) = \frac{B}{\alpha + i\omega}, \tag{41}$$

where A and α are real positive constants. In (18) [10]

$$\begin{aligned} f_n &= \frac{1}{2\pi i}\int_I A^{n+1}J^{n+1}(i\omega)e^{i\omega t}di\omega \\ &= Be^{-\alpha t}(B^n t^n/n!). \end{aligned} \tag{42}$$

$$s(t) = Be^{-\alpha t}(1 + Bt + B^2t^2/2! + \cdots). \tag{43}$$

The successive terms f_0, f_1, etc., represent the impressed wave and the successive round trips. The whole series is the total current.

It is suggested that the reader should sketch the first few terms graphically for $B = \alpha$, and sketch the admittance diagrams for $B < \alpha$, and $B > \alpha$.

The expression in parentheses equals e^{Bt} and

$$s(t) = Be^{(B-\alpha)t}. \tag{44}$$

This expression will be seen to converge to 0 as t increases or fail to do so according to whether $B < \alpha$ or $B \geq \alpha$. This will be found to check the rule as applied to the admittance diagram.

2. Let the network be as in 1 except that the amplifier is so arranged that there is a reversal. Then

$$AJ(i\omega) = \frac{-B}{\alpha + i\omega}. \tag{45}$$

$$f_n = (-1)^{n+1}Be^{-\alpha t}(B^n t^n/n!). \tag{46}$$

The solution is the same as in 1 except that every other term in the series has its sign reversed:

$$\begin{aligned} s(t) &= -Be^{-\alpha t}(1 - Bt + B^2t^2/2! + \cdots) \\ &= -Be^{(-\alpha-B)t}. \end{aligned} \tag{47}$$

[10] Campbell, loc. cit. Pair 105.

This converges to 0 as t increases regardless of how great B may be taken. If the admittance diagram is drawn this is again found to check the rule.

3. Let the network be as in 1 except that there are two separated condensers bridged across resistance circuits. Then

$$AJ(i\omega) = \frac{B^2}{(\alpha + i\omega)^2} \cdot \tag{48}$$

The solution for $s(t)$ is obtained most simply by taking every other term in the series obtained in 1.

$$\begin{aligned} s(t) &= Be^{-\alpha t}(Bt + B^3t^3/3! + \cdots) \\ &= Be^{-\alpha t} \sinh Bt. \end{aligned} \tag{49}$$

4. Let the network be as in 3 except that there is a reversal. Then

$$AJ(i\omega) = \frac{-B^2}{(\alpha + i\omega)^2} \cdot \tag{50}$$

The solution is obtained most directly by reversing the sign of every other term in the series obtained in 3.

$$\begin{aligned} s(t) &= -Be^{-\alpha t}(Bt - B^3t^3/3! + \cdots) \\ &= -Be^{-\alpha t} \sin Bt. \end{aligned} \tag{51}$$

This is a most instructive example. An approximate diagram has been made in Fig. 5, which shows that as the gain is increased the

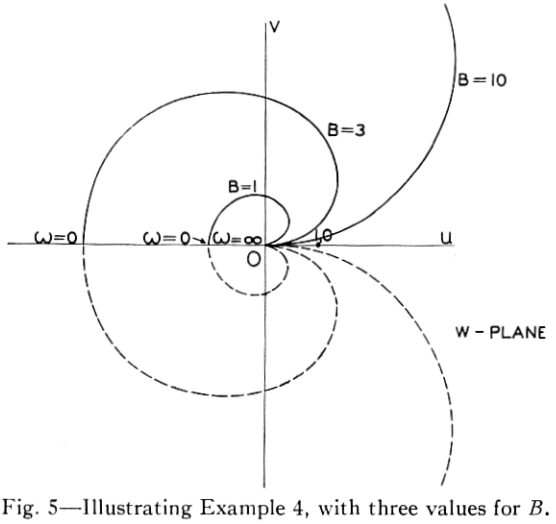

Fig. 5—Illustrating Example 4, with three values for B.

feed-back ratio may be made arbitrarily great and the angle arbitrarily small without the condition being unstable. This agrees with the expression just obtained, which shows that the only effect of increasing the gain is to increase the frequency of the resulting transient.

5. Let the conditions be as in 1 and 3 except for the fact that four separated condensers are used. Then

$$AJ(i\omega) = \frac{B^4}{(\alpha + i\omega)^4}. \tag{52}$$

The solution is most readily obtained by selecting every fourth term in the series obtained in 1.

$$\begin{aligned} s(t) &= Be^{-\alpha t}(B^3t^3/3! + B^7t^7/7! + \cdots) \\ &= \tfrac{1}{2}Be^{-\alpha t}\ (\sinh Bt - \sin Bt). \end{aligned} \tag{53}$$

This indicates a condition of instability when $B \geqq \alpha$, agreeing with the result deducible from the admittance diagram.

6. Let the conditions be as in 5 except that there is a reversal. Then

$$Y = \frac{-B^4}{(\alpha + i\omega)^4}. \tag{54}$$

The solution is most readily obtained by changing the sign of every other term in the series obtained in 5.

$$s(t) = Be^{-\alpha t}(-B^3t^3/3! + B^7t^7/7! - \cdots). \tag{55}$$

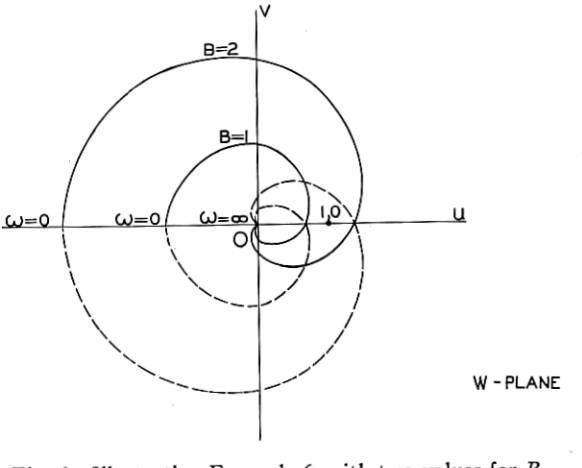

Fig. 6—Illustrating Example 6, with two values for B.

For large values of t this approaches

$$s(t) = -\tfrac{1}{2}Be^{(B/\sqrt{2}-\alpha)t}\sin(Bt/\sqrt{2} - \pi/4). \tag{56}$$

This example is interesting because it shows a case of instability although there is a reversal. Fig. 6 shows the admittance diagram for $B\sqrt{2} - \alpha < 0$ and for $B\sqrt{2} - \alpha > 0$.

7. Let

$$AG(t) = f_0(t) = A(1 - t), \qquad 0 \leq t \leq 1. \tag{57}$$

$$AG(t) = f_0(t) = 0, \qquad -\infty < t < 0, \qquad 1 < t < \infty. \tag{57'}$$

We have

$$AJ(i\omega) = A\int_0^1 (1 - t)e^{-t\omega t}dt$$

$$= A\left(\frac{1 - e^{-t\omega}}{\omega^2} + \frac{1}{i\omega}\right). \tag{58}$$

Fig. 7 is a plot of this case for $A = 1$.

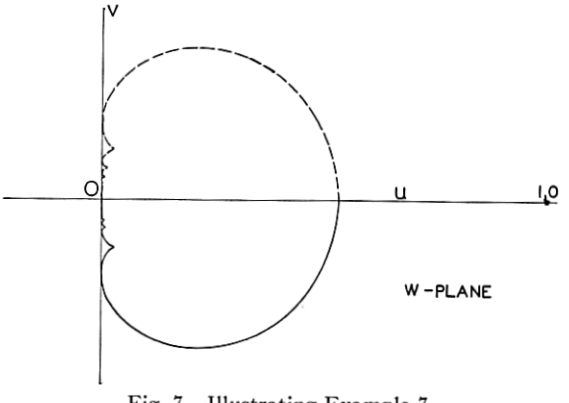

Fig. 7—Illustrating Example 7.

8. Let

$$AJ(i\omega) = \frac{A(1 + i\omega)}{(1 + i2\omega)}. \tag{59}$$

This is plotted on Fig. 8 for $A = 3$. It will be seen that the point 1 lies outside of the locus and for that reason we should expect that the system would be stable. We should expect from inspecting the diagram that the system would be stable for $A < 1$ and $A > 2$ and that it would be unstable for $1 \leq A \leq 2$. We have overlooked one fact, however; the expression for $AJ(i\omega)$ does not approach zero as ω

increases indefinitely. Therefore, it does not come within restriction (BI) and consequently the reasoning leading up to the rule does not apply.

The admittance in question can be made up by bridging a capacity in series with a resistance across a resistance line. This admittance

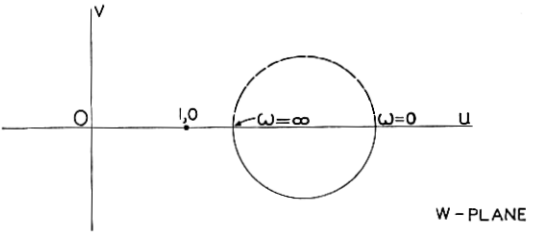

Fig. 8—Illustrating Example 8, without distributed constants.

obviously does not approach zero as the frequency increases. In any actual network there would, however, be a small amount of distributed capacity which, as the frequency is increased indefinitely, would cause the transmission through the network to approach zero. This is shown graphically in Fig. 9. The effect of the distributed capacity is

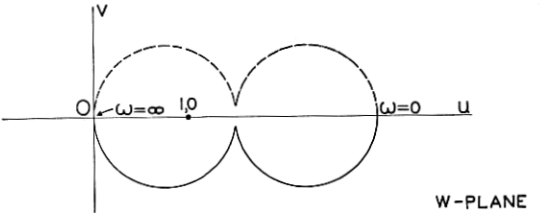

Fig. 9—Illustrating Example 8, with distributed constants.

essentially to cut a corridor from the circle in Fig. 8 to the origin, which insures that the point lies inside the locus.

Appendix I

Alternative Procedure

In some cases $AJ(i\omega)$ may be given as an analytic expression in $(i\omega)$. In that case the analytic expression may be used to define w for all values of z for which it exists. If the value for $AJ(i\omega)$ satisfies all the restrictions the value thus defined equals the w defined above for $0 \leq x < \infty$ only. For $-\infty < x < 0$ it equals the analytic continuation of the function w defined above. If there are no essential

singularities anywhere including at ∞, the integral in (33) may be evaluated by the theory of residues by completing the path of integration so that all the poles of the integrand are included. We then have

$$s(t) = \sum_{j=1}^{j=n} \sum_{k=1}^{r_j} A_{jk} t^{k-1} e^{z_j t}. \tag{60}$$

If the network is made up of a finite number of lumped constants there is no essential singularity and the preceding expression converges because it has only a finite number of terms. In other cases there is an infinite number of terms, but the expression may still be expected to converge, at least, in the usual case. Then the system is stable if all the roots of $1 - w = 0$ have $x < 0$. If some of the roots have $x \geq 0$ the system is unstable.

The calculation then divides into three parts:

1. The recognition that the impedance function is $1 - w$.[11]

2. The determination of whether the impedance function has zeros for which $x \geq 0$.[12]

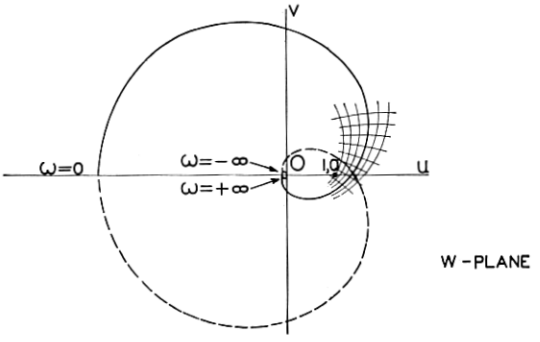

Fig. 10—Network of loci x = const., and y = const.

3. A deduction of a rule for determining whether there are roots for which $x \geq 0$. The actual solution of the equation is usually too laborious.

To proceed with the third step, plot the locus $x = 0$ in the w plane, i.e., plot the imaginary part of w against the real part for all the values of y, $-\infty < y < \infty$. See Fig. 10. Other loci representing

$$x = \text{const.} \tag{61}$$

and

$$y = \text{const.} \tag{62}$$

[11] Cf. H. W. Nichols, *Phys. Rev.*, vol. 10, pp. 171–193, 1917.
[12] Cf. Thompson and Tait, "Natural Philosophy," vol. I, § 344.

may be considered and are indicated by the network shown in the figure in fine lines. On one side of the curve x is positive and on the other it is negative. Consider the equation

$$w(z) - 1 = 0$$

and what happens to it as A increases from a very small to a very large value. At first the locus $x = 0$ lies wholly to the left of the point. For this case the roots must have $x < 0$. As A increases there may come a time when the curve or successive convolutions of it will sweep over the point $w = 1$. For every such crossing at least one of the roots changes the sign of its x. We conclude that if the point $w = 1$ lies inside the curve the system is unstable. It is now possible to enunciate the rule as given in the main part of the paper but there deduced with what appears to be a more general method.

Appendix II

Discussion of Restrictions

The purpose of this appendix is to discuss more fully the restrictions which are placed on the functions defining the network. A full discussion in the main text would have interrupted the main argument too much.

Define an additional function

$$n(z) = \frac{1}{2\pi i}\int_I \frac{AJ(i\lambda)}{i\lambda - z}d(i\lambda), \qquad -\infty < x < 0. \tag{63}$$

$$n(iy) = \lim_{x\to 0} n(z).$$

This definition is similar to that for $w(z)$ given previously. It is shown in the theorem [13] referred to that these functions are analytic for $x \neq 0$ if $AJ(i\omega)$ is continuous. We have not proved, as yet, that the restrictions placed on $G(t)$ necessarily imply that $J(i\omega)$ is continuous. For the time being we shall assume that $J(i\omega)$ may have finite discontinuities. The theorem need not be restricted to the case where $J(i\omega)$ is continuous. From an examination of the second proof it will be seen to be sufficient that $\int_I J(i\omega)d(i\omega)$ exist. Moreover, that proof can be slightly modified to include all cases where conditions (AI)–(AIII) are satisfied.

[13] Osgood, loc. cit.

For, from the equation at top of page 298 [13]

$$\left|\frac{w(z_0-\Delta z)-w(z_0)}{\Delta z}-\frac{1}{2\pi i}\int_I \frac{AJ(i\lambda)}{(i\lambda-z_0)^2}d(i\lambda)\right| \leq |\Delta z|\left|\frac{1}{2\pi i}\int_I \frac{AJ(i\lambda)d(i\lambda)}{(i\lambda-z_0-\Delta z)(i\lambda-z_0)^2}\right|, \qquad x_0>0. \quad (64)$$

It is required to show that the integral exists. Now

$$\int_I \frac{AJ(i\lambda)d(i\lambda)}{(i\lambda-z_0-\Delta z)(i\lambda-z_0)^2} = \int_I \frac{AJ(i\lambda)d(i\lambda)}{(i\lambda-z_0)^3}\left(1+\frac{\Delta z}{i\lambda-z_0}+\frac{\Delta z^2}{i\lambda-z_0}+\text{etc.}\right) \quad (65)$$

if Δz is taken small enough so the series converges. It will be sufficient to confine attention to the first term. Divide the path of integration into three parts,

$$-\infty<\lambda<-|z_0|-1, \qquad -|z_0|-1<\lambda<|z_0|+1, \qquad |z_0|+1<\lambda<\infty.$$

In the middle part the integral exists because both the integrand and the range of integration are finite. In the other ranges the integral exists if the integrand falls off sufficiently rapidly with increasing λ. It is sufficient for this purpose that condition (BI) be satisfied. The same proof applies to $n(z)$.

Next, consider $\lim_{x\to 0} w(z) = w(iy)$. If iy is a point where $J(iy)$ is continuous, a straightforward calculation yields

$$w(iy) = AJ(iy)/2 + P(iy). \quad (66a)$$

Likewise,

$$n(iy) = -AJ(iy)/2 + P(iy) \quad (66b)$$

where $P(iy)$ is the principal value [14] of the integral

$$\frac{1}{2\pi i}\int_I \frac{AJ(i\lambda)}{i\lambda-iy}d(i\lambda).$$

Subtracting

$$w(iy) - n(iy) = AJ(iy) \quad (67)$$

If (iy) is a point of discontinuity of $J(iy)$

$$|w| \text{ and } |n| \text{ increase indefinitely as } x\to 0. \quad (68)$$

Next, evaluate the integral

$$\frac{1}{2\pi i}\int_{x+I} w(z)e^{zt}dz,$$

[14] E. W. Hobson, "Functions of a Real Variable," vol. I, 3d edition, § 352.

where the path of integration is from $x - i\infty$ to $x + i\infty$ along the line $x = \text{const}$. On account of the analytic nature of the integrand this integral is independent of x (for $x > 0$). It may be written then

$$\lim_{x\to 0}\frac{1}{2\pi i}\int_{x+I} w(z)e^{zt}dz = \lim_{x\to 0}\frac{1}{2\pi i}\int_{x+I}\frac{1}{2\pi i}\int_{I}\frac{AJ(i\lambda)}{i\lambda - z}e^{zt}d(i\lambda)dz$$

$$= \lim_{x\to 0}\frac{1}{2\pi i}\int_{x+I}\frac{1}{2\pi i}\lim_{M\to\infty}\left[\int_{-tM}^{ty-t\delta} + \int_{ty-t\delta}^{ty+t\delta} + \int_{ty+t\delta}^{tM}\right]\frac{AJ(i\lambda)}{i\lambda - z}e^{zt}d(i\lambda)dz$$

$$= \lim_{x\to 0}\left[\frac{1}{2\pi i}\int_{x+I}\frac{1}{2\pi i}\int_{ty-t\delta}^{ty+t\delta}\frac{AJ(i\lambda)}{i\lambda - z}e^{zt}d(i\lambda)dz + Q(t,\,\delta)\right],\ x>0, \quad (69)$$

where δ is real and positive. The function Q defined by this equation exists for all values of t and for all values of δ. Similarly,

$$\lim_{x\to 0}\frac{1}{2\pi i}\int_{x+I} n(z)e^{zt}dz$$

$$= \left[\lim_{x\to 0}\frac{1}{2\pi i}\int_{x+I}\frac{1}{2\pi i}\int_{ty+t\delta}^{ty+t\delta}\frac{AJ(i\lambda)}{i\lambda - z}e^{zt}d(i\lambda)dz + Q(t,\,\delta)\right],\ x<0, \quad (70)$$

Subtracting and dropping the limit designations

$$\frac{1}{2\pi i}\int_{x+I} w(z)e^{zt}dz - \frac{1}{2\pi i}\int_{x+I} n(z)e^{zt}dz = \frac{1}{2\pi i}\int_{I} AJ(i\lambda)e^{i\lambda t}d(i\lambda). \quad (71)$$

The first integral is zero for $t < 0$ as can be seen by taking x sufficiently large. Likewise, the second is equal to zero for $t > 0$. Therefore,

$$\frac{1}{2\pi i}\int_{x+I} w(z)e^{zt}dz = \frac{1}{2\pi i}\int_{I} AJ(i\omega)e^{i\omega t}d(i\omega) = AG(t),\quad 0 < t < \infty \quad (72)$$

$$-\frac{1}{2\pi i}\int_{x+I} n(z)e^{zt}dz$$

$$= \frac{1}{2\pi i}\int_{I} AJ(i\omega)e^{i\omega t}d(i\omega) = AG(t) \quad -\infty < t < 0. \quad (73)$$

We may now conclude that

$$\int_{I} n(iy)e^{iyt}d(iy) = 0, \qquad -\infty < t < \infty \quad (74)$$

provided

$$G(t) = 0, \qquad -\infty < t < 0. \quad \text{(AII)}$$

But (74) is equivalent to

$$n(z) = 0, \quad (74')$$

which taken with (67) gives

$$w(iy) = AJ(iy). \tag{BIII}$$

(BIII) is, therefore, a necessary consequence of (AII). (74′) taken with (68) shows that

$$J(iy) \text{ is continuous.} \tag{BII}$$

It may be shown [15] that (BI) is a consequence of (AI). Consequently all the B conditions are deducible from the A conditions.

Conversely, it may be inquired whether the A conditions are deducible from the B conditions. This is of interest if $AJ(i\omega)$ is given and is known to satisfy the B conditions, whereas nothing is known about G.

Condition AII is a consequence of BIII as may be seen from (67) and (74). On the other hand AI and AIII cannot be inferred from the B conditions. It can be shown by examining (5), however, that if the slightly more severe condition

$$\lim_{y \to \infty} y^{\gamma} J(iy) \text{ exists}, \qquad (\gamma > 1), \tag{BIa}$$

is satisfied then

$$G(t) \text{ exists}, \qquad -\infty < t < \infty, \tag{AIa}$$

which, together with AII, insures the validity of the reasoning.

It remains to show that the measured value of $J(i\omega)$ is equal to that defined by (6). The measurement consists essentially in applying a sinusoidal wave and determining the response after a long period. Let the impressed wave be

$$E = \text{real part of } e^{i\omega t}, \qquad t \geq 0. \tag{75}$$
$$E = 0, \qquad t < 0. \tag{75′}$$

The response is

$$\text{real part of } \int_0^t AG(\lambda) e^{i\omega(t-\lambda)} d\lambda$$
$$= \text{real part of } Ae^{i\omega t} \int_0^t G(\lambda) e^{-i\omega\lambda} d\lambda. \tag{76}$$

For large values of t this approaches

$$\text{real part of } Ae^{i\omega t} J(i\omega). \tag{77}$$

Consequently, the measurements yield the value $AJ(i\omega)$.

[15] See Hobson, loc. cit., vol. II, 2d edition, § 335. It will be apparent that K depends on the total variation but is independent of the limits of integration.