## Approximability status of Survivable Network problems

by Zeev Nutov

In Survivable Network problems (a.k.a. Survivable Network Design Problem – SNDP) we are given a graph with costs/weights on edges and/or nodes and prescribed connectivity requirements/demands. Among the subgraphs of G that satisfy the requirements, we seek to find one of minimum cost. Formally, the problem is defined as follows. Given a graph G = (V, E) and  $Q \subseteq V$ , the Q-connectivity  $\lambda_G^Q(uv)$  of uv in G is the maximum number of edge-disjoint uv-paths such that no two of them have a node in  $Q - \{u, v\}$  in common. The case  $S = \emptyset$  is just the edge-connectivity when the paths should be edge-disjoint, and the case S = V is just the node-connectivity when the paths should be internally node-disjoint.

## Survivable Network

Instance: A (possibly directed) graph G = (V, E) with edge/node-costs, a node subset  $Q \subseteq V$ , and a nonnegative integer requirements  $\{r_{uv} : uv \in D\}$  on a set D of demand pairs on a set  $S \subseteq V$  of terminals.

Objective: Find a minimum cost subgraph G' of G such that  $\lambda_{G'}^Q(uv) \geqslant r_{uv}$  for all  $uv \in D$ .

Extensively studied particular choices of Q are edge-connectivity  $(Q = \emptyset)$ , node-connectivity (Q = V), and element-connectivity  $(r_{uv} = 0 \text{ whenever } u \in Q \text{ or } v \in Q)$ .

Given an instance of Survivable Network let  $k = \max_{uv \in D} r_{uv}$  denote the maximum connectivity requirement, and let k-Survivable Network be the restriction of Survivable Network to instances with  $\max_{uv \in D} r_{uv} = k$ .

Survivable Network has received considerable attention in the past, c.f. surveys in [17, 28, 34]. The edge-connectivity version admits an elegant 2-approximation algorithm via the seminal iterative rounding method by Jain [27] (see also [39] for an elegant and short proof). On the other hand, the only known nontrivial ratio for the node-connectivity version is  $O(k^3 \log |D|)$  [12] due to Chuzhoy and Khanna; the problem also admits a folklore ratio |D|.

The following classification of Survivable Network problems is widely used, c.f. [34]. We may assume that the input graph G is complete (edges that do not appear in G can be added to G and assigned infinite costs). Under this assumption, the edge costs are categorized as follows:

- $\{0,1\}$ -costs (known also as "augmentation problems"): here we are given an initial graph  $G_0$  (formed by the edges of cost 0), and the goal is to find a min-size augmenting edge set F of new edges (any edge is allowed and has cost 1) such that the graph  $G' = G_0 + F$  satisfies the requirements.
- $\{1,\infty\}$ -costs (known also as "min-size subgraph problems" or "uniform costs"): given a graph H (formed by the edges of cost 1 of G, while edges not in H have cost  $\infty$ ) find a min-size spanning subgraph G' of H that satisfies the requirements.
- Metric Costs: here we assume that the edge costs satisfy the triangle inequality.
- General (non-negative) costs.

For each type of costs, the following four types of requirements were studied extensively:

- Uniform requirements:  $r_{uv} = k$  for every pair  $u, v \in V$ . The corresponding edge-connectivity and node-connectivity versions are the k-Edge-Connected Subgraph and the k-Connected Subgraph problems, respectively.
- Rooted (single source) requirements: there is  $s \in V$  such that  $r_{uv} > 0$  implies u = s; this gives the Rooted Survivable Network problem.
- Subset uniform requirements:  $r_{uv} = k$  for every pair  $u, v \in U \subseteq V$ . The corresponding edge-connectivity and node-connectivity versions are the Subset k-Edge-Connected Subgraph and the Subset k-Connected Subgraph problems.
- Arbitrary requirements.

Many fundamental problems are particular cases of Survivable Network. When there is only one pair uv with  $r_{uv} > 0$  (namely, when |D| = 1) we get the (uncapacitated) Min-Cost k-Flow problem, which is solvable in polynomial time (cf., [50]). The undirected 1-Connected Subgraph is just the MST problem; however, the directed 1-Connected Subgraph is NP-hard. The 1-Survivable Network problem (the case  $r_{uv} \in \{0,1\}$ ) is the Steiner Forest problem which admits ratio 2 for undirected graphs [1, 23] and ratio  $O(n^{2/3+\epsilon})$  for directed graphs [2]. Rooted 1-Survivable Network is the extensively studied Steiner Tree problem; c.f. [49, 3] for the undirected case and [4] for the directed case; the undirected Steiner Tree problem can also be casted as the undirected Subset 1-Connected Subgraph problem. Several other fundamental problems are also particular cases of the Survivable Network problem.

For directed graphs, many Survivable Network problems with node-costs are equivalent to those with edge-costs, but for undirected graphs the node-costs problems are usually harder to approximate. For example, Steiner Tree with edge-costs admits a constant ratio, while the version with node-costs is Set-Cover hard [31]. We will consider mainly Survivable Network problems with edge-costs. For Survivable Network problems and some other Network Design problems with node-costs see, for example, [31, 24, 42, 43, 47, 51, 5, 20].

In low connectivity Survivable Network problems, k=1,2, among them: Directed Steiner Tree, Directed Steiner Forest, Tree Augmentation, Directed Rooted 2-Survivable Network, and others. Examples of high connectivity Survivable Network problems are k-Connected Subgraph and the general Survivable Network with edge/node costs. Table 1 summarizes the current approximability status for high edge/node-connectivity Survivable Network problems. See also surveys in [16, 28, 34]. We mention some additional results not appearing in the table.

Element connectivity: Element-Connectivity Survivable Network admits ratio 2 [15, 10]. For  $\{0,1\}$ -costs the problem is NP-hard even for  $r(u,v) \in \{0,2\}$  [30]. For  $\{0,1\}$ -costs the best known ratio is 7/4 [40].

Rooted requirements: A graph G = (V, E) is k-edge-outconnected from s (k-outconnected from s) if it contains k edge disjoint (k internally disjoint) sv-paths for every  $v \in V \setminus \{s\}$ . In the corresponding k-Edge-Outconnected Subgraph and the k-Outconnected Subgraph problems,  $r_{sv} = k$  for every  $v \in V$ . For directed graphs, both problems can be solved in polynomial time, see [14]

c,r	Edge-Connectivity		Node-Connectivity	
	Undirected	Directed	Undirected	Directed
$\{0,1\},U$	in P [52]	in P [16]	$\min\{2, 1 + \frac{k^2}{2opt}\}\ [18, 26]$	in P [18]
$\{0,1\},R$	in P [16]	$O(\ln n)$ [35]	$O(\min\{\ln^2 k, \ln n\})$ [44, 35]	$O(\ln n)$ [35]
		$\Omega(\ln n)$ [16]	$\Omega(\ln n)$ [40]	$\Omega(\ln n)$ [16]
$\{0,1\},S$	in P [16]	$O(\ln n) [35]$	$\frac{ S }{ S -k} \cdot O(\min\{\ln^2 k, \ln n\}) $ [45]	$\frac{ S }{ S -k} \cdot O(\ln n) \ [45]$
		$\Omega(\ln n) \ [16]$	$\Omega(2^{\ln^{1-\varepsilon}n})$ [38]	$\Omega(2^{\ln^{1-\varepsilon}n})$ [13]
$\{0,1\},G$	in P [16]	$O(\ln n)$ [35]	$k \cdot O(\min\{\ln^2 k, \ln n\})$ [44, 35]	$O(k \ln n) [35]$
		$\Omega(\ln n) \ [16]$	$\Omega(2^{\ln^{1-\varepsilon}n}) [41]$	$\Omega(2^{\ln^{1-\varepsilon}n}) [41]$
	$1 + \frac{2}{k} [22, 8]$	$1 + \frac{1}{k} [37]$	$1 - \frac{1}{k} + \frac{n}{opt} \ [8] \ ([46])$	$1 - \frac{1}{k} + \frac{2n}{opt} \ [8] \ ([46])$
$\{1,\infty\},R$	2 [27] ([39])		$O(k \ln k)$ [43]	
		$\Omega(\ln^{2-\varepsilon} n)$ [25]	$\Omega(\ln^{2-\varepsilon} n)$ [38]	$\Omega(\ln^{2-\varepsilon} n)$ [25]
$\{1,\infty\},S$	2 [27]	D	$\frac{ S }{ S -k} \cdot O(k \ln k)$ [45]	$\mid \mid D \mid$
		$\frac{\Omega(2^{\ln^{1-\varepsilon}n})}{ D } [13]$	$\Omega(2^{\ln^{1-\varepsilon}n})$ [32] $O(k^3 \ln  S )$ [12]	$\frac{\Omega(2^{\ln^{1-\varepsilon}n})}{ D } [13]$
$\{1,\infty\},G$	2 [27]		$O(k^3 \ln  S )$ [12]	
		$\Omega(2^{\ln^{1-\varepsilon}n}) [13]$	$\Omega(2^{\ln^{1-\varepsilon}n})$ [32]	$\Omega(2^{\ln^{1-\varepsilon}n}) [13]$
MC,U	2 [29]	2 [29]	2 + (k-1)/n [33]	2 + k/n [33]
MC,R	2 [27]		$O(\ln k)$ [11]	D
		$\Omega(\ln^{2-\varepsilon} n) \ [25]$		$\Omega(\ln^{2-\varepsilon} n) [25]$
MC,S	2 [27]	D	24 [11]	D
		$\Omega(2^{\ln^{1-\varepsilon}n}) [13]$		$\frac{\Omega(2^{\ln^{1-\varepsilon}n})}{ D } [13]$
MC,G	2 [27]		$O(\ln k)$ [11]	D
		$\Omega(2^{\ln^{1-\varepsilon}n}) [13]$		$\Omega(2^{\ln^{1-\varepsilon}n}) [13]$
GC,U	2 [29]	2 [29]	$O\left(\ln\frac{n}{n-k}\cdot\ln k\right)$ [48]	$O\left(\ln\frac{n}{n-k}\cdot\ln k\right)$ [48]
			$6 \text{ if } n \geqslant k^3 \text{ [9] ([21])}$	,
			$\lceil (k+1)/2 \rceil$ if $k \le 8 \lceil 33 \rceil$	$k+1 \text{ if } k \leqslant 6 \text{ [33]}$
GC,R	2 [27]		$O(k \ln k)$ [43]	
		$\Omega(\max\{k^{1/2},  D ^{1/4}\})$ [36]	$\Omega(\max\{k^{1/10},  D ^{1/4}\})$ [36]	$\Omega(\max\{k^{1/2},  D ^{1/4}\})$ [36]
$_{\mathrm{GC,S}}$	2 [27]	D	$\frac{ S }{ S -k} \cdot O(k \ln k)$ [45]	D
		$\Omega(\max\{k^{1/2},  D ^{1/4}\})$ [36]	$\Omega(\max\{k^{1/10},  D ^{1/4}\})$ [36]	$\frac{\Omega(\max\{k^{1/2},  D ^{1/4}\}) [36]}{ D }$
$_{\mathrm{GC,G}}$	2 [27]	D	$O(k^3 \ln  S )$ [12]	
		$\Omega(\max\{k^{1/2},  D ^{1/4}\})$ [36]	$\Omega(\max\{k^{1/6},  D ^{1/4}\})$ [36]	$\Omega(\max\{k^{1/2},  D ^{1/4}\})$ [36]

TABLE 1. Known approximability status of Survivable Network problems. MC and GC stand for metric and general costs, U, R, S, and G stand for uniform, rooted, subset uniform, and general requirements, respectively.  $k = \max_{uv \in D} r_{uv}$  is the maximum requirement and S is the set of terminals; |D| = |S| - 1 in the case of rooted requirements and  $|D| = \Theta(|S|^2)$  in the case of subset-uniform requirements. Ratio |D| is obtained by computing a min-cost  $r_{uv}$ -flow for every  $uv \in D$ . References in brackets either contain a simplified proof, or a slight improvement of the main result needed to achieve the approximation ratio or threshold stated.

and [19], respectively. This implies a 2-approximation algorithm for undirected graphs. This fact is widely used for designing approximation algorithms for k-Edge-Connected Subgraph and k-Connected Subgraph problems. For additional literature see [4, 14, 17, 19, 40, 6, 12, 43, 7].

Relation between directed and undirected Node-Connectivity Survivable Network problems: In [38] it is shown that for k = n/2 + k' the approximability of the undirected Node-Connectivity Survivable Network variant is the same (up to factor of 2) as that of the directed one with maximum requirement k'.

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