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NEW PROOFS OF KONIG – EGERVARY THEOREM AND MAXIMAL FLOW – MINIMAL CUT CAPACITY THEOREM USING O.R. TECHNIQUES

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The Konig-Egervary theorem on binary matrices and the maximal flow-minimal cut capacity theorem on network flows were proved using graph theoretic and combinatoric techniques. In this paper we give new proofs of these two theorems using unimodularity of some coefficient matrices and the fundamental theorem of duality in linear programming.

1. INTRODUCTION

The Konig-Egervary theorem on binary matrices was proved by Konig[1] (see Harary[2]). Another famous result known as the maximal flow-minimal cut capacity theorem on networks was proved by Ford and Fulkerson[3,4]. It has been proved that these theorems are equivalent to Menger's theorem on graphs[2,5], Dilworth's theorem on finite lattices[2,6] and Hall's marriage theorem[7]. The equivalance of these five theorems are proved using graph theoretic and combinatoric techniques. These theorems (except Hall's marriage theorem) equates the maximum value of one set of real numbers with the minimum value of another set of real numbers.

The fundamental theorem of duality in linear programming[8,9] states that the maximum value of a linear programming problem is equal to the minimum value of the dual problem. We prove in sections 3 and 4 the Konig-Egervary theorem and the maximal flow-minimal cut capacity theorem using the fundamental theorem of duality in linear programming. In section 2 we give the preliminary ideas necessary for the proof of these theorems. For ready reference we give below the statements of all the five theorems.

Hall's Marriage Theorem

A necessary and sufficient condition for a solution of the marriage problem is that every set of k boys collectively know atleast k girls ($1 \leq k \leq m$ where m is the total number of boys).

Konig-Egervary Theorem

In a binary matrix, the maximum number of independent zeros is equal to the minimum number of lines containing all the zeros.

Maximal Flow-Minimal Cut Capacity Theorem

In any network the maximal value of any flow is equal to the minimal capacity of any cut.

Menger's Theorem

The maximum number of edge-disjoint chains connecting two distinct vertices v and w of a connected graph G is equal to the minimum number of edges in a vw-disconnecting set.

Dilworth's Theorem

In any finite lattice, the maximum number of incomparable elements is equal to the minimum number of chains which include all the elements.

2. Preliminaries

Definition 2.1 — A matrix A over real numbers is said to be unimodular if every subsquare matrix of A has determinant equal to 0, 1 or -1.

Definition 2.2 — If the primal linear programming problem (LPP) is given by

$$\begin{aligned} &\text{Maximise} && c^T X \\ &\text{Subject to} && AX \leq b \\ & && X \geq 0. \end{aligned} \qquad \ldots(1)$$

Where A is an $m \times n$ matrix, c, X and b are column vectors of order n, n and m and c^T denotes the transpose of c; then the dual LPP is given by

$$\begin{aligned} &\text{Minimise} && b^T Y \\ &\text{Subject to} && A^T Y \geq c \\ & && Y \geq 0 \end{aligned} \qquad \ldots(2)$$

where Y is a column vector of order m.

Lemma 2.1 — For the LPP (1), if A is a unimodular matrix and b is integral then some optimal solution is integral.

Proof : Let E be the coefficient matrix of the LPP obtained after converting all inequalities of the given LPP into equations using slack variables. Then $E = (A, D)$ where D is a matrix in which every column is a unit vector. It can be easily verified that if A is unimodular, then E is also unimodular.

Now consider the LPP

$$\begin{aligned} &\text{Maximise} && c^T X \\ &\text{Subject to} && E X = b \\ & && X \geq 0. \end{aligned}$$

An optimal basic feasible solution is given by $X_B = B^{-1} b$ where B is a basis matrix. Since B is a nonsingular square submatrix of E, $|B| = 1$ or -1. $B^{-1} = (\text{Adj } B)/|B|$. Since E is unimodular, Adj B is an integral matrix and consists of elements 0, 1 or -1. Since b is integral, $B^{-1}\, b$ is also integral.

Note : The fundamental theorem of duality in linear programming states that the maximum value of the LPP (1) is equal to the minimum value of the dual LPP (2).

Definition 2.3 — A matrix A in which every entry is 0 or 1 is called a binary matrix.

Definition 2.4 — A set of zeros in a binary matrix is said to be independent if no two of these zeros lie in the same row or column.

Definition 2.5 — A network N is a digraph (V, E) with vertex set V and arc set E such that with each arc (i, j) is associated a non-negative real number C_{ij}. C_{ij} is called the capacity of the arc (i, j).

Definition 2.6 — Given a network N, a flow in N from a vertex s (called source) to a vertex t (called sink) is a vector $X = (x_{ij})$ where x_{ij} is the flow in arc (i, j) satisfying the conditions; (i) $x_{ij} \geq 0$ for all (i, j); (ii) $x_{ij} \leq c_{ij}$ for all (i, j); and (iii) the total flow into any vertex (other than the specified vertices s and t) is equal to the total flow out of this vertex.

It is easy to see that the net flow out of the source s is equal to the net flow into the sink t and this value is called the value of the flow from source s to sink t. Those flows whose value is as large as possible are called maximal flows.

Definition 2.7 — A cut in a network N is a partition $(S, \bar{S})$ of the vertex set V such that $S \cup \bar{S} = V$, $S \cap \bar{S} = \phi$, $s \in S$, $t \in \bar{S}$.

Definition 2.8 — The capacity of a cut $(S, \bar{S})$ denoted by $C(S, \bar{S})$ is $\Sigma\, \{C_{ij}/i \in S, j \in \bar{S}\}$.

Those cuts whose capacity is as small as possible are called minimal cuts.

3. Proof of Konig-Egervary Theorem

Theorem 3.1 — In a binary matrix, the maximum number of independent zeros is equal to the minimum number of lines (a line is a row or column) containing all the zeros.

Proof : Let $M = (m_{ij})$ be an $m \times n$ binary matrix. Let $S = \{(i, j)/m_{ij} = 0\}$. We shall assume without loss of generality that S is nonempty and $|S| = r$. We shall convert the problem of finding the maximum number of independent zeros in M into a mathematical programming problem (P) as follows.

For every (i, j) in S let

$x_{ij} = 1$ if the zero in cell (i, j) is chosen.

$\quad = 0$ if the zero in cell (i, j) is not chosen.

Then the problem (P) is

$$\text{Maximize} \quad \Sigma\, \{x_{ij}/(i, j) \in S\} \qquad \ldots(3)$$

Subject to $\sum_j x_{ij} \leq 1$ for all i ...(4)

$\sum_i x_{ij} \leq 1$ for all j ...(5)

$x_{ij} = 0$ or 1 for all $(i, j) \in S$. ...(6)

The conditions (4) and (5) imply that atmost one zero can be chosen from each row and column if the zeros are to be independent.

Consider the inequalities

$x_{ij} \geq 0$ for all $(i, j) \in S$. ...(7)

Let A be the coefficient matrix of the corresponding LPP (P^1) given by (3), (4), (5) and (7). By Lemma 3.1, A is unimodular. Hence by Lemma 2.1 there exists an optimal solution X of the LPP (P^1) which is integral. Obviously in the optimal solution X, every $x_{ij} = 0$ or 1. Hence optimal value of the LPP (P^1) is equal to the optimal value of the problem (P), which is equal to the maximum number of independent zeros in M.

Now we shall convert the problem of finding the minimum number of lines containing all the zeros in M into a mathematical programming problem (Q) as follows :

Let $p_i = 1$ if ith row is chosen

$= 0$ if ith row is not chosen

$q_j = 1$ if jth column is chosen

$= 0$ if jth column is not chosen.

The problem (Q) is

Minimise $\sum_i p_i + \sum_j q_j$. ...(8)

Subject to $p_i + q_j \geq 1$ for all $(i, j) \in S$ ...(9)

$$\left.\begin{aligned} p_i &= 0 \text{ or } 1 \text{ for all } i \\ q_j &= 0 \text{ or } 1 \text{ for all } j \end{aligned}\right\}. \quad ...(10)$$

$$\text{Consider } \left.\begin{aligned} p_i &\geq 0 \text{ for all } i \\ q_j &\geq 0 \text{ for all } j \end{aligned}\right\}. \quad ...(11)$$

Let B be the coefficient matrix of the corresponding LPP (Q^1) given by (8), (9) and (11). B has r rows and $m + n$ columns. It is easy to verify that matrix B is the transpose of matrix A. Since A is unimodular (Lemma 3.1), B is also unimodular. Therefore by Lemma 2.1, some optimal solution of the LPP (Q^1) is integral. It is easy to see that in this optimal solution all p_i and q_j are equal to 0 or 1. Hence the optimal value of the problem (Q^1) is equal to the optimal value of the problem (Q), which is equal to the minimum number of lines containing all zeros in the matrix M.

It is easy to see that the LPP (Q^1) is the dual of the LPP (P^1). By fundamental theorem of Duality in linear programming (P^1) and (Q^1) have the same optimal value. Hence the maximum number of independent zeros in M is equal to the minimum number of lines containing all the zeros in M. Hence the theorem.

Lemma 3.1 – The coefficient matrix X of the LPP (P^1) given by (3), (4), (5) and (7) is unimodular.

PROOF : A has $m + n$ rows and r columns. Each column of A has exactly two '1', one in the first m rows and the other '1' in the last n rows and all other elements of A being 'O'. Let D be a square submatrix of order k. We apply induction on k. If $k = 1$, $|D| = 0$ or 1 since each element of A is 0 or 1. Assume that all square submatrices of order $k - 1$ have determinant equal to 0, 1 or −1.

(a) If D has atleast one column containing only zeros, then $|D| = 0$.

(b) If D has atleast one column containing exactly a single '1' then $|D| = \pm |E|$ where E is the square submatrix of D got by deleting the corresponding column and the row containing the single '1'. By induction assumption, $|E| = 0, 1$, or −1 and hence $|D| = 0$, 1 or −1.

(c) If neither (a) nor (b) holds, then every column of D has exactly two '1'. In this case the sum of the rows of D from the first m rows of A is equal to the sum of the rows of D from the last n rows of A and hence the rows of D are linearly dependent and $|D| = 0$. Hence A is unimodular.

4. Proof of Maximal Flow – Minimal Cut Capacity Theorem

Theorem 4.1 – In any network the value of any maximal flow is equal to the capacity of any minimal cut.

PROOF : Let N be any network with vertex set V and arc set E. Let $|V| = n$ and $|E| = m$. We shall convert the problem of finding the maximal flow into an LPP (R) as follows.

Let C_ij be the capacity of arc (i, j). We want to find the maximal flow that can be sent from a specified vertex s (source) to a specified vertex t (sink). Let v be the value of any flow and x_{ij} be the flow that is sent along arc (i, j). Let the vertices be labelled using integers 1 to n such that source s corresponds to the vertex 1 and sink t corresponds to the vertex n. Then the LPP(R) corresponding to the maximum flow problem is,

$$\text{Maximise } v \quad \ldots(12)$$

$$\text{Subject to} \quad \sum_{(i,j)\in E} x_{ij} - \sum_{(k,i)\in E} x_{ki} - v = 0 \text{ if } i = 1 \quad \ldots(13)$$

$$\sum_{(i,j)\in E} x_{ij} - \sum_{(k,i)\in E} x_{ki} + v = 0 \text{ if } i = n \quad \ldots(14)$$

$$\sum_{(i,j)\in E} x_{ij} - \sum_{(k,i)\in E} x_{ki} = 0 \text{ if } i = 2,3,\ldots,n-1 \quad \ldots(15)$$

$$x_{ij} \le C_{ij} \text{ for all } (i, j) \in E \quad \ldots(16)$$

$$\left.\begin{array}{l} x_{ij} \ge 0 \text{ for all } (i, j) \in E \\ v \ge 0 \end{array}\right\} . \quad \ldots(17)$$

(13) imply that the net flow out of vertex 1 is equal to the value of the flow v, (14) imply that the net flow into vertex n is equal to v and (15) imply that total flow into any intermediate vertex is equal to the total flow out of this vertex.

Consider the inequalities

$$\sum_{(i,j)\in E} x_{ij} - \sum_{(k,i)\in E} x_{ki} - v \le 0 \text{ if } i = 1 \qquad \ldots(18)$$

$$\sum_{(i,j)\in E} x_{ij} - \sum_{(k,i)\in E} x_{ki} + v \le 0 \text{ if } i = n \qquad \ldots(19)$$

$$\sum_{(i,j)\in E} x_{ij} - \sum_{(k,i)\in E} x_{ki} \le 0 \text{ if } i = 2, 3,\ldots, n-1 \qquad \ldots(20)$$

and the LPP (R^1) given by (12), (18), (19), (20),(16) and (17). In any feasible solution of (R^1), the inequalities; (18), (19) and (20) hold as equalities; otherwise by adding LHS and RHS we get $0 < 0$, a contradiction. Therefore problems R and R^1 are equal are equivalent.

Now we shall convert the problem of finding the minimal cut capacity into a mathematical programming problem (T) as follows:

Corresponding to a cut $(S, \bar{S})$ of the network N, let

$$u_1 = 0 \text{ if vertex } i \in S$$
$$= 1 \text{ if vertex } i \in \bar{S}$$
$$y_{ij} = 1 \text{ if } i \in S, j \in \bar{S}$$
$$= 0 \text{ otherwise.}$$

Then the mathematical programming problem (T) is

Minimise $\quad \Sigma\ \{c_{ij}\, y_{ij} / (i, j) \in E\}$ ...(21)

Subject to $\quad u_i - u_j + y_{ij} \ge 0 \ (i, j) \in E$ ...(22)

$$-u_1 + u_n \ge 1 \qquad \ldots(23)$$

$$\left.\begin{aligned} u_i &= 0 \text{ or } 1 \ \forall\, i \\ y_{ij} &= 0 \text{ or } 1 \ \forall\, i, j \end{aligned}\right\}. \qquad \ldots(24)$$

Consider the inequalities

$$\left.\begin{aligned} u_i &\ge 0 \ \forall\, i \\ y_{ij} &\ge 0 \ \forall\, i, j \end{aligned}\right\}. \qquad \ldots(25)$$

It is clear that in any optimal solution of (T), $(u_i, u_j) = (0, 0)$ or $(1, 0)$ or $(1,1)$ will imply $y_{ij} = 0$ and $(u_i, u_j) = (0, 1)$ will imply $y_{ij} = 1$ and hence (21) gives the capacity of the cut $(S, \bar{S})$.

Now consider the corresponding LPP (T^1) given by (21), (22), (23) and (25). The coefficient matrix B of (T^1) has $m + 1$ rows and $m + n$ columns and B is the transpose of the coefficient matrix A of LPP (R^1). It is easy to see that LPP (T^1) is the dual of LPP (R^1), B is unimodular and that there exists an optimal integral solution of (T^1) in which each u_i and each y_{ij} is equal to 0 or 1. Hence the optimal values of (T) and (T^1) are equal.

By fundamental theorem of duality is linear programming, optimal values of LPP (R^1) and LPP (T^1) are equal. Optimal value of (R^1) is the maximal flow in the network and optimal value of the problem (T^1) is the minimal cut capacity of the network. Hence the theorem.

The relation between duality theory in Linear programming and Maximal flow – Minimal cut capacity theorem of networks has been discussed by Ford and Fulkerson[4,10]. But in this paper we follow a different approach by using duality theory and unimodularity property to prove maximal flow-minimal cut capacity theorem. In fact this proof is much simplet than that given in Ford and Fulkerson[10].

5. Conclusion

In this paper we have proved the Konig – Egervary theorem on binary matrices and the maximal flow – minimal cut capacity theorem on a network using unimodularity property of some coefficient matrices and the fundamental theorem of duality in linear programming. There are many theorems in mathematics where the maximum value of one set of real numbers is equated with the minimum value of another related set of real numbers. One can explore the possibilities of proving such theorems using the duality theorem in linear programming and other O.R. techniques.

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