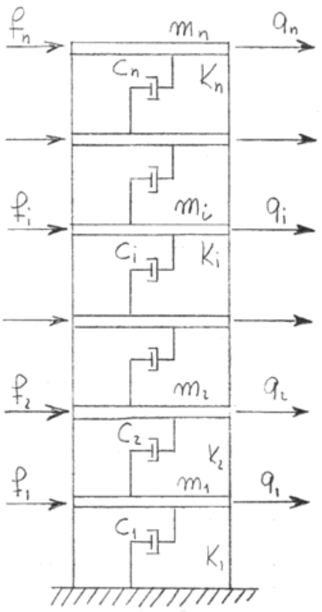
EQUATIONS OF MOTION

Shear-type system – damped forced vibrations



Equation of motion of the i-th mass; 2nd Newton law:

$$F_i = m_i a_i$$

$$\begin{aligned} F_i = & -k_i(q_i - q_{i-1}) - k_{i+1}(q_i - q_{i+1}) - && \text{restoring elastic forces} \\ & - c_i(\dot{q}_i - \dot{q}_{i-1}) - c_{i+1}(\dot{q}_i + \dot{q}_{i+1}) + && \text{viscous damping forces} \\ & + f_i && \text{external force} \end{aligned}$$

$a_i = \ddot{q}_i$ absolute acceleration

$$1) \quad m_1\ddot{q}_1 \qquad + (c_1 + c_2)\dot{q}_1 - c_2\dot{q}_2 - \\ + (k_1 + k_2)q_1 - k_2 q_2 = f_1$$

$$i) \quad m_i\ddot{q}_i - c_i\dot{q}_{i-1} + (c_i + c_{i+1})\dot{q}_i - c_{i+1}\dot{q}_{i+1} - \\ - k_i q_{i-1} + (k_i + k_{i+1})q_i - k_{i+1}q_{i+1} = f_i$$

$$n) \quad m_n\ddot{q}_n - c_n\dot{q}_{n-1} + c_n\dot{q}_n - k_n q_{n-1} + k_n q_n = f_n$$

In matrix form:

$$\boxed{\begin{array}{l} \mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{C}\dot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{f}(t) \\ \mathbf{q}(0) = \mathbf{q}_o \ ; \ \dot{\mathbf{q}}(0) = \dot{\mathbf{q}}_o \end{array}}$$

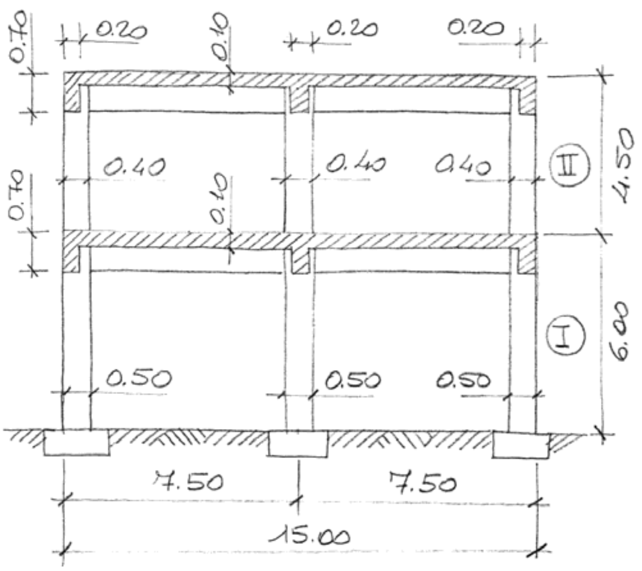
$$
\mathbf{q}=\begin{Bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{Bmatrix} \quad ; \quad \mathbf{f}=\begin{Bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{Bmatrix} \quad ; \quad \mathbf{M}=\begin{bmatrix} m_1 & 0 & \cdots & 0 \\ 0 & m_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & m_n \end{bmatrix}
$$

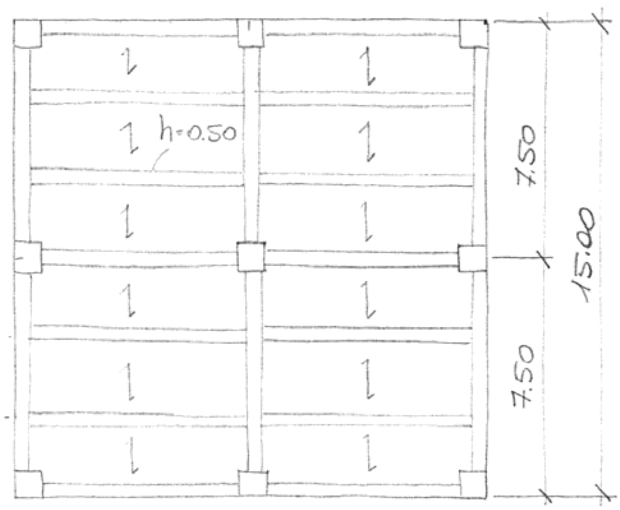
$$
\mathbf{K}=\begin{bmatrix} k_1+k_2 & -k_2 & & & & \\ -k_2 & k_2+k_3 & -k_3 & & 0 & \\ & & & -k_{n-1} & k_{n-1}+k_n & -k_n \\ & 0 & & & -k_n & k_n \end{bmatrix}
$$

$$
\mathbf{C}=\begin{bmatrix} c_1+c_2 & -c_2 & & & & \\ -c_2 & c_2+c_3 & -c_3 & & 0 & \\ & & & -c_{n-1} & c_{n-1}+c_n & -c_n \\ & 0 & & & -c_n & c_n \end{bmatrix}
$$

In this case **M, K**, **C** are real, symmetric, positive definite matrices; **M** is also diagonal.

<u>Example: Shear-Type building with 2 D.O.F.</u>





Mass of the column per unit length

1st order: $m_I = 0.50 \times 0.50 \times 2500 = 625\ \mathrm{kg/m}$

2nd order: $m_{II} = 0.40 \times 0.40 \times 2500 = 400\ \mathrm{kg/m}$

Mass of the beams (outside the slab) per each floor

Main beams: $m_{tp} = 0.20 \times 0.60 \times 7.50 \times 2500 = 2250\ \mathrm{kg}$

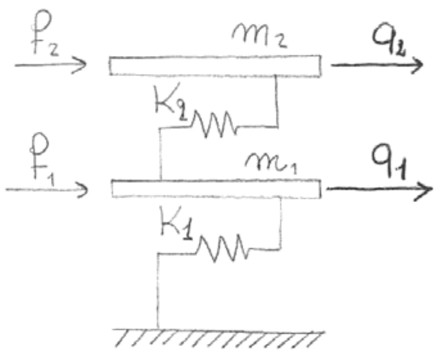
Secundary beams: $m_{tps} = 0.20 \times 0.40 \times 7.50 \times 2500 = 1500\ \mathrm{kg}$

Total mass: $m_t = 2500 \times 12 + 1500 \times 8 = 39000\ \mathrm{kg}$

Mass of the slab per each floor

$m_s = 0.10 \times 15 \times 15 \times 2500 = 56250\ \mathrm{kg}$

Structural scheme



$$\mathbf{q} = \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} \quad ; \quad \mathbf{f} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} \quad ; \quad \mathbf{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \quad ; \quad \mathbf{K} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}$$

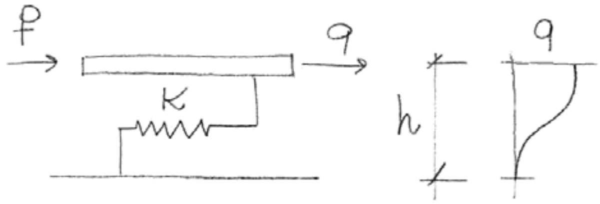
Mass at the 1st level

– columns:	(625 × 3 + 400 × 2.25) × 9	=	24975 kg
– beams:		=	39000 “
– slab:		=	56250 “
– sottofondo:	0.03 × 15 × 15 × 1700	=	11475 “
– pavement :	15 × 15 × 40	=	9000 “
– walls :	80 × 15 × 15	=	18000 “
– accidental load:	500 × 15 × 15	=	112500 “
		m_1 =	271200 kg

Mass at the 2nd level

– columns:	400 × 2.25 × 9	=	8100 kg
– beams:		=	39000 “
– slab:		=	56250 “
– sottofondo:		=	11475 “
– pavement:		=	9000 “
– accidental load:	100 × 15 × 15	=	22500 “
		m_2 =	146325 kg

Inter-storey stiffness



$$f = \sum_{1}^{N} {}_k f_k = \sum_{1}^{N} {}_k \left(\frac{12EJ}{h^3} \right) q = N \frac{12EJ}{h^3} q \Rightarrow$$
$$k = \frac{f}{q} = N \frac{12EJ}{h^3}$$

N = number of columns = 9
$E = .3 \times 10^{11}$ N/m^2
$J_I = .5 \times .\overline{5}^3 / 12 = 5.2083 \times 10^{-3}\ m^4$; $J_{II} = .4 \times .\overline{4}^3 / 12 = 2.1333 \times 10^{-3}\ m^4$
$h_I = 5.65$ m ; $h_{II} = 4.50$ m
$$k_1 = \frac{9 \times 12 \times 3 \times 10^{11} \times 5.2083 \times 10^{-3}}{\overline{5.65}^3} = 0.9356 \times 10^8\ N/m$$
$$k_2 = \frac{9 \times 12 \times 3 \times 10^{11} \times 2.1333 \times 10^{-3}}{\overline{4.50}^3} = 0.7585 \times 10^8\ N/m$$

$$\mathbf{M} = \begin{bmatrix} 271200 & 0 \\ 0 & 146325 \end{bmatrix} kg \quad ; \quad \mathbf{K} = \begin{bmatrix} 16941 \times 10^8 & -0.7585 \times 10^8 \\ -0.7585 \times 10^8 & 0.7585 \times 10^8 \end{bmatrix} N/m$$

UNDAMPED FREE VIBRATIONS

Let us consider the damped forced vibrations of a NDOF system:

$$\boxed{\begin{array}{l}\mathbf{M}\ddot{\mathbf{q}}(t)+\mathbf{C}\dot{\mathbf{q}}(t)+\mathbf{K}\mathbf{q}(t)=\mathbf{f}(t)\\ \mathbf{q}(0)=\mathbf{q}_o\ ;\ \dot{\mathbf{q}}(0)=\dot{\mathbf{q}}_o\end{array}}$$

Assuming **C** = **0** and **f** = **0** it follows:

$$\boxed{\begin{array}{l}\mathbf{M}\,\ddot{\mathbf{q}}(t)+\mathbf{K}\,\mathbf{q}(t)=0\\ \mathbf{q}(0)=\mathbf{q}_o\ ;\ \dot{\mathbf{q}}(0)=\dot{\mathbf{q}}_o\end{array}}$$

The above equation of motion admits the solution:

$$\mathbf{q}(t)=\boldsymbol{\psi}f(t)$$

where $\boldsymbol{\psi}$ is a vector of n constant components and f is a function of time, provided that $\boldsymbol{\psi}$ and f satisfies the following equations:

$$\boxed{\ddot{f}(t)+\lambda f(t)=0} \qquad (1)$$

$$\boxed{(\mathbf{K}-\lambda\mathbf{M})\boldsymbol{\psi}=\mathbf{0}} \qquad (2)$$

Let us consider first Eq. (1). It represents a system of n linear homogeneous equations in the n unknowns ψ_i (i = 1, 2, … n). Obviously, it involves the trivial solution $\boldsymbol{\psi}=\mathbf{0}$.
In order to obtain non-trivial solutions, $\boldsymbol{\psi}\neq\mathbf{0}$, it is necessary that the determinant of the matrix of the coefficients is null:

$$D=\det(\mathbf{K}-\lambda\mathbf{M})=0$$

This leads to an algebraic equation of order n in λ, called <u>characteristic equation</u>, from which n roots may be obtained, called <u>characteristic values or eigenvalues</u> $(\lambda=\lambda_1,\lambda_2,\ldots\lambda_n)$.

Since **K** and **M** are real, symmetric and positive definite matrices, then the eigenvalues $\lambda_1,\lambda_2,\ldots\lambda_n$ are real and positive. Let us assume, for sake of simplicity, that they are also distinct. In addition, let us order them in ascending order: $\lambda_1<\lambda_2<\ldots<\lambda_n$.

For each eigenvalue, the system (2) involves a non-trivial solution called <u>characteristic vector or eigenvector</u> $(\boldsymbol{\psi}=\boldsymbol{\psi}_1,\boldsymbol{\psi}_2,\ldots\boldsymbol{\psi}_n)$. Each eigenvector is defined unless an arbitrary factor.
The eigenvectors are linearly independent. So they constitute a basis in the space of the Lagrangian coordinates. They are real.

Let us define as k-th modal mass and k-th modal stiffness the positive quantities:

$$\boxed{m_k = \psi_k^T \mathbf{M} \psi_k \ ; \ k_k = \psi_k^T \mathbf{K} \psi_k} \quad \boxed{\lambda_k = \frac{k_k}{m_k}}$$

It is possible to demonstrate that, for $\lambda_i \neq \lambda_j$:

$$\boxed{\psi_j^T \mathbf{M} \psi_i = 0} \ ; \ \boxed{\psi_j^T \mathbf{K} \psi_i = 0}$$

Thus, the eigenvectors are orthogonal with respect to the matrices **M** and **K**.

Let us define as modal matrix or the matrix of the eigenvectors the matrix:

$$\boldsymbol{\Psi} = \left[\psi_1 \, \psi_2 \ldots \psi_n \right]$$

Let us define as the matrices of the modal masses and modal stiffnesses the matrices:

$$\mathbf{L} = \boldsymbol{\Psi}^T \mathbf{M} \boldsymbol{\Psi} = \mathrm{diag}\left[m_k \right] = \begin{bmatrix} m_1 & 0 & \cdots & 0 \\ 0 & m_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & m_n \end{bmatrix} \quad ; \quad \mathbf{N} = \boldsymbol{\Psi}^T \mathbf{K} \boldsymbol{\Psi} = \mathrm{diag} \left[k_k \right] = \begin{bmatrix} k_1 & 0 & \cdots & 0 \\ 0 & k_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & k_n \end{bmatrix}$$

Let us define as the matrix of the eigenvalues the matrix:

$$\boldsymbol{\Lambda} = \mathbf{L}^{-1} \mathbf{N} = \mathrm{diag} \left[k_k / m_k \right] = \mathrm{diag} \left[\lambda_k \right] = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

Since the eigenvectors are defined unless an arbitrary factor, it is possible to assume $m_k = 1$; thus, $k_k = \lambda_k$. Accordingly, the eigenvectors are said orthonormal with respect to **M** and **K**:

$$\mathbf{L} = \mathbf{I} \quad ; \quad \mathbf{N} = \boldsymbol{\Lambda}$$

where **I** is the identity matrix.

Let us assume $\lambda_k = \omega_k^2$ and let us examine Eq. (1). It becomes:

$$\ddot{f}(t) + \omega_k^2 f(t) = 0$$

from which:

$$f(t) = f_k(t) = A_k \cos \omega_k t + B_k \sin \omega_k t$$

where A_k, B_k are real constants. Thus, Eq. (1) represents a harmonic motion and the eigenvalue λ_k has a fundamental mechanical meaning: it is the square of the circular frequency ω_k.
Thus, Eq. (1) involves n linearly independent solutions:

$$\mathbf{q}(t)=\mathbf{q}_k(t)=\psi_k\, f_k(t)=\psi_k\left(A_k\cos\omega_k t+B_k\sin\omega_k t\right)$$

The general solution of Eq. (1) is a linear combination of the n solutions defined above::

$$\boxed{\mathbf{q}(t)=\sum_{1}^{n}{}_{k}\,\psi_k\left(A_k\cos\omega_k t+B_k\sin\omega_k t\right)}$$

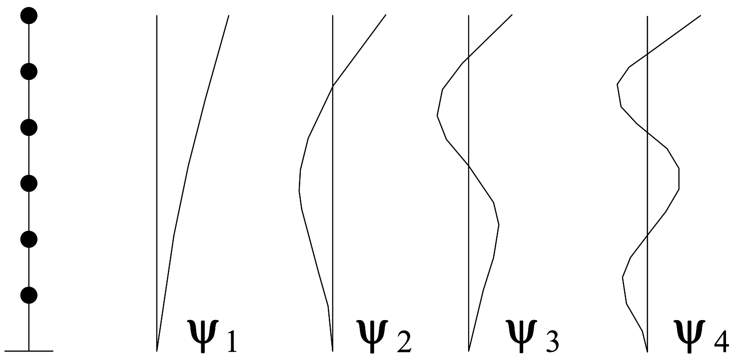
where the 2n arbitrary constants $A_k, B_k\ (k=1,2,..n)$ shall be set based on the initial conditions:
The physical meaning of the eigenvector can be explained assuming:

$$\mathbf{q}_0=\psi_j \quad ; \quad \dot{\mathbf{q}}_0=\mathbf{0}$$

i.e. deformating the structure in accordance with its j-th eigenvector and leaving its free to oscillate.
It results:

$$\mathbf{q}(t)=\psi_j\cos\omega_j t$$

Thus, the k-th eigenvector is a special pattern of the initial displacement that causes the oscillation of all the DOFs of the structure with the same circular frequency ω_k. For this reason the eigenvectors represent proper/natural/elementary modes/shapes of vibration. Each eigenvalue is the square of a proper/natural/elementary circular frequency of vibration.



<u>eigenvectors</u>	ψ_1	ψ_2	ψ_3	ψ_4
<u>eigenvalues</u>	λ_1	λ_2	λ_3	λ_4
<u>circular frequencies</u>	ω_1	ω_2	ω_3	ω_4
<u>frequencies</u>	n_1	n_2	n_3	n_4
<u>periods</u>	T_1	T_2	T_3	T_4

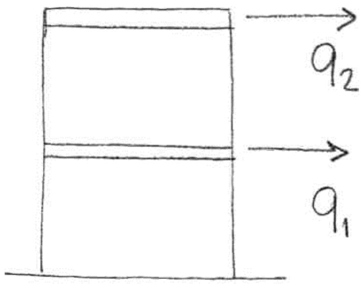
$$\omega_k^2=\lambda_k\ ;\ n_k=\omega_k/2\pi\ ;\ T_k=1/n_k\ \left(k=1,2,..n\right)$$

Any free vibration may be regarded as a linear combination of proper/natural oscillations. More generally, since the set of the eigenvectors represents a basis in the space of the Lagrangian coordinates, **q**(t) may be expressed as a linear combinations of the modes $\boldsymbol{\psi}_k$:

$$\boxed{\mathbf{q}(t)=\sum_{1}^{n}{}_{k}\,\boldsymbol{\psi}_k\,p_k(t)}$$

This expression is called principal transformation rule.

Example: 2 D.O.F. shear-type building



$$\mathbf{M}=\begin{bmatrix}271200 & 0\\ 0 & 146325\end{bmatrix}\quad(\text{kg})\ ;\quad \mathbf{K}=\begin{bmatrix}1.6941 & -0.7585\\ -0.7585 & 0.7585\end{bmatrix}\times10^{8}\quad(\text{N/m})$$

$$D=\det\left(\mathbf{K}-\omega^2\mathbf{M}\right)=0\Rightarrow$$

$$D=\det\left(\begin{bmatrix}1.6941 & -0.7585\\ -0.7585 & 0.7585\end{bmatrix}\times10^{8}-\omega^2\begin{bmatrix}271200 & 0\\ 0 & 146325\end{bmatrix}\right)=$$

$$=\det\begin{bmatrix}1.6941\times10^8-\omega^2\cdot271200 & -0.7585\times10^8\\ -0.7585\times10^8 & 0.7585\times10^8-\omega^2\cdot271200\end{bmatrix}=$$

$$=\left(1.6941\times10^8-271200\,\omega^2\right)\left(0.7585\times10^8-146325\,\omega^2\right)-\overline{0.7585\times10^8}^{\,2}=$$

$$=3.968\times10^{10}\,\omega^4-4.5359\times10^{13}\,\omega^2+7.0965\times10^{15}=0\Rightarrow$$

$$\omega^2=\frac{4.5359\times10^{13}\mp\sqrt{\left(4.5359\times10^{13}\right)^2-4\times3.968\cdot10^{10}\times7.0965\cdot10^{15}}}{2\times3.968\cdot10^{10}}\Rightarrow$$

$\omega_1^2=187.063\ ;\ \omega_2^2=956.056\Rightarrow$

$\omega_1=13.677\ \text{rad/s}\ ;\ \omega_2=30.920\ \text{rad/s}$

$n_1=2.177\,\text{Hz}\ ;\ n_2=4.921\,\text{Hz}$

$T_1=0.459\,\text{s}\ ;\ T_2=0.203\,\text{s}$

$$\left(\mathbf{K}-\omega_k^2\mathbf{M}\right)\boldsymbol{\psi}_k=\mathbf{0}$$

k = 1

$$\begin{bmatrix} 1.6941\times10^8-\omega_1^2\times271200 & -0.7585\times10^8 \\ -0.7585\times10^8 & 0.7585\times10^8-\omega_1^2\times146325 \end{bmatrix}\begin{Bmatrix}\psi_{11}\\ \psi_{12}\end{Bmatrix}=\begin{Bmatrix}0\\0\end{Bmatrix}$$

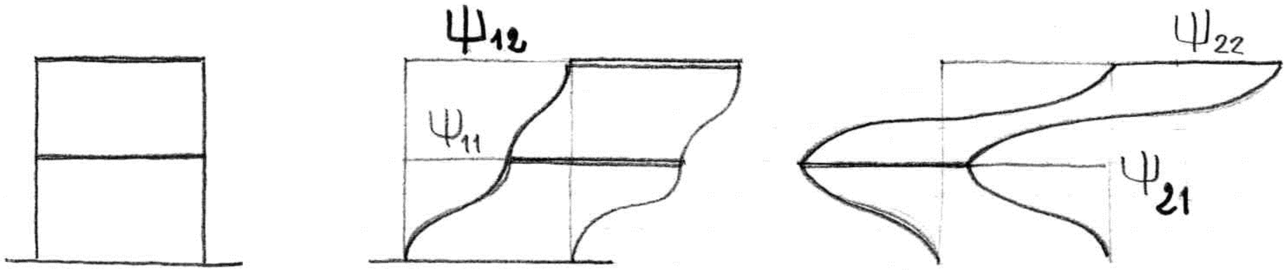
$$\psi_{12}=1\Rightarrow -0.7585\times10^8\ \psi_{11}+0.4848\times10^8=0\Rightarrow\psi_{11}=0.639$$

<u>k = 2</u>

$$\begin{bmatrix} 1.6941\times10^8-\omega_2^2\times271200 & -0.7585\times10^8 \\ -0.7585\times10^8 & 0.7585\times10^8-\omega_2^2\times146325 \end{bmatrix}\begin{Bmatrix}\psi_{21}\\ \psi_{22}\end{Bmatrix}=\begin{Bmatrix}0\\0\end{Bmatrix}$$

$$\psi_{22}=1\Rightarrow -0.7585\times10^8\ \psi_{21}+0.6404\times10^8=0\Rightarrow\psi_{21}=0.844$$

$$\boldsymbol{\Psi}=\left[\psi_1\ \psi_2\right]=\begin{bmatrix}\psi_{11} & \psi_{21}\\ \psi_{12} & \psi_{22}\end{bmatrix}=\begin{bmatrix}0.639 & -0.844\\ 1. & 1.\end{bmatrix}$$



$$\boldsymbol{\Psi}^T\mathbf{M}\boldsymbol{\Psi}=\begin{bmatrix}0.639 & 1.\\ -0.844 & 1.\end{bmatrix}\begin{bmatrix}271200 & 0\\ 0 & 146325\end{bmatrix}\begin{bmatrix}0.639 & -0.844\\ 1. & 1.\end{bmatrix}=$$

$$=\begin{bmatrix}257062 & 0\\ 0 & 339511\end{bmatrix}\Rightarrow$$

$$m_1=\boldsymbol{\Psi}_1^T\mathbf{M}\boldsymbol{\Psi}_1=257062$$

$$m_2=\boldsymbol{\Psi}_2^T\mathbf{M}\boldsymbol{\Psi}_2=339511$$

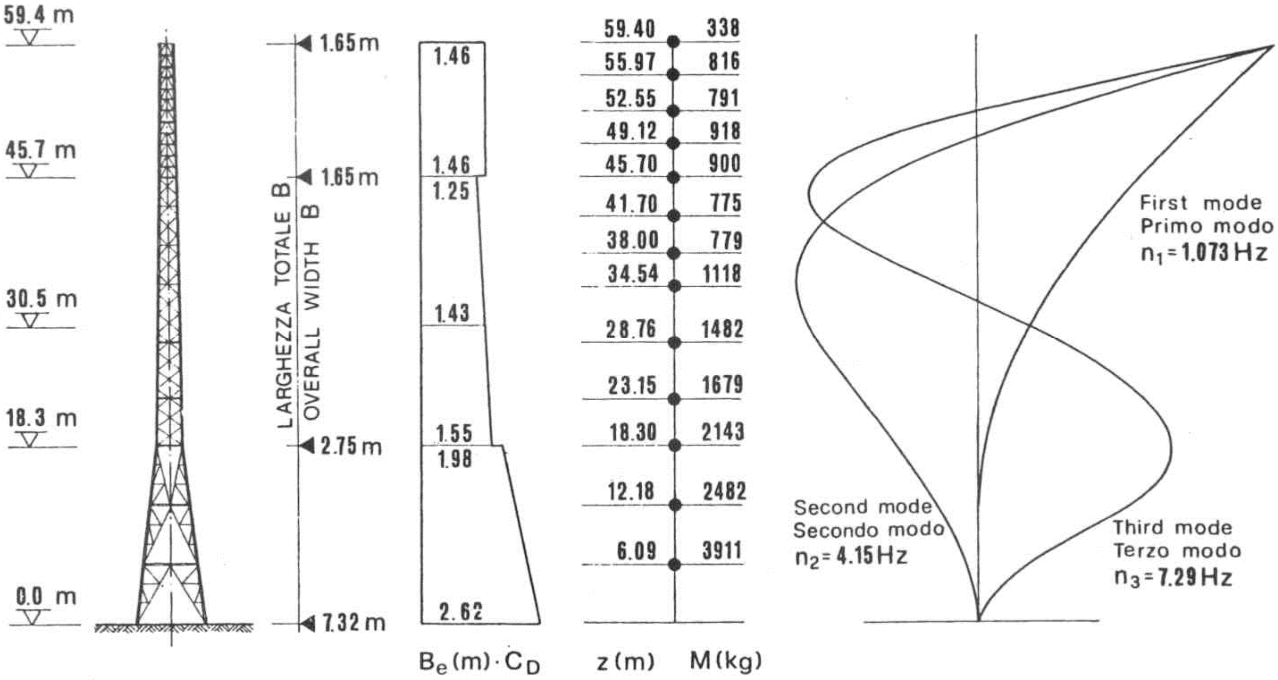
In order to make the eigenvectors orthonormal:

$$\boldsymbol{\Psi}_1=\frac{1}{\sqrt{m_1}}\begin{Bmatrix}0.639\\ 1.\end{Bmatrix}=\begin{Bmatrix}1.260\times10^{-3}\\ 1.972\times10^{-3}\end{Bmatrix}$$

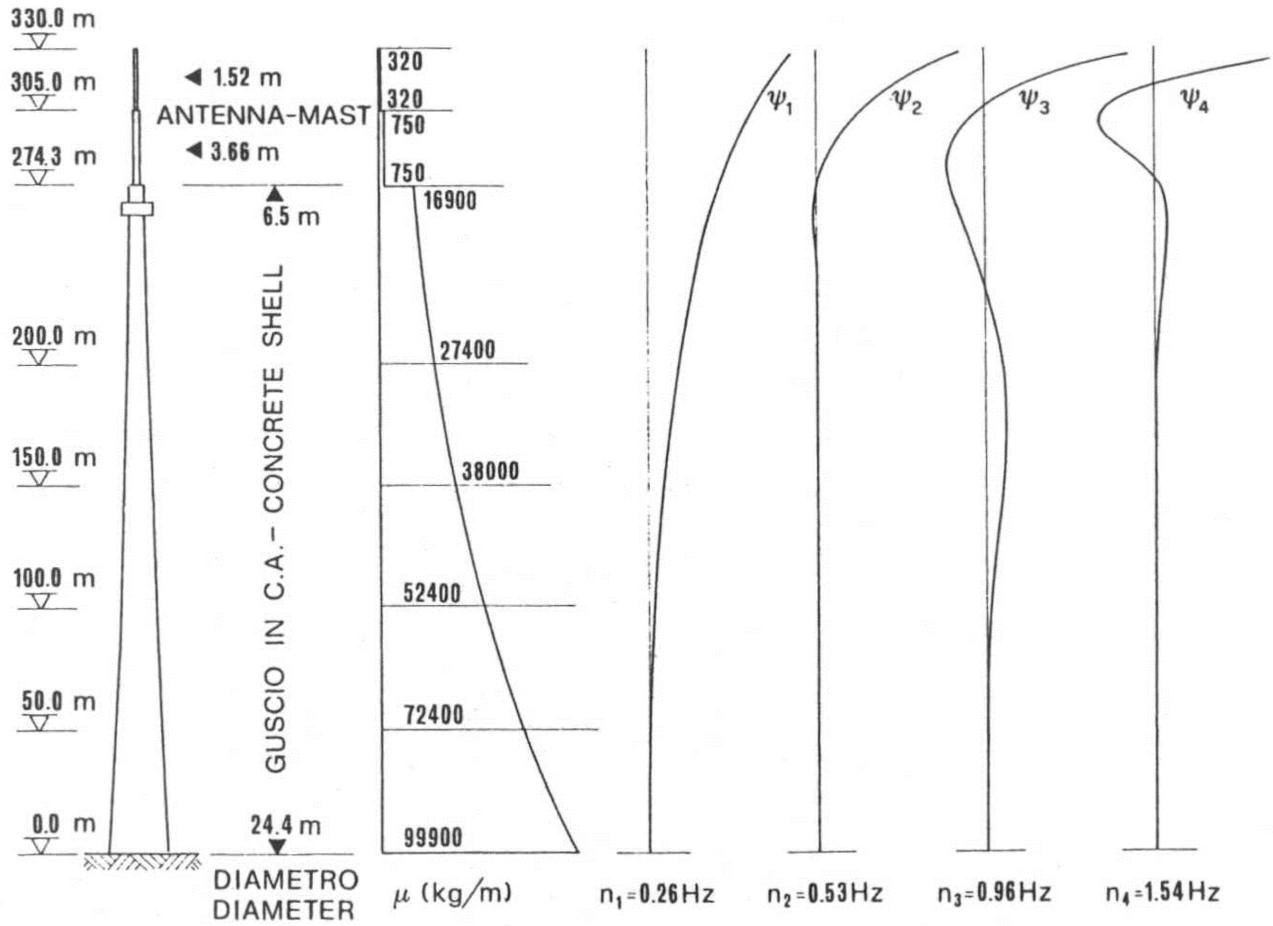
$$\boldsymbol{\Psi}_2=\frac{1}{\sqrt{m_2}}\begin{Bmatrix}-0.844\\ 1.\end{Bmatrix}=\begin{Bmatrix}-1.448\times10^{-3}\\ 1.716\times10^{-3}\end{Bmatrix}$$

$$\boldsymbol{\Psi}=\left[\psi_1\ \psi_2\right]=\begin{bmatrix}1.260 & -1.448\\ 1.972 & 1.716\end{bmatrix}\times10^{-3}\Rightarrow\boldsymbol{\Psi}^T\mathbf{M}\boldsymbol{\Psi}=\mathbf{I}$$

$$\boldsymbol{\Psi}^T\mathbf{M}\boldsymbol{\Psi}=\boldsymbol{\Lambda}=\begin{bmatrix}187.063 & 0\\ 0 & 956.056\end{bmatrix}$$



59.4 m
45.7 m
30.5 m
18.3 m
0.0 m
1.65 m
1.65 m
2.75 m
7.32 m
LARGHEZZA TOTALE B
OVERALL WIDTH B
1.46
1.46
1.25
1.43
1.55
1.98
2.62
Be(m)·CD
59.40 338
55.97 816
52.55 791
49.12 918
45.70 900
41.70 775
38.00 779
34.54 1118
28.76 1482
23.15 1679
18.30 2143
12.18 2482
6.09 3911
z(m) M(kg)
First mode
Primo modo
n1 = 1.073 Hz
Second mode
Secondo modo
n2 = 4.15 Hz
Third mode
Terzo modo
n3 = 7.29 Hz



330.0 m
305.0 m
274.3 m
200.0 m
150.0 m
100.0 m
50.0 m
0.0 m
1.52 m
ANTENNA-MAST
3.66 m
6.5 m
GUSCIO IN C.A. - CONCRETE SHELL
24.4 m
DIAMETRO
DIAMETER
320
320
750
750
16900
27400
38000
52400
72400
99900
μ (kg/m)
ψ1
ψ2
ψ3
ψ4
n1 = 0.26 Hz
n2 = 0.53 Hz
n3 = 0.96 Hz
n4 = 1.54 Hz

FORCED VIBRATIONS

Undamped forced vibrations

Since the eigenvectors constitute a base in the space of Lagrangian coordinates, the displacement **q** at the time t can be expressed as a linear combination of the modes ψ_k:

$$\boxed{\mathbf{q}(t) = \sum_{1}^{n} {}_k\psi_k p_k(t)}$$

$$\boxed{\mathbf{q}(t) = \boldsymbol{\Psi}\mathbf{p}(t)}$$

where $\boldsymbol{\Psi} = \{\psi_1 \; \psi_2 \; ... \; \psi_n\}$ is the modal matrix and $\mathbf{p}(t) = \{p_1(t) \, p_2(t) .. \, p_n(t)\}^T$ is the vector of the principal coordinates. Eqs. (1) and (2) are referred to as the principal transformation law.

Let us consider now the equation of motion:

$$\boxed{\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{f}(t)}$$

and let us apply the principal transformation law. It follows:

$$\ddot{\mathbf{p}}(t) + \boldsymbol{\Lambda}\mathbf{p}(t) = \mathbf{L}^{-1}\boldsymbol{\Psi}^T\mathbf{f}(t)$$

This equation represents a set un uncoupled equations:

$$\boxed{\ddot{\mathbf{p}}_k(t) + \omega_k^2 p_k(t) = \frac{1}{m_k}\boldsymbol{\Psi}_k^T\mathbf{f}(t) = \frac{1}{m_k}\sum_{1}^{n} {}_i\psi_{ki} f_i(t) \quad (k = 1, 2, ...n)}$$

where $\boldsymbol{\Psi}_k^T\mathbf{f}(t)$ is the k-th modal force. It is the k-th component of the generalised forces in the principal system.

Thus, the undamped forced vibrations of a NDOF system may be studied as the undamped forced vibrations of n SDOF systems. The DOF of the k-th oscillator is the k-th principal coordinate. The fundamental circular frequency of the k-th oscillator is the k-th principal circular frequency. The mass is the k-th modal mass. The external force is the k-th modal force.

Damped forced vibrations

Let us consider the equation of motion:

$$\boxed{\mathbf{M}\ddot{\mathbf{q}}(t)+\mathbf{C}\dot{\mathbf{q}}(t)+\mathbf{K}\mathbf{q}(t)=\mathbf{f}(t)}$$

and let us apply the principal transformation law:

$$\mathbf{q}(t)=\boldsymbol{\Psi}\mathbf{p}(t)$$

It follows:

$$\ddot{\mathbf{p}}(t)+\boldsymbol{\Gamma}\dot{\mathbf{p}}(t)+\boldsymbol{\Lambda}\mathbf{p}(t)=\mathbf{L}^{-1}\boldsymbol{\Psi}^{T}\mathbf{f}(t)$$

where:

$$\boldsymbol{\Gamma}=\mathbf{L}^{-1}\boldsymbol{\Psi}^{T}\mathbf{C}\boldsymbol{\Psi}$$

Since $\boldsymbol{\Psi}^{T}\mathbf{C}\boldsymbol{\Psi}$ is in general not diagonal, then also $\boldsymbol{\Gamma}$ is in general not diagonal. Thus, the equation of motion becomes a set of coupled differential equations:

$$\boxed{\begin{array}{l}\ddot{p}_k(t)+\sum_{1}^{n}{}_{\ell}\,\gamma_{k\ell}\,\dot{p}_{\ell}(t)+\omega_k^2\,p_k(t)=\dfrac{1}{m_k}\psi_k^T\,\mathbf{f}(t)\\ p_k(0)=p_{k0}\,;\,\dot{p}_k(0)=\dot{p}_{k0}\qquad(k=1,..n)\end{array}} \tag{3}$$

Obviously, the above equation is a set of decoupled equations for **C** = **0**.
In other words, if the structural system is damped, the principal transformation generally does not decouple the equations of motion.

Decoupling conditions

Let us assume that **C** is such that $\boldsymbol{\Gamma}$ is diagonal, i.e. $\gamma_{k\ell}=0$ for $k\neq\ell$. Thus Eq. (3) is decoupled and may be rewritten as:

$$\ddot{p}_k(t)+\gamma_{kk}\,\dot{p}_k(t)+\omega_k^2\,p_k(t)=\frac{1}{m_k}\psi_k^T\,\mathbf{f}(t)$$

$$p_k(0)=p_{k0}\,;\,\dot{p}_{k0}=\dot{p}_{k0}\qquad(k=1,2,\,..n)$$

Setting:

$$\gamma_{kk}=2\,\xi_k\omega_k$$

it follows:

$$\boxed{\begin{array}{l} \ddot{p}_k(t)+2\,\xi_k\,\omega_k\,\dot{p}_k(t)+\omega_k^2\,p_k(t)=\dfrac{1}{m_k}\,\psi_k^T\,\mathbf{f}(t) \\ p_k(0)=p_{k0}\ ;\ \dot{p}_{k0}(0)=\dot{p}_{k0} \qquad (k=1,2,\,..\,n) \end{array}}$$

Thus, if $\mathbf{\Gamma}$ is diagonal, the damped forced vibrations of a n-D.O.F. system may be studied (likewise the undamped vibrations) as the forced vibrations on n S.D.O.F., each characterised by a modal damping ratio $\xi_k = \gamma_{kk} / (2\,\omega_k)$.

The systems endowed with such a property are called classically damped. The structures which do not satisfy this property are not classically damped.

Analogously, the damping is said to be classic if $\mathbf{C}$ is such that $\mathbf{\Gamma}$ is diagonal. The damping is not classic if $\mathbf{C}$ is such that $\mathbf{\Gamma}$ is not diagonal.

The necessary and sufficient condition which makes $\mathbf{\Gamma}$ not diagonal is (Canghey e O'Kelly, 1965):

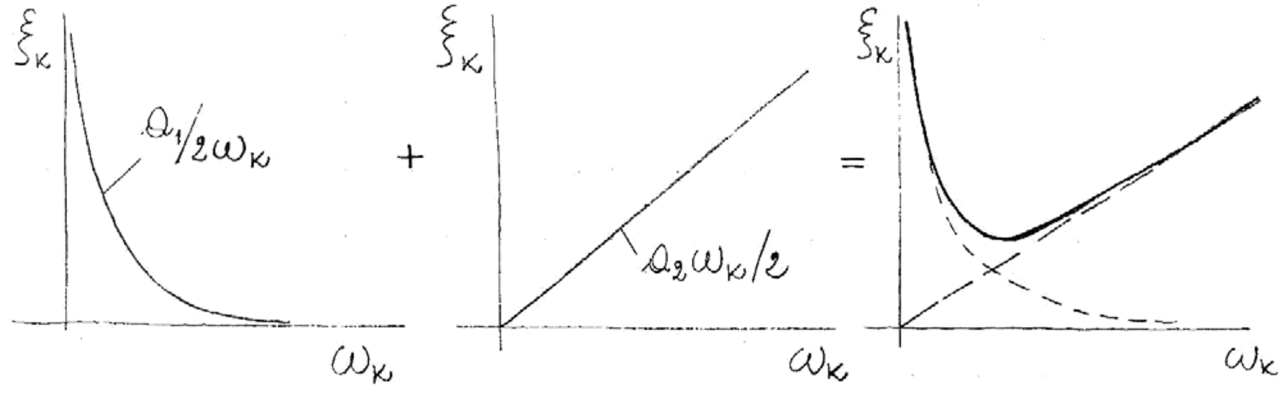
$$\mathbf{C}=\mathbf{M}\sum_{1}^{n}{}_{k}\,a_k\left(\mathbf{M}^{-1}\mathbf{K}\right)^{k-1}$$

being a_k ($k = 1, \ldots n$) suitable constants.
In particular, assuming $a_k = 0$ for $k > 2$, the following sufficient (not necessary) condition results:

$$\mathbf{C}=a_1\mathbf{M}+a_2\mathbf{K}$$

A structural system that satisfies the above equation has a Rayleigh damping or a proportional damping. In such a case:

$$\xi_k=\frac{a_1}{2\,\omega_k}+\frac{a_2\,\omega_k}{2}$$



In reality, the structures do not possess a classical or proportional damping. Nevertheless, being the definition of $\mathbf{C}$ very uncertain or difficult to evaluate, it is usual to avoid to evaluation of $\mathbf{C}$, writing the equations of motion in their decoupled form, giving to ξ_k values suggested by experience.

Modal truncation

Let us consider the principal transformation law:

$$\mathbf{q}(t) = \psi\, \mathbf{p}(t) = \sum_{1}^{n} {}_k\psi_k p_k(t)$$

where the k-th principal coordinate $p_k(t)$ is given by the solution of the differential equation:

$$\ddot{p}_k(t) + 2\xi_k\omega_k\dot{p}_k(t) + \omega_k^2 p_k(t) = \frac{1}{m_k}\psi_k^T\mathbf{f}(t) \quad (k = 1,..n)$$

The modal truncation is a technique that replaces the rigorous equation of motion by the approximate one:

$$\mathbf{q}(t) \cong \sum_{1}^{\bar{n}} {}_k\psi_k p_k(t)$$

being $\bar{n} < n$.

The above equation involves two fundamental advantages:

a) it allows to solve a number $\bar{n} < n$ of differential equations;
b) it allows to calculate only the first $\bar{n} < n$ eigenvalues and eigencetors (very useful applying iterative algorithms).

Experience shows that in most cases the choice $\bar{n} << n$ provides excellent approximations.