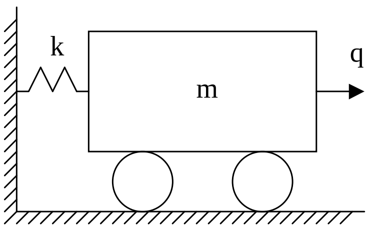
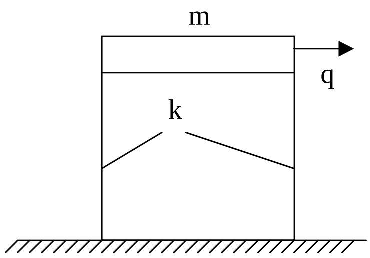
EQUATIONS OF MOTION

Undamped free vibrations





2nd Newton law $\quad F(t) = ma(t)$

$F(t) = -kq(t)$ $\quad$ restoring elastic force

$a(t) = \ddot{q}(t)$ $\quad$ absolute acceleration

$\boxed{m\ddot{q}(t) + kq(t) = 0} \Rightarrow$ (dividing both members by m)

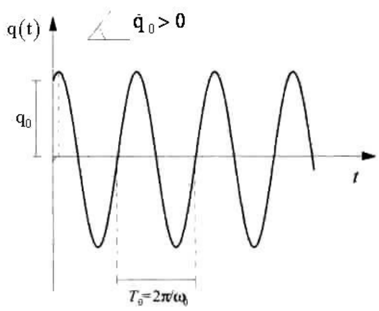
$\ddot{q}(t) + \frac{k}{m} q(t) = 0 \Rightarrow$ Defining $\boxed{\omega_0^2 = \frac{k}{m}}$

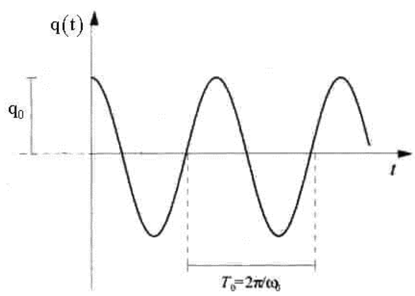
$\boxed{\ddot{q}(t) + \omega_0^2 q(t) = 0} \Rightarrow$ 2nd order, homogeneous, linear differential equation with constant coefficients

$q(0) = q_0 \; ; \; \dot{q}(0) = \dot{q}_0$ $\quad$ initial conditions

$q(t) = A \cos \omega_0 t + B \sin \omega_0 t \Rightarrow A = q_0 ; \quad B = \frac{\dot{q}_0}{\omega_0}$

$\boxed{q(t) = q_0 \cos \omega_0 t + \frac{\dot{q}_0}{\omega_0} \sin \omega_0 t}$ $\qquad \dot{q}_0 = 0 \Rightarrow q(t) = q_0 \cos \omega_0 t$





$\omega_0 =$ fundamental circular frequency

$n_0 = \omega_0 / 2\pi =$ fundamental frequency

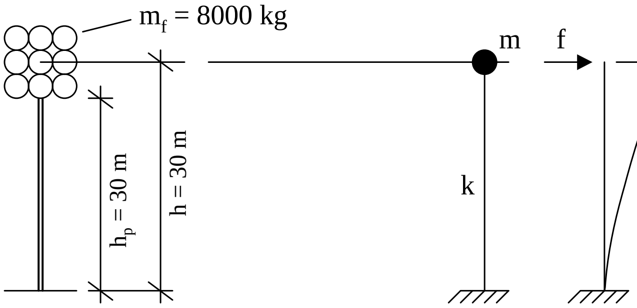
$T_0 = 1/n_0 = 2\pi/\omega_0 =$ fundamental period

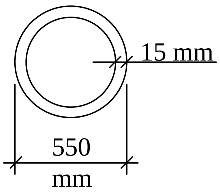
Indicatively:

$n_0 < 1\,\text{Hz}\,(T_0 > 1\text{s})$ - Dynamically flexible structure

$n_0 > 1\,\text{Hz}\,(T_0 < 1\text{s})$ - Dynamically rigid structure

Example: Arc lamp





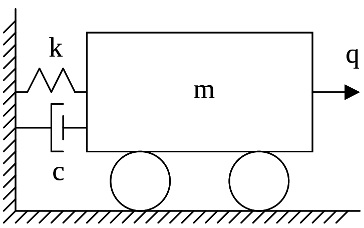
$$E = 2.100.000\ \text{kgf}/\text{cm}^2 = 0.21\times10^{12}\ \text{N}/\text{m}^2$$
$$J = 9.027\times10^{-4}\ \text{m}^4$$
$$A = 0.0252\,\text{m}^2$$
$$k = \frac{f}{q} = \frac{f}{\frac{f\,h^3}{3EJ}} = \frac{3EJ}{h^3} = 21063\,\text{N}/\text{m}$$

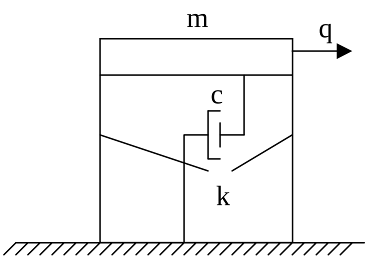
Mass of the pole $m_p = 0.0252\times27\times7850 = 5343$ kg

It is assumed $m = m_f + m_p/2 = 10671$ kg

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{21063}{10671}} = 1.405\ \text{rad}/\text{s}$$
$$n_0 = \omega_0/2\pi = 0.223\ \text{Hz}$$
$$T_0 = 1/n_0 = 4.47\ \text{s}$$

Damped free vibrations





2nd Newton law $\quad F(t) = ma(t) \quad ; \quad a(t) = \ddot{q}(t)$

$F(t) = -kq(t) - c\dot{q}(t) \quad ; \quad -c\dot{q}(t) =$ damping viscous force

$\boxed{m\ddot{q}(t) + c\dot{q}(t) + kq(t) = 0} \Rightarrow$ (dividing both members by m)

$$\ddot{q}(t) + \frac{c}{m}\dot{q}(t) + \frac{k}{m}q(t) = 0 \qquad \frac{k}{m} = \omega_0^2$$

$$\frac{c}{m} = \frac{c\cdot2\sqrt{k}}{\sqrt{m}\sqrt{m}\cdot2\sqrt{k}} = 2\underbrace{\frac{c}{2\sqrt{k\,m}}}_{\xi}\underbrace{\sqrt{\frac{k}{m}}}_{\omega_0}$$

<u>Damping ratio or damping coefficient</u>

$$\boxed{\xi = \frac{c}{2\sqrt{km}}}$$

$$\boxed{\begin{array}{l}\ddot{q}(t) + 2\xi\omega_0\dot{q}(t) + \omega_0^2 q(t) = 0 \\ q(0) = q_0 \ ; \ \dot{q}(0) = \dot{q}_0\end{array}}$$

This equation admits three distinct solutions depending on whether $\xi < 1, \xi > 1, \xi = 1$. In structural engineering not only $\xi < 1$ but, even more $\xi \ll 1$. A structure with $\xi < 1$ is said "under-damped". In this case the solution of the above equation is given by:

$$\boxed{q(t) = e^{-\xi\omega_0 t}\left(a_1 \cos \omega_0\sqrt{1-\xi^2}\ t + a_2 \sin \omega_0\sqrt{1-\xi^2}\ t\right)}$$

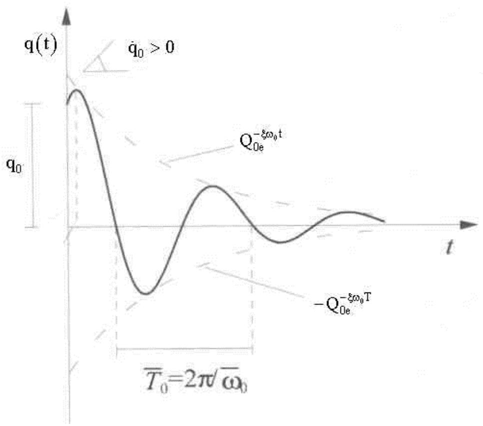
where a_1 and a_2 are constants depending on the initial conditions:

$$\boxed{a_1 = q_0 \ ; \quad a_2 = \frac{\dot{q}_0 + \xi\,\omega_0 q_0}{\omega_0\sqrt{1-\xi^2}}}$$

Eq. (2) may be rewritten as:

$$\boxed{q(t) = Q_0\, e^{-\xi\omega_0 t} \cos\left(\omega_0\sqrt{1-\xi^2}\ t + \phi\right)} \qquad (3)$$

$$Q_0 = \sqrt{q_0^2 + \left(\frac{\dot{q}_0 + \xi\,\omega_0 q_0}{\omega_0\sqrt{1-\xi^2}}\right)^2} \ ; \quad \phi = -\mathrm{arctg}\left(\frac{\dot{q}_0 + \xi\,\omega_0 q_0}{\omega_0 q_0\sqrt{1-\xi^2}}\right)$$



$$\overline{\omega}_0 = \omega_0\sqrt{1-\xi^2}$$

Remarks

1. $q(t)$ defines a damped vibratory motion for which the relative maximum and minimum values occur every $T = 2\pi / \omega_0 \sqrt{1-\xi^2} \simeq 2\pi / \omega_0$; they lie on the symmetric curves $\pm Q_0 e^{-\xi\omega_0 t}$.
2. The absolute values of the relative maxima and minima correspond to a series with rate $e^{-\xi\omega_0 t}$; the logarithmic decrement is defined as:

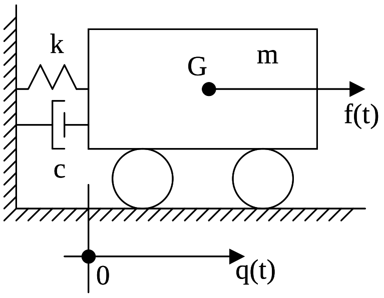
$$\delta = \ell n\left[\frac{Q_0 e^{-\xi\omega_0 t}}{Q_0 e^{-\xi\omega_0 (t+T)}}\right] \Rightarrow \delta = \xi\omega_0 T = \frac{2\pi\xi}{\sqrt{1-\xi^2}} \simeq 2\pi\xi$$

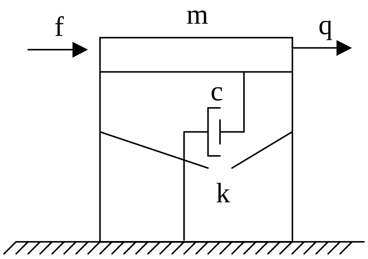
3. The vibratory motion tends to vanish on increasing the time:

$$\ell\mathrm{im}_{t\to\infty} q(t) = 0$$

This tendence becomes faster on increasing the damping ratio ξ.

Forced damped vibrations





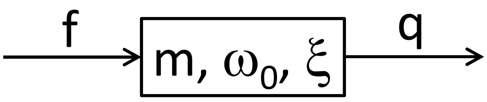
2nd Newton law $F = ma$; $a = \ddot{q}$

$F = -kq - c\dot{q} + f$; $f = f(t)$ = external force; $q = q(t)$.

$\boxed{m\ddot{q}(t) + c\dot{q}(t) + kq(t) = f(t)}$ ⇒ (dividing both members by m)

$$\ddot{q}(t) + \frac{c}{m}\dot{q}(t) + \frac{k}{m}q(t) = \frac{f}{m}(t) \Rightarrow$$

$$\boxed{\begin{aligned}&\ddot{q}(t) + 2\xi\omega_0\dot{q}(t) + \omega_0^2 q(t) = \frac{1}{m} f(t)\\ &q(0) = q_0 \;;\; \dot{q}(0) = \dot{q}_0\end{aligned}}$$



Dynamics, quasi-statics and statics

$\boxed{m\ddot{q}(t)+c\dot{q}(t)+kq(t)=f(t)}$ Equation of dynamics

Assuming that $q(t)$ varies slowly with time: $\dot{q}(t)\approx\ddot{q}(t)\approx 0 \Rightarrow$

$\boxed{kq(t)=f(t)}$ Equation of quasi-statics $\boxed{q(t)=\frac{f(t)}{m}}$

Assuming that f does not depend on time: $f(t)=f \Rightarrow$

$\boxed{kq=f}$ Equation of statics $\boxed{q=\frac{f}{m}}$

Decomposition of the equation of motion

Due to the linearity of the equation of motion its solution may be expressed as:

$$\boxed{q(t)=q'(t)+q''(t)}$$

where:

$$\boxed{\begin{array}{l}\ddot{q}'(t)+2\xi\omega_0\dot{q}'(t)+\omega_0^2q'(t)=0\\ q'(0)=q_0\ ;\ \dot{q}'(0)=\dot{q}_0\end{array}}$$

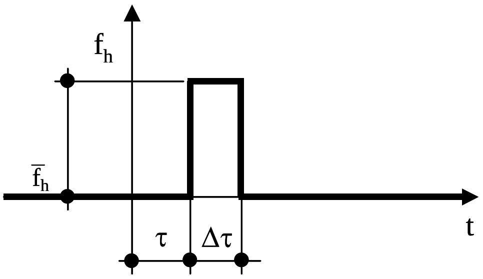
$$\boxed{\begin{array}{l}\ddot{q}''(t)+2\xi\omega_0\dot{q}''(t)+\omega_0^2q''(t)=\frac{1}{m}f(t)\\ q''(0)=0\ ;\ \dot{q}''(0)=0\end{array}}$$

The first equation defines the problem of the free vibrations with an initial perturbation. The second equation defines the problem of the forced vibrations without an initial perturbation.

Since $q'(t)$ tends to zero on increasing the time, sufficiently far from t = 0 $q(t)=q''(t)$.

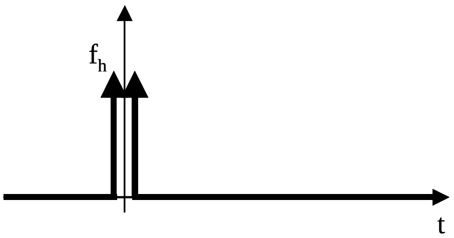
TIME-DOMAIN ANALYSIS

Impulsive force



$$f_h(t) = \begin{cases} \bar{f}_h & \text{per } \tau < t < \tau + \Delta\tau \\ 0 & \text{elsewhere} \end{cases}$$; <u>Impulse</u>: $I = \bar{f}_h \, \Delta\tau$.

<u>Elementary impulsive force</u>: $\tau = 0$, $\Delta\tau \to 0$, $I = 1 \left(\bar{f}_h \to \infty\right)$.

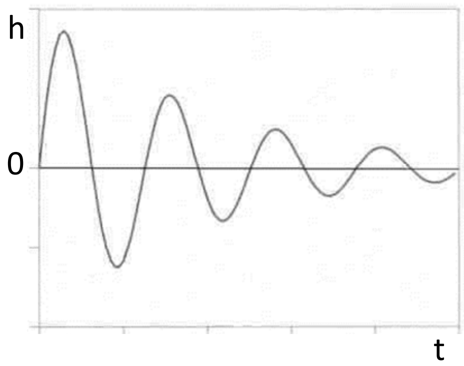


$f_h(t) = \delta(t) =$ <u>Dirac function</u>

$$\begin{cases} \delta(t-a) = 0 \quad \text{per } t \neq a \\ \int_{-\infty}^{\infty} \varepsilon(t)\, \delta(t-a) dt = \varepsilon(a) \end{cases}$$

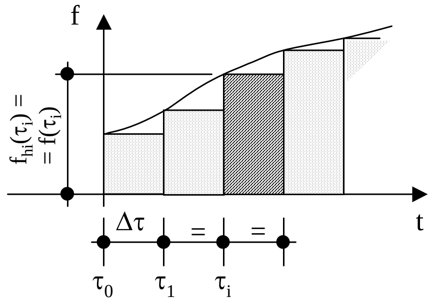
Dynamic response to an elementary impulsive force:
<u>Impulse response function</u>

$$\boxed{h(t) = e^{-\xi\omega_0 t} \frac{1}{m\,\omega_0 \sqrt{1-\xi^2}} \sin \omega_0 \sqrt{1-\xi^2}\; t}$$



Impulsive force method

A generic force f(t) may be approximated by a superposition of a series of suitable impulsive forces $f_{hi}(t)$ (i = 1, 2, …): $f(t) \cong \sum_{0} {}_i f_{hi}(t)$.

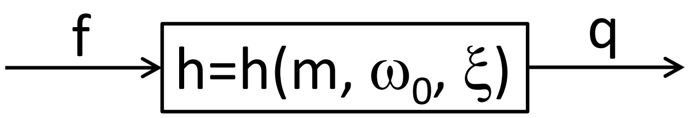


Therefore, the response q(t) can be expressed as the superposition of the responses $q_i(t)$ to each impulsive component force $f_{hi}(t)$: $q(t) \cong \sum_{0} {}_i q_i(t)$, where $q_i(t) = I_1 h(t - \tau_i)$.

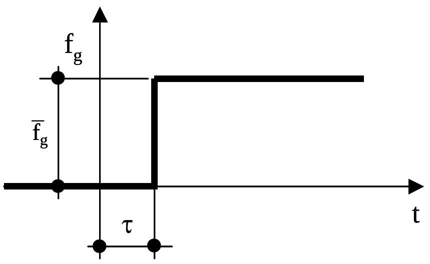
At the limit for $\Delta t \to 0$:
Duhamel integral of the first type

$$\boxed{q(t) = \int_0^t f(\tau)\, h(t-\tau)\, d\tau}$$

This equation represent the convolution integral of f(t) and h(t): $q(t) = f(t) * h(t)$

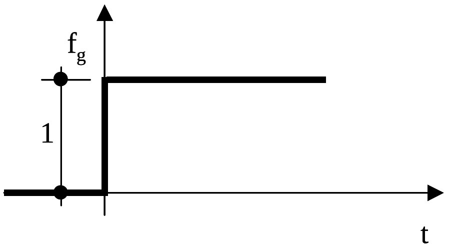


Step force



$$f_g(t) = \begin{cases} 0 & \text{for } t < \tau \\ \bar{f}_g & \text{for } t > \tau \end{cases}$$

Elementary step force: $\tau = 0$, $\bar{f}_g = 1$.

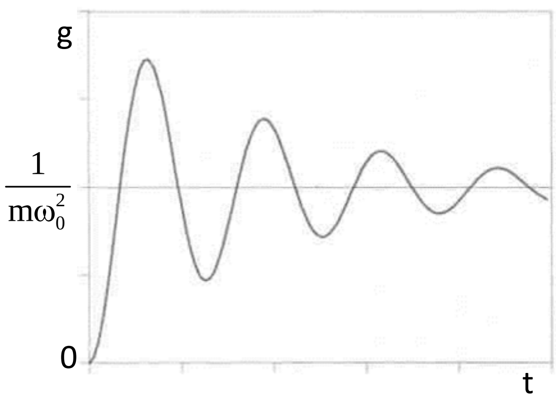


$f_g(t) = \mathbf{1}(t) =$ Heaviside function

$$\mathbf{1}(t-a) = \begin{cases} 0 & \text{for } t < a \\ 1 & \text{for } t \geq a \end{cases}$$

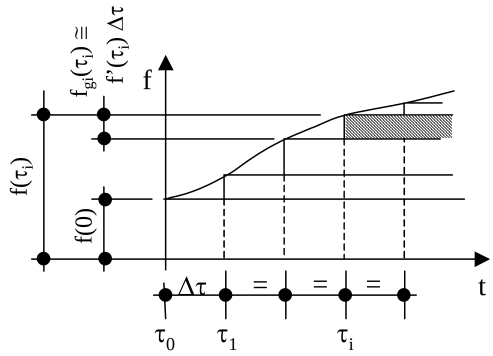
Dynamic response to an elementary step force:
Step response function

$$\boxed{g(t) = \frac{1}{m\omega_0^{\,2}}\left[1 - e^{-\xi\omega_0 t}\left(\frac{\xi}{\sqrt{1-\xi^2}}\sin\omega_0\sqrt{1-\xi^2}\,t + a_2\cos\omega_0\sqrt{1-\xi^2}\,t\right)\right]}$$



Step force method

A generic force f(t) may be approximated by a superposition of a series of suitable step forces $f_{gi}(t)$ (i = 1, 2, …): $f(t) \cong \sum_{0}^{i} f_{gi}(t)$.

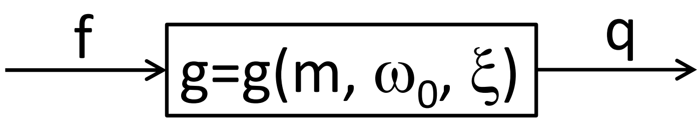


Therefore, the response q(t) can be expressed as the superposition of the responses $q_i(t)$ to each step component force $f_{gi}(t)$: $q(t) \cong \sum_{0}^{i} q_i(t)$, dove $q_i(t) = \bar{f}_{gi}(\tau_i)\, g(t-\tau_i) \Rightarrow$

At the limit for $\Delta t \to 0$:
Duhamel integral of the second type

$$\boxed{q(t) = f(0)g(t) + \int_0^t f'(\tau)\, g(t-\tau)\, d\tau}$$

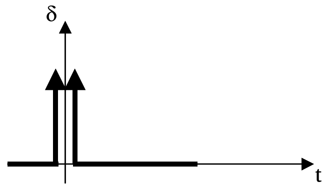
Therefore: $q(t) = f(0) \cdot g(t) + f'(t) * g(t)$.

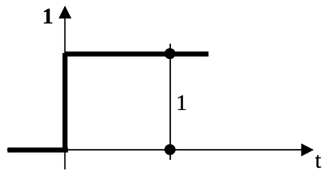


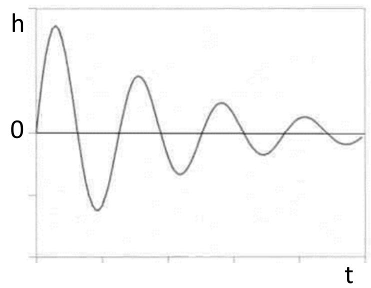
Summary

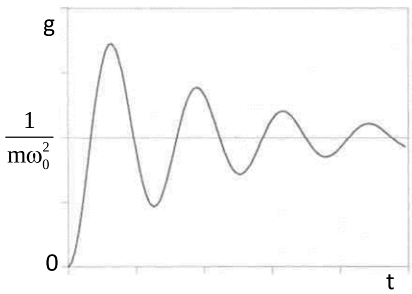
$$\ddot{h}(t) + 2\xi\omega_0\dot{h}(t) + \omega_0^2 h(t) = \frac{1}{m}\delta(t)$$

$$\ddot{g}(t) + 2\xi\omega_0\dot{g}(t) + \omega_0^2 g(t) = \frac{1}{m}\mathbf{1}(t)$$









$$h(t) = e^{-\xi\omega_0 t}\,\frac{1}{m\,\omega_0\sqrt{1-\xi^2}}\,\sin\omega_0\sqrt{1-\xi^2}\,t$$

$$g(t) = \frac{1}{m\omega_0{}^2}\left[1 - e^{-\xi\omega_0 t}\left(\frac{\xi}{\sqrt{1-\xi^2}}\sin\omega_0\sqrt{1-\xi^2}\,t + a_2\cos\omega_0\sqrt{1-\xi^2}\,t\right)\right]$$

Duhamel’s integrals:

$$q(t) = \int_0^t f(\tau)\,h(t-\tau)\,d\tau$$

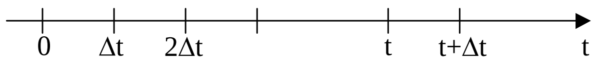
$$q(t) = f(0)g(t) + \int_0^t f'(\tau)\,g(t-\tau)\,d\tau$$

$$\boxed{\begin{aligned}\delta(t) &= \frac{d}{dt}\mathbf{1}(t)\\[1em] h(t) &= \frac{d}{dt}g(t)\end{aligned}}$$

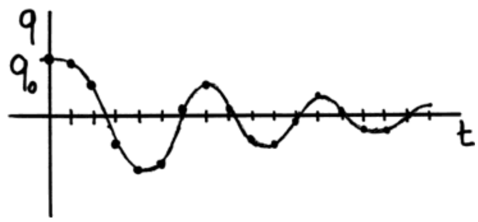
Numerical integration

$$\ddot{q}(t) + 2\xi\omega_0\dot{q}(t) + \omega_0^2 q(t) = \frac{1}{m} f(t)$$
$$q(0) = q_0 \;\; ; \;\; \dot{q}(0) = \dot{q}_0$$

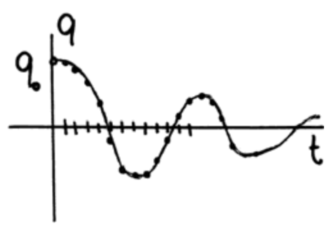
- Instead of solving the equation of motion at any time t, it is satisfied at discrete time intervals Δt. So, the dynamic balance is imposed in a finite number of points along the time axis.
- The solution is searched by recursive algorithms. Knowing the solution at times 0, Δt, $2\Delta t$, … t, the algorithm provides the solution at time + Δt.

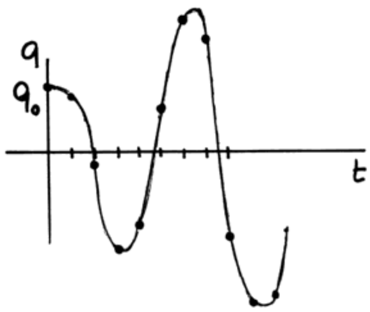


- An explicit integration method is a method formulated by imposing the dynamic balance at time t. An implicit integration method is a method formulated by imposing the dynamic balance at time t + Δt.
- The accuracy, the stability and the burdensome of the algorithm depend on the choice of the time interval Δt and by the way in which $q, \dot{q}, \ddot{q}$ are assumed to vary within Δt.
- A numerical integration method is defined as unconditionally stable if the solution to any initial condition does not increse without limits on increasing t, for any choice of Δt.



- A numerical integration method is said conditionally stable if the above condition holds for $\Delta t < \Delta t_{critico}$, where $\Delta t_{critico}$ is a stability limit.





Well-known methods for the numerical integration of the equation of motion are:
- Finite difference method
- Houbolt method
- Wilson “ϑ” method
- Newmark “β” method

FREQUENCY-DOMAIN ANALYSIS

Elementary harmonic force

A harmonic force f(t) is defined as elementary when it has a unit amplitude. This condition is satisfied by the real expression $f(t) = \sin\omega t$ and by the complex expression $f(t) = e^{i\omega t}$:

$$\boxed{f(t) = e^{i\omega t} = \cos\omega t + i\sin\omega t}$$

$$\Rightarrow \left|e^{i\omega t}\right| = \sqrt{(\sin\omega t + i\cos\omega t)\ (\sin\omega t - i\cos\omega t)} = \sqrt{\sin^2\omega t + \cos^2\omega t} = 1$$

Dynamic response to a elementary harmonic force

$$\boxed{\begin{array}{l} \ddot{q}(t) + 2\xi\omega_0\dot{q}(t) + \omega_0^2 q(t) = \dfrac{1}{m} f(t) = \dfrac{1}{m} e^{i\omega t} = \dfrac{1}{m}\ (\cos\omega t + i\sin\omega t) \\ q(0) = q_0\ \ ;\ \dot{q}(0) = \dot{q}_0 \end{array}}$$

$$q(t) = q'(t) + q''(t)$$

After a sufficiently long time $q'(t) \approx 0$ and $q(t) \approx q''(t)$. It follows that:

$$\boxed{q(t) = H(\omega) e^{i\omega t}}$$

where $H(\omega)$ is the <u>complex frequency response function</u>:

$$\boxed{H(\omega) = \frac{1}{m\omega_0^2} \frac{1}{1 - \dfrac{\omega^2}{\omega_0^2} + 2i\xi\dfrac{\omega}{\omega_0}}}$$

$$H(\omega) = R(\omega) + i\, I(\omega)$$

$$H(\omega) = |H(\omega)| e^{i\psi(\omega)}$$

$$R(\omega) = \frac{1}{m\omega_0^2} \frac{1 - \omega^2/\omega_0^2}{\left(1 - \omega^2/\omega_0^2\right) + 4\xi\omega^2/\omega_0^2}\ ;\ \ I(\omega) = \frac{1}{m\omega_0^2} \frac{-2\zeta\omega/\omega_0}{\left(1 - \omega^2/\omega_0^2\right)^2 + 4\xi^2\omega^2/\omega_0^2}$$

$$|H(\omega)| = \frac{1}{m\omega_0^2} \cdot \frac{1}{\sqrt{\left(1 - \omega^2/\omega_0^2\right)^2 + 4\xi\omega^2/\omega_0^2}}\ \ ;\ \ \psi(\omega) = \mathrm{arctg}\left(-\frac{2\xi\omega/\omega_0}{1 - \omega^2/\omega_0^2}\right)$$

The <u>magnification factor</u> $N(\omega)$ is defined as the ratio between the amplitude $|H(\omega)|$ of the dynamic response and the amplitude $H(0) = 1/m\omega_0^2 = 1/k$ of the static response:

$$\boxed{N(\omega) = \frac{|H(\omega)|}{H(0)} \frac{1}{\sqrt{\left(1 - \frac{\omega^2}{\omega_0^2}\right)^2 + 4\xi^2 \frac{\omega^2}{\omega_0^2}}}}$$

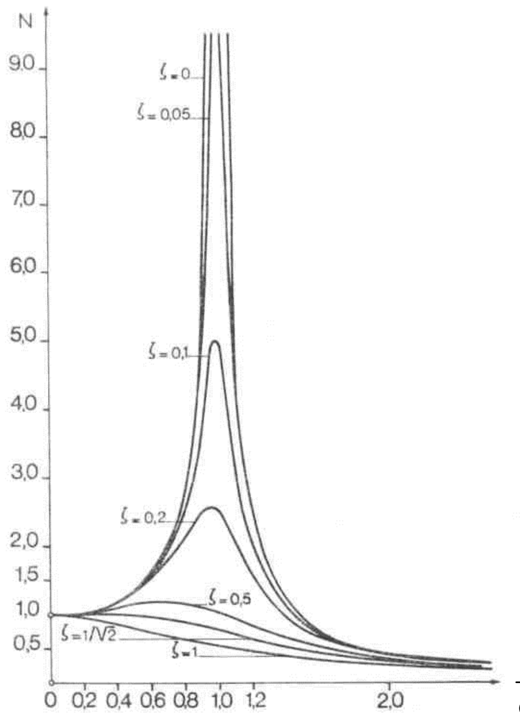
force

dynamic phase delay

In conclusion: $q(t) = N(\omega) \frac{1}{m\omega_0^2} e^{i\omega t} e^{i\psi(\omega)}$

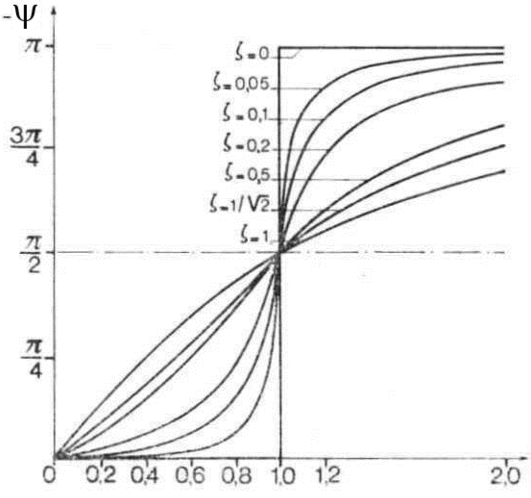
dynamic magnification
of response amplitude

static response



$\frac{\omega}{\omega_0}$

-Ψ



$\frac{\omega}{\omega_0}$

S Q.S. A. D.

Deamplification N ~> 0

Amplification N ~> ½ ξ

Quasi-static N ~ 1

Static N = 1

Periodic force

A function f(t) is defined as periodic with period T when f(t) = f(t + T) for $\forall t \in \mathbf{R}$, with T > 0. The period, is the minimum value of T for which above condition is satisfied.

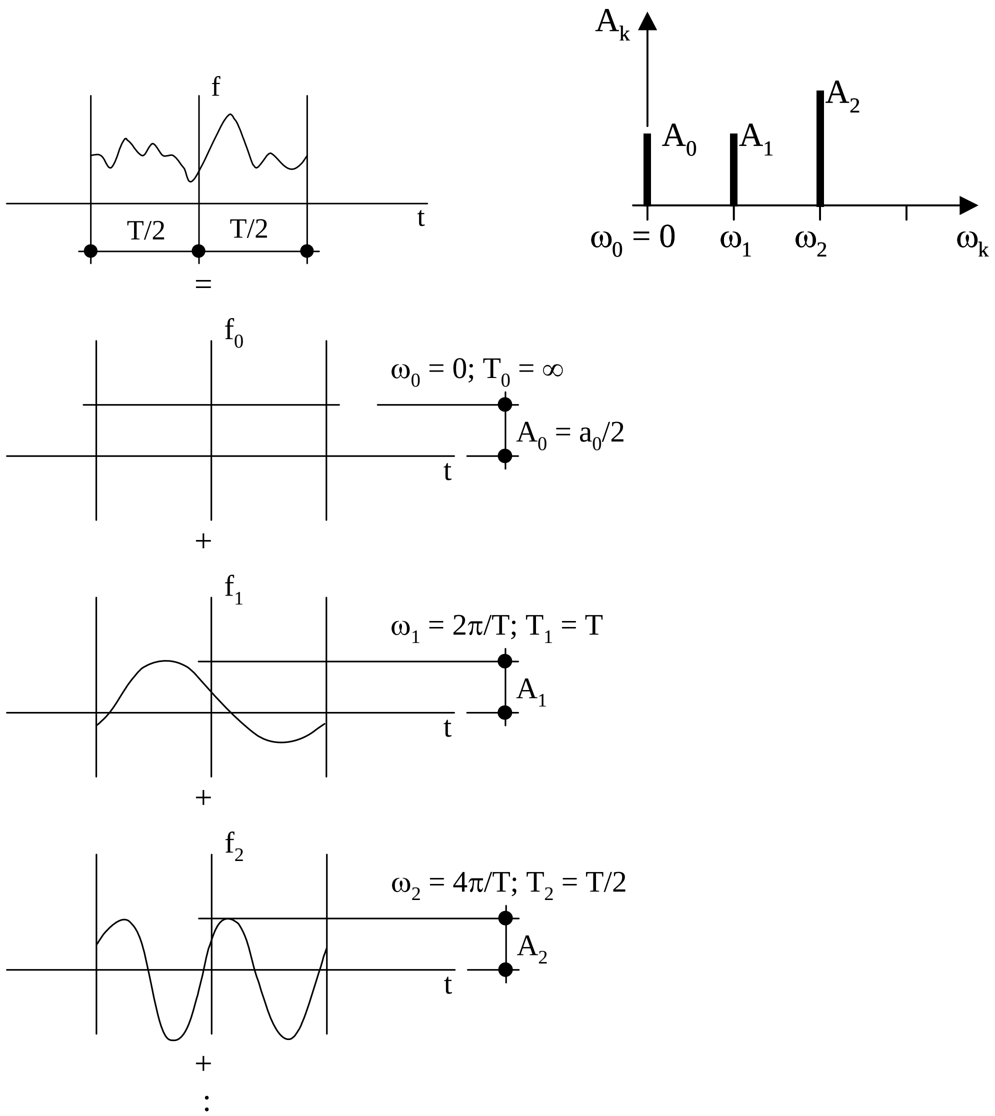
Under general conditions, a periodic function f(t) can be expanded in the following Fourier series:

$$\boxed{f(t) = \frac{a_0}{2} + \sum_{1}^{\infty}{}_k \left(a_k \cos\omega_k t + b_k \sin\omega_k t \right)}$$

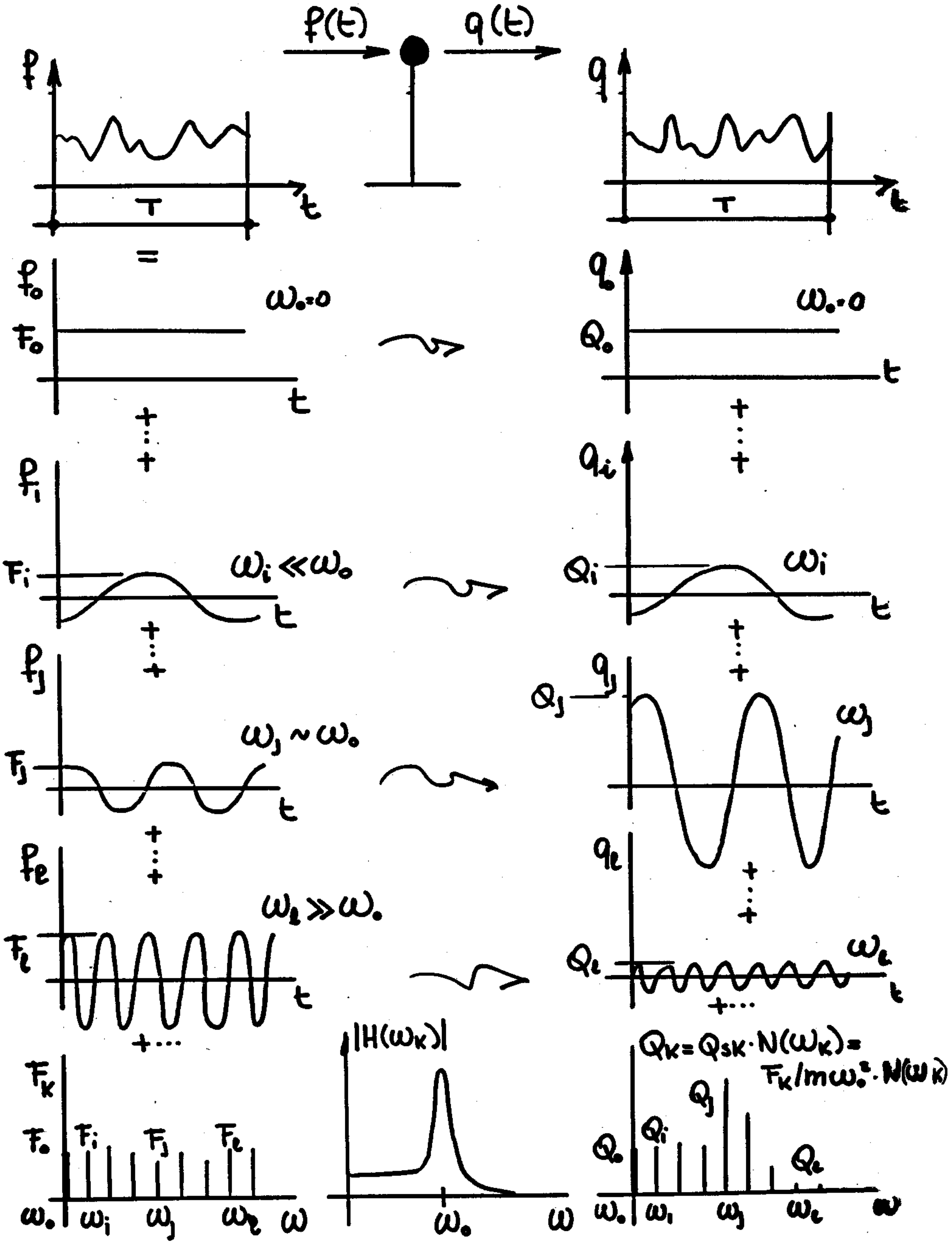
$$\boxed{\begin{aligned} a_k &= \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos\omega_k t\,dt \quad k = 0,1,2,\ldots \\ b_k &= \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin\omega_k t\,dt \quad k = 1,2,\ldots \\ \omega_k &= k\frac{2\pi}{T} \quad k = 0,1,2,\ldots \end{aligned}}$$

$$\boxed{f(t) = \frac{a_0}{2} + \sum_{1}^{\infty}{}_k A_k \sin\left(\omega_k t + \varphi_k \right)}$$

$$\boxed{\begin{aligned} A_k &= \sqrt{a_k^2 + b_k^2} \\ \varphi_k &= \mathrm{arc\,tg}\left(\frac{a_k}{b_k} \right) \end{aligned}}$$



Dynamic response to a periodic force



Generic force

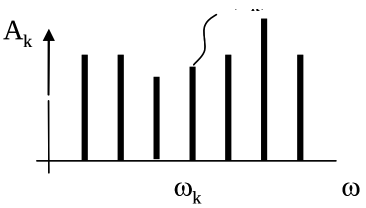
Under general conditions, a generic function f(t) can be expanded in the following Fourier integral:

$$\boxed{\begin{aligned} f(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}\, d\omega \\ F(\omega) &= \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\, dt \end{aligned}}$$

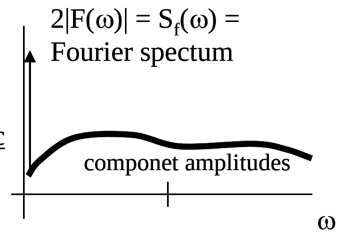
$F(\omega)$ is a complex function called Fourier transform; $f(t)$ is consequently called inverse Fourier transform. The uniqueness of a Fourier couple, $f(t)$ and $F(\omega)$, is demonstrated under general conditions. $F(\omega)$ exists provided that:

$\int_{-\infty}^{\infty} |f(t)| dt$ is finite.

f periodic



f not periodic



Dynamic response to a generic force

$$\ddot{q}(t) + 2\xi\omega_0\dot{q}(t) + \omega_0^2 q(t) = \frac{1}{m} f(t)$$

The steady-state response q(t) to a generic force $f(t)$ can be expressed as the integral of the elementary component responses to the elementary component harmonic forces:

$$\begin{aligned} f(t) &= e^{i\omega t} \Rightarrow & q(t) &= H(\omega)e^{i\omega t} \\ f(t) &= F(\omega)e^{i\omega t} \Rightarrow & q(t) &= F(\omega)\, H(\omega)e^{i\omega t} \end{aligned}$$

$$\boxed{\begin{aligned} f(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}d\omega \Rightarrow \\ q(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, H(\omega)e^{i\omega t}d\omega \end{aligned}}$$

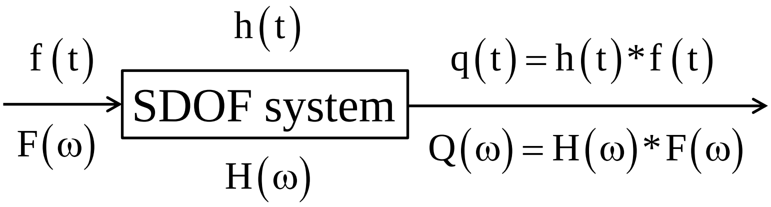
Moreover, using the definition of Fourier transform and inverse Fourier response of the response:

$$\boxed{\begin{aligned} q(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} Q(\omega)e^{i\omega t}d\omega \\ Q(\omega) &= \int_{-\infty}^{\infty} q(t)e^{-i\omega t}dt \end{aligned}}$$

Comparing the above equations:

$$\boxed{Q(\omega) = H(\omega)\,F(\omega)}$$

Summarising, the frequency domain analysis consists of 4 steps:



(1) Starting from $f(t)$ its Fourier transform is calculated $F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, dt$;

(2) The structural system is characterised by its complex frequency response function:

$$H(\omega) = \frac{1}{m\,\omega_o^2} \frac{1}{\left(1 - \omega^2 / \omega_o^2\right) + 2i\xi\omega / \omega_o};$$

(3) The Fourier transform of $q(t)$ is determined: $Q(\omega) = H(\omega)\,F(\omega)$;

(4) The inverse Fourier transform of $Q(\omega)$ is calculated: $q(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Q(\omega) e^{i\omega t} d\omega$.

It is easy to demonstrate that:

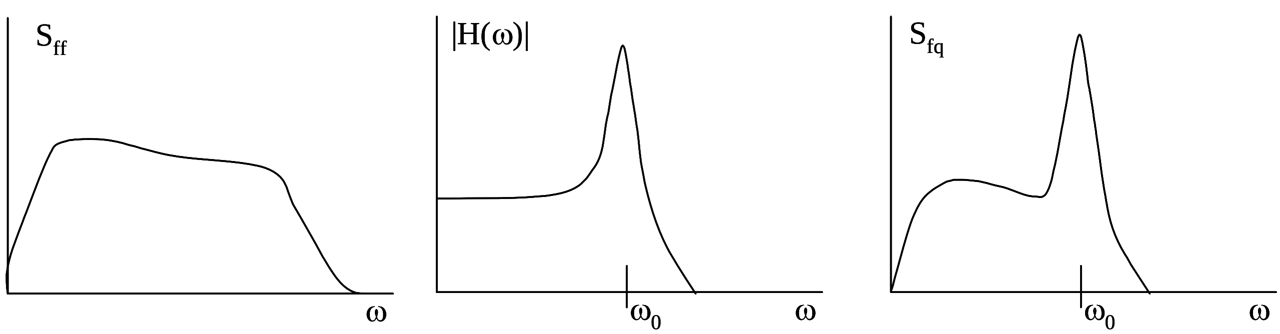
$$\boxed{\left|Q(\omega)\right| = \left|H(\omega)\right| \left|F(\omega)\right|}$$

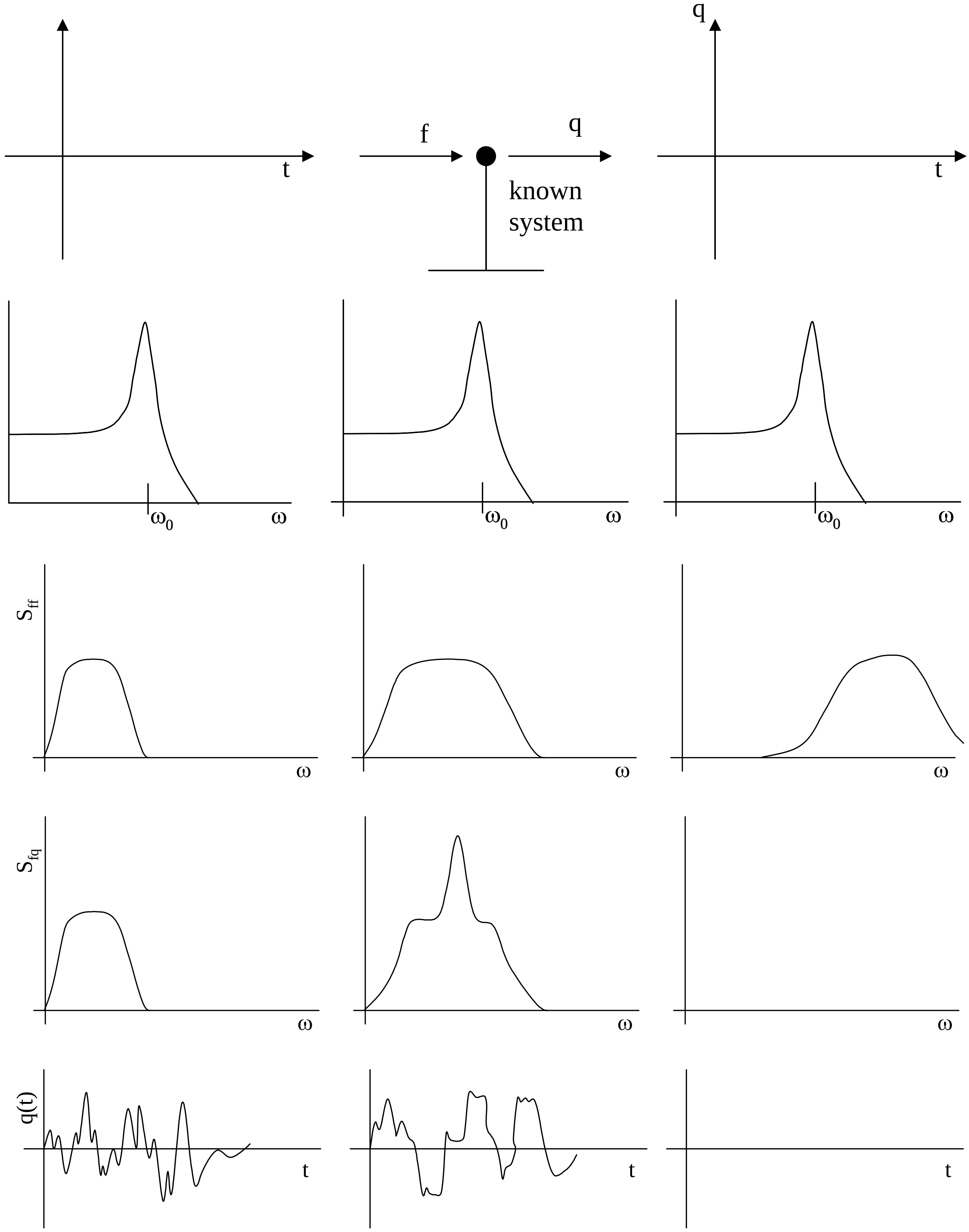
$S_{ff}(\omega) = 2\left|F(\omega)\right|$ = Fourier spectrum of the force =
= amplitude of the harmonic components of $f(t)$

$S_{fq}(\omega) = 2\left|Q(\omega)\right|$ = Fourier spectrum of the response =
= amplitude of the harmonic components of $q(t)$

$\left|H(\omega)\right| = N(\omega)/m\omega_o^2$ = Ratio between the amplitudes of the harmonic components of the response and of the force

Thus: $S_{fq}(\omega) = \left|H(\omega)\right| \cdot S_{ff}(\omega)$




q
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q(t)
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t