

RANDOM DYNAMICS

Maximum value of the response

Definitions

Let us consider the random stationary process $Q'(t)$ that represents the fluctuating part of the response $Q(t)$. Let \hat{Q}' be the random variable that expresses the maximum value of $Q'(t)$ in the temporal interval T . \hat{Q}' is the set of the random occurrences of the maximum values of the sample functions $q'^{(i)}(t)$ in T (Fig. 1).

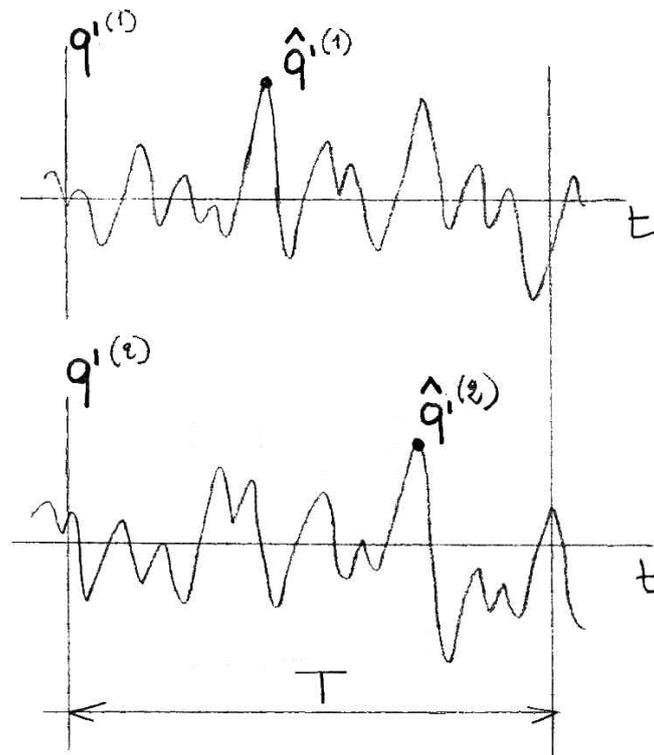


Fig. 1

The evaluation of the distribution of the random variable \hat{Q}' , i.e. of the maximum value of the fluctuation of the response, involves the preliminary analysis of the threshold up-crossing of the process $Q'(t)$.

For sake of simplicity, this treatment will be carried omitting the apex prime. Thus, Q' is replaced by Q , q' by q , and \hat{Q}' by \hat{Q} .

Threshold up-crossing of a stationary process

Let us consider a deterministic quantity $\zeta \geq 0$ defined as threshold. Let $K^+(\zeta, T)$ be the discrete random variable that counts the number of the up-crossings (+) of the threshold ζ in the time interval T (Fig. 2).

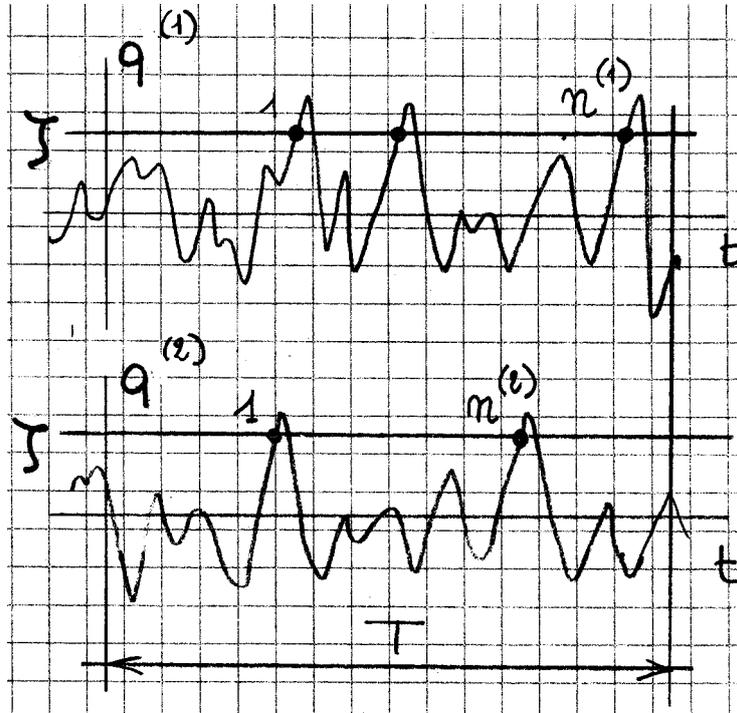


Fig. 2

It is demonstrated (Rice 1944, 1945) that the mean value of the up-crossings of the threshold ζ in the time interval T is given by:

$$\mu_{N^+}(\zeta, T) = E[K^+(\zeta, T)] = T \int_0^{\infty} |\dot{q}| p_{Q\dot{Q}}(\zeta, \dot{q}) d\dot{q} \quad (1)$$

where $p_{Q\dot{Q}}(q, \dot{q})$ is the joint density function of $Q(t)$ and $\dot{Q}(t)$ (at the time lag $\tau = 0$).

From Eq. (1) it follows that the mean number of the up-crossings of the threshold ζ in the unit time is given by:

$$\mu_{N^+}(\zeta) = E[N^+(\zeta)] = \frac{\mu_{N^+}(\zeta, T)}{T} \Rightarrow$$

$$\boxed{\mu_{N^+}(\zeta) = \int_0^{\infty} |\dot{q}| p_{Q\dot{Q}}(\zeta, \dot{q}) d\dot{q}} \quad (2)$$

The solution of Eq. (2) is simple in the fundamental case in which $Q(t)$ is a normal process. Since any stationary process $Q(t)$ and its time derivative $\dot{Q}(t)$ are uncorrelated for $\tau = 0$, i.e. $C_{Q\dot{Q}}(0) = E[Q(t)\dot{Q}(t)] = 0$, if $Q(t)$ is a normal process, $Q(t)$ and $\dot{Q}(t)$ are independent (for $\tau = 0$). So:

$$p_{Q\dot{Q}}(q, \dot{q}) = p_Q(q)p_{\dot{Q}}(\dot{q}) = \exp\left\{-\frac{q^2}{2\sigma_Q^2} - \frac{\dot{q}^2}{2\sigma_{\dot{Q}}^2}\right\} \quad (3)$$

Substituting Eq. (3) into Eq. (2), it follows that:

$$\mu_{N^+}(\zeta) = \nu_Q \exp\left\{-\frac{\zeta^2}{2\sigma_Q^2}\right\} \quad (4)$$

where:

$$\nu_Q = \mu_{N^+}(0) = \frac{1}{2\pi} \frac{\sigma_{\dot{Q}}}{\sigma_Q} \quad (5)$$

is the mean number of the up-crossings of the threshold $\zeta = 0$ in the unit time. This quantity is referred to as the expected frequency of the process.

It is easy to show that the expected frequency ν_Q of a narrow band process is the mean number of the cycles made by $Q(t)$ in the unit time (Fig. 3).

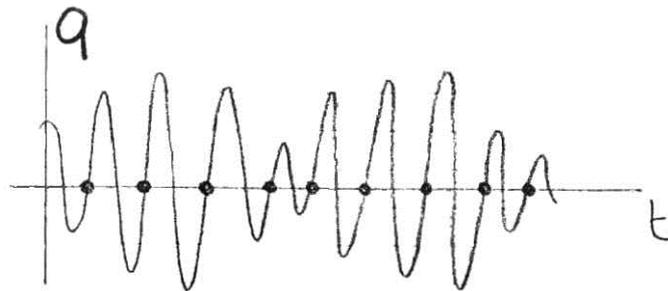


Fig. 3

Distribution of the maximum (independent threshold up-crossings)

Assuming that the threshold ζ is high enough, the threshold up-crossings are rare and independent events (Fig. 4). As such, they are characterised by a Poisson distribution.

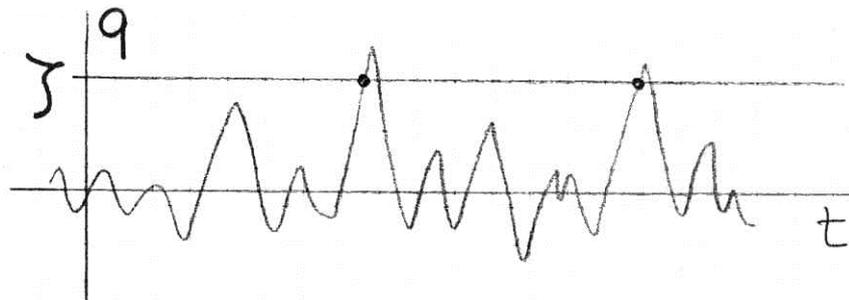


Fig. 4

In this case, the probability that the threshold ζ is up-crossed $N^+ = n$ times in the time interval T is given by:

$$P_{N^+}(n, \zeta, T) = \exp\{-\mu_{N^+}(\zeta, T)\} \frac{\{\mu_{N^+}(\zeta, T)\}^n}{n!} \quad (6)$$

Thus, due to Eq. (2):

$$P_{N^+}(n, \zeta, T) = \exp\{-T\mu_{N^+}(\zeta)\} \frac{\{T\mu_{N^+}(\zeta)\}^n}{n!} \quad (7)$$

where, if $Q(t)$ is a normal process, $\mu_{N^+}(\zeta)$ is given by Eq. (4).

Applying Eq. (7) and Eq. (4), the probability that the threshold ζ is up-crossed $N^+ = 0$ times in the time interval T is given by:

$$P_{N^+}(0, \zeta, T) = \exp\left\{-v_Q T \exp\left[-\frac{\zeta^2}{2\sigma_Q^2}\right]\right\} \quad (8)$$

But the probability that the threshold ζ is up-crossed $N^+ = 0$ times in the time interval T is equal to the probability that the maximum value \hat{Q} of $Q(t)$ in T is less or equal ζ . Thus:

$$P[\hat{Q} \leq \zeta] = P_{N^+}(0, \zeta, T) \quad (9)$$

being $P[\hat{Q} \leq \zeta] = F_{\hat{Q}}(\zeta)$. It follows that the distribution function of the maximum is given by the relationship (Davenport, 1963):

$$F_{\hat{Q}}(\hat{q}) = \exp\left\{-v_Q T \exp\left[-\frac{\hat{q}^2}{2\sigma_Q^2}\right]\right\} \quad (10)$$

from which, deriving with respect to \hat{q} , the density function of the maximum value is obtained:

$$P_{\hat{Q}}(\hat{q}) = v_Q T \frac{\hat{q}}{\sigma_Q^2} \exp\left[-\frac{\hat{q}^2}{2\sigma_Q^2}\right] \exp\left\{-v_Q T \exp\left[-\frac{\hat{q}^2}{2\sigma_Q^2}\right]\right\} \quad (11)$$

From Eq. (11) it is possible to determine the mean value $\mu_{\hat{Q}}$ and the standard deviation $\sigma_{\hat{Q}}$ of \hat{Q} :

$$\mu_{\hat{Q}} = \int_0^{\infty} \hat{q} p_{\hat{Q}}(\hat{q}) d\hat{q} \Rightarrow$$

$$\mu_{\hat{Q}} \approx \sigma_Q \left[\sqrt{2\ln(v_Q T)} + \frac{0.5772}{\sqrt{2\ln(v_Q T)}} \right] \quad (12)$$

$$\sigma_{\hat{Q}}^2 = \int_0^{\infty} (\hat{q} - \mu_{\hat{Q}})^2 p_{\hat{Q}}(\hat{q}) d\hat{q} \Rightarrow$$

$$\sigma_{\hat{Q}} \approx \sigma_Q \left[\frac{\pi}{\sqrt{6}} \frac{1}{\sqrt{2 \ln(v_Q T)}} \right] \quad (13)$$

Fig. 5 shows the density function of the response process $Q(t)$ and of its maximum \hat{Q} (on varying the non-dimensional parameter $v_Q T$).

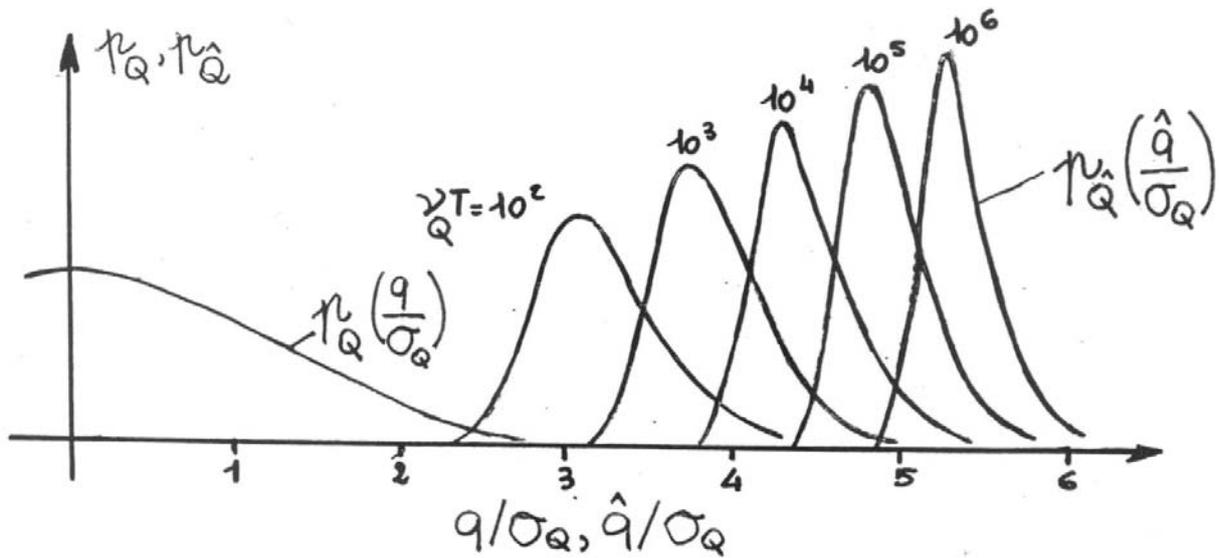


Fig. 5

On increasing $v_Q T$, the density function of the maximum translates towards the largest values of q (since $\mu_{\hat{Q}}$ increases), and its shape becomes more and more narrow and sharp (since $\sigma_{\hat{Q}}$ diminishes). In other words, on increasing $v_Q T$, \hat{Q} tends to assume deterministic properties.

Therefore, when $v_Q T$ is small (e.g. in the seismic case), \hat{Q} is definitely a random variable. When $v_Q T$ is large (e.g. in the wind case), \hat{Q} may be considered, roughly, as a deterministic quantity equal to the mean value $\mu_{\hat{Q}}$ of \hat{Q} . Using Eq. (12) this value becomes:

$$\mu_{\hat{Q}} = g_Q \sigma_Q \quad (14)$$

where g_Q is a non-dimensional quantity referred to as the peak coefficient:

$$g_Q = \sqrt{2 \ln(v_Q T)} + \frac{0.5772}{\sqrt{2 \ln(v_Q T)}} \quad (15)$$

Finally, let us remember that all the above analyses refer to the fluctuating part of the response $Q'(t)$ (indicated by $Q(t)$). Considering also the mean value μ_Q of the response process, the actual maximum value is given by (Fig. 6):

$$\mu_{\hat{Q}} = \mu_Q + g_Q \sigma_Q \quad (16)$$

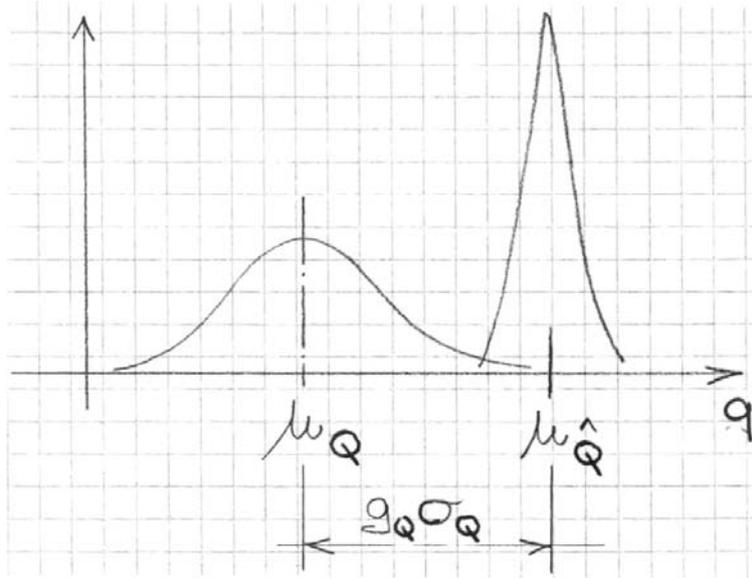


Fig. 6

Bilateral threshold crossings and absolute maximum value of a normal zero mean process

The above analyses examined the up-crossings of ζ (Fig. 2). However, in the case of the normal zero mean processes (e.g. the seismic response), it is relevant to count both the up-crossings of ζ and the down-crossings of $-\zeta$, i.e. the number of times in which the process out-crosses the domain $(-\zeta, \zeta)$ (Fig. 7). This problem is referred to as the bilateral threshold crossing.

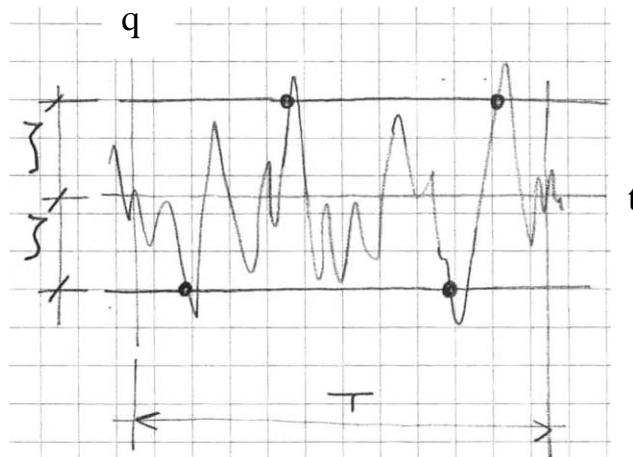


Fig. 7

Let N^- be the discrete random variable that counts the down-crossings of $-\zeta$ in the unit time: $\mu_{N^-} = \mu_{N^+}$. Thus, setting $N = N^+ + N^-$, then $\mu_N = \mu_{N^+} + \mu_{N^-} = 2\mu_{N^+}$. From Eq. (4) it follows that:

$$\mu_N(\zeta) = 2\nu_Q \exp\left\{-\frac{\zeta^2}{2\sigma_Q^2}\right\} \quad (17)$$

Based on this new formulation, the distribution function and the density function of the maximum absolute value $|\hat{Q}|$ of $Q(t)$ are given by the relationships:

$$F_{|\hat{Q}|}(|\hat{q}|) = \exp\left\{-2\nu_Q T \exp\left[-\frac{|\hat{q}|^2}{2\sigma_Q^2}\right]\right\} \quad (18)$$

$$p_{|\hat{Q}|}(|\hat{q}|) = 2\nu_Q T \frac{|\hat{q}|}{\sigma_Q^2} \exp\left\{-\frac{|\hat{q}|^2}{2\sigma_Q^2}\right\} \exp\left\{-2\nu_Q T \exp\left[-\frac{|\hat{q}|^2}{2\sigma_Q^2}\right]\right\} \quad (19)$$

Finally, the mean value and the standard deviation of $|\hat{Q}|$ are given by:

$$\mu_{|\hat{Q}|} = g_{|\hat{Q}|} \sigma_Q \quad (20)$$

$$g_{|\hat{Q}|} = \sqrt{2\ell n(2\nu_Q T)} + \frac{0.5772}{\sqrt{2\ell n(2\nu_Q T)}} \quad (21)$$

$$\sigma_{|\hat{Q}|} = \sigma_Q \left[\frac{\pi}{\sqrt{6}} \frac{1}{\sqrt{2\ell n(2\nu_Q T)}} \right] \quad (22)$$

Crossings in clumps and envelope process of a narrow band process

Fig. 8 shows a typical sample function of a random stationary normal zero-mean narrow-band process. It is apparent that the up-crossings of a threshold ζ tend to occur in clumps. In other words, any time a sample function up-crosses the threshold, the up-crossing tend to repeat also in the following cycles.

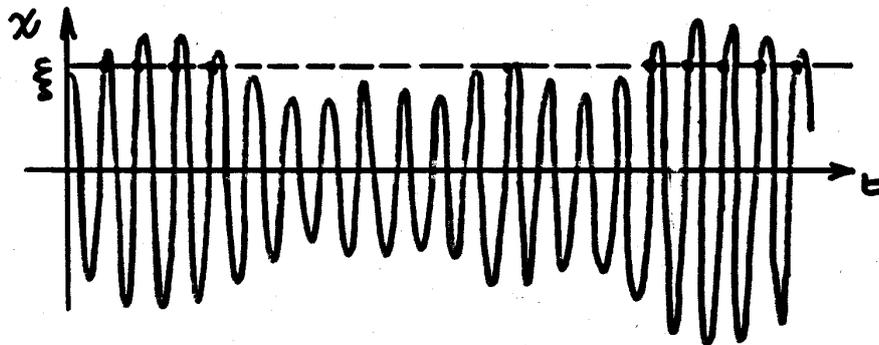


Fig. 8

In order to understand better this concept, let us consider a new process $A(t)$, the envelope process of the process $Q(t)$, constituted by the sample functions $a(t)$ (Fig. 9a).

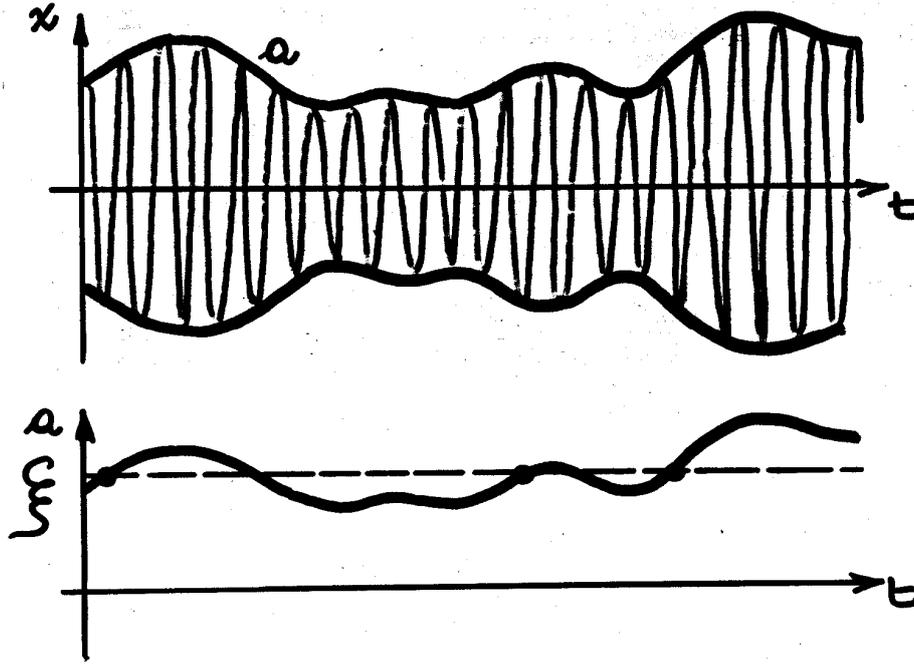


Fig. 9

Let $N_A^+(\zeta)$ be the discrete random variable that counts the up-crossings of the threshold ζ made by the process $A(t)$ (Fig. 9b) in the unit time. It is demonstrated that the mean value of $N_A^+(\zeta)$ results:

$$\mu_{N_A^+}(\zeta) = \frac{\zeta q_Q \sigma_Q}{\sqrt{2\pi\sigma_Q^2}} \exp\left\{-\frac{\zeta^2}{2\sigma_Q^2}\right\} = \frac{\sqrt{2\pi}\zeta q_Q}{\sigma_Q} \mu_{N^+}(\zeta) \quad (23)$$

where q_Q is the spectral bandwidth parameter of $Q(t)$.

The mean number of the up-crossings in a clump is referred to as the average clump size. A good approximation of this quantity is given by the relationship:

$$\langle cs \rangle = \frac{\mu_{N^+}(\zeta)}{\mu_{N_A^+}(\zeta)} = \frac{\sigma_Q}{\sqrt{2\pi}\zeta q_Q} \quad (24)$$

It is worth noting that $\langle cs \rangle$ diminishes on increasing ζ . Thus, on increasing ζ , the threshold up-crossings tend to occur independently.

In the meanwhile, $\langle cs \rangle$ is linked with the spectral bandwidth parameter q_Q . When the process is sinusoidal, q_Q tends to 0 and $\langle cs \rangle$ tends to infinite; in other words, each time the process up-crosses ζ , the up-crossing repeats infinite other times. On the contrary, when the harmonic content of the process becomes wider, q_Q increases, $\langle cs \rangle$ diminishes and the threshold up-crossings tend to become independent events.

It is worth noting, however, that the limit case in which q_Q tends to 1 has no practical meaning in this context: it is so far from the hypothesis of a narrow band process that all the above treatment does not apply.

Distribution of the maximum (threshold up-crossings in clumps)

The distribution of the maximum value derived above is based on the hypothesis that the threshold up-crossings are rare and independent events; thus they can be characterised by a Poisson distribution. Further to this formulation, however, it was noted that this hypothesis is not appropriate in the case of the narrow band processes, where the threshold up-crossings tend to occur in clumps. This remark is very important since the dynamic response of the structural systems are often narrow-band processes. Therefore, it is necessary to generalise the evaluation of the distribution of the maximum, developed considering independent up-crossings, to the case of up-crossings in clumps.

The most simple way of dealing with this problem is observing that the envelope $A(t)$ up-crosses the threshold ζ once per each clump of up-crossings of $Q(t)$ (Fig. 9). So, the threshold up-crossings of $A(t)$ are really independent events and can be schematised by a Poisson distribution (Lin, 1967). Paraphrasing the above demonstration (Eq. (7)), the probability that the process $A(t)$ up-crosses the threshold ζ $K_A^+ = n$ times in the time interval T is given by:

$$P_{N_A^+}(n, \zeta, T) = \exp\{-T\mu_{N_A^+}(\zeta)\} \frac{\{T\mu_{N_A^+}(\zeta)\}^n}{n!} \quad (25)$$

where $\mu_{N_A^+}(\zeta)$ is given by Eq. (23). Thus, the probability that $A(t)$ up-crosses the threshold ζ $K_A^+ = 0$ times in the time interval T coincides with the distribution function of the maximum value and is given by:

$$F_{\hat{Q}}(\hat{q}) = \exp\left\{-v_Q T \frac{\sqrt{2\pi}q_Q\hat{q}}{\sigma_Q} \exp\left[-\frac{\hat{q}^2}{2\sigma_Q^2}\right]\right\} \quad (26)$$

Vanmarcke (1975) perfected this treatment arriving at the following expression of the distribution function of the maximum value:

$$F_{\hat{Q}}(\hat{q}) = \exp\left\{-v_Q T \frac{1 - \exp\left[-\frac{\sqrt{\pi}q_Q\hat{q}}{\sigma_Q}\right]}{1 - \exp\left[-\frac{\hat{q}^2}{2\sigma_Q^2}\right]} \exp\left[-\frac{\hat{q}^2}{2\sigma_Q^2}\right]\right\} \quad (27)$$

The comparison of Eqs. (10), (26) and (27), considering Eq. (24), points out that Eqs. (26) and (27) tend to Eq. (10) when $\langle cs \rangle$ tends to 1; this is increasingly true on increasing the threshold and the maximum values considered. Moreover, it is easy to show that Eqs. (26) and (27) tend to Eq. (10) on increasing $v_Q T$. Under this point of view, Eq. (10) provides better reasonable results in the wind case, less appropriate solutions in the seismic case.

Finally, it is worth noting that, differently from Eq. (10), Eq. (27) does not allow the derivation of closed form solutions of the mean value and of the standard deviation of the maximum value. Der Kiureghian (1980) solved this problem semi-empirically, obtaining the following formulae:

$$\mu_{\hat{Q}} \approx \sigma_Q \left[\sqrt{2 \ln(v_{Qe} T)} + \frac{0.5772}{\sqrt{2 \ln(v_{Qe} T)}} \right] \quad (28)$$

$$\sigma_{\hat{Q}} = \begin{cases} \sigma_Q \left\{ \frac{1.2}{\sqrt{2 \ln(v_{Qe} T)}} - \frac{5.4}{13 + \left[\sqrt{2 \ln(v_{Qe} T)} \right]^{6.4}} \right\} & \text{per } v_{Qe} T > 2.1 \\ 0.65 \sigma_Q & \text{per } v_{Qe} T \leq 2.1 \end{cases} \quad (29)$$

where v_{Qe} is referred to as the reduced expected frequency:

$$v_{Qe} = \begin{cases} (1.63 q_Q^{0.45} - 0.38) v_Q & \text{per } 0.10 \leq q_Q < 0.69 \\ v_Q & \text{per } q_Q \geq 0.69 \end{cases} \quad (30)$$

Eqs. (28)-(30) apply for $2.5 \leq v_Q T \leq 500$. Eq. (28) coincides with Eq. (12) for $q_Q \geq 0.69$. This confirms that, if the process is broad band, the threshold up-crossings tend to occur independently.

Bilateral threshold up-crossings (in clumps) and absolute maximum values

Generalising Eq. (27), it is easy to show that the distribution of the absolute maximum value $|\hat{Q}|$ is given by the relationship:

$$F_{|\hat{Q}|}(|\hat{q}|) = \exp \left\{ -2v_Q T \frac{1 - \exp \left[-\frac{\sqrt{\pi} q_Q |\hat{q}|}{\sqrt{2} \sigma_Q} \right]}{1 - \exp \left[-\frac{|\hat{q}|^2}{2\sigma_Q^2} \right]} \exp \left[-\frac{|\hat{q}|^2}{2\sigma_Q^2} \right] \right\} \quad (31)$$

Analogously, generalising Eqs. (28-30), it is easy to show that the mean value and the standard deviation of $|\hat{Q}|$ are furnished by the semi-empirical formulae:

$$\mu_{|\hat{Q}|} \approx \sigma_Q \left[\sqrt{2 \ln(2v_{Qe} T)} + \frac{0.5772}{\sqrt{2 \ln(2v_{Qe} T)}} \right] \quad (32)$$

$$\sigma_{|\hat{Q}|} = \begin{cases} \sigma_Q \left\{ \frac{1.2}{\sqrt{2 \ln(2v_{Qe} T)}} - \frac{5.4}{13 + \left[\sqrt{2 \ln(2v_{Qe} T)} \right]^{6.4}} \right\} & \text{per } 2v_{Qe} T > 2.1 \\ 0.65 \sigma_Q & \text{per } 2v_{Qe} T \leq 2.1 \end{cases} \quad (33)$$

where v_{Qe} is the reduced expected frequency given by Eq. (30).