RANDOM DYNAMICS

Single-Degree-Of-Freedom Systems

Equations of motion

The equation of motion $q(t)$ of a S.D.O.F. system subjected to a deterministic force $f(t)$ and to deterministic initial conditions q_0 and $\dot{q}_0$ is given by:

$$\begin{aligned} &\ddot{q}(t) + 2\xi\omega_0\dot{q}(t) + \omega_0^2 q(t) = \frac{1}{m} f(t) \\ &q(0) = q_0 \ ; \ \dot{q}(0) = \dot{q}_0 \end{aligned} \qquad (1)$$

Let us assume that f(t) is a sample function of a loading process F(t). Analogously, the initial conditions q_0 and $\dot{q}_0$ are random occurrences of a couple of random variables Q_0 and $\dot{Q}_0$. The equation of motion q(t) is the sample function of the response process Q(t) corresponding to f(t).

In this case Eq. (1) is the deterministic relationship that expresses q(t) as a function of f(t), q_0 and $\dot{q}_0$. Analogously, the relationship that expresses Q(t) as a function of F(t), Q_0 and $\dot{Q}_0$ assumes the form (Fig. 1):

$$\boxed{\begin{aligned} &\ddot{Q}(t) + 2\xi\omega_0\dot{Q}(t) + \omega_0^2 Q(t) = \frac{1}{m} F(t) \\ &Q(0) = Q_0 \ ; \ \dot{Q}(0) = \dot{Q}_0 \end{aligned}} \qquad (2)$$

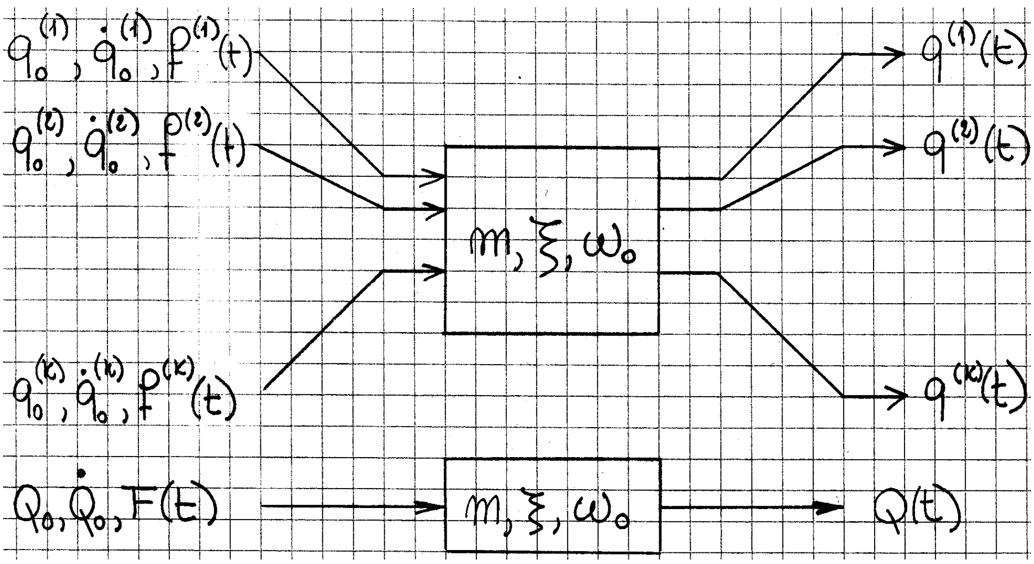


Fig. 1

Using this model the loading is random and the response is random. Instead, the structural system remains deterministic. This is because the loading is usually characterised by greater uncertainties than the structural system. Dealing with the structure as random (thus defining M, Ξ, Ω_0 as random variables of which m, ξ, ω_0 are random occurrences) makes the treatment much more complex.

Please also note that assuming the initial conditions as deterministic is equivalent to set:

$$P\left[Q(t_0)=q_0 \cap \dot{Q}(t_0)=\dot{q}_0\right]=1$$

Under the hypothesis of quasi-steady vibrations (i.e. loosing the memory of the initial conditions) the deterministic response of a S.D.O.F. system (Eq. 1) is given by:

$$q(t)=\int_0^t f(\tau)h(t-\tau)d\tau \tag{3}$$

where $h(\bullet)$ is the impulse response function.
Analogously, the quasi-steady response process of a S.D.O.F. system (Eq. 2) excited by the loading process F(t) is given by:

$$\boxed{Q(t)=\int_0^t F(\tau)h(t-\tau)d\tau} \tag{4}$$

Let us assume that F(t) is a random stationary process. The lower integration limit in Eq. (4) is the time instant at which the loading application begins. Since the loading is stationary, its beginning occurs at t = -∞. It follows that Eq. (4) should be rewritten as:

$$Q(t)=\int_{-\infty}^t F(\tau)h(t-\tau)d\tau$$

Moreover, let us observe that the function $h(t-\tau)$ is identically null for $t<\tau$ (Fig. 2), so also for $\tau>t$. Thus, the upper integration limit in Eq. (4) can be put equal to +∞ without introducing any mistake. Thus, the equation of motion becomes:

$$\boxed{Q(t)=\int_{-\infty}^{\infty} F(\tau)h(t-\tau)d\tau} \tag{5}$$

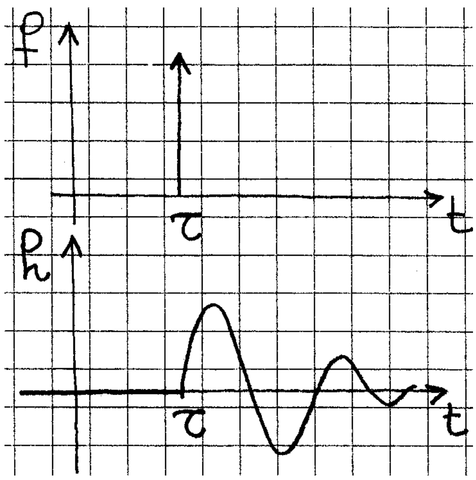


Fig. 2

Mean value and fluctuation of the response

Let us express the loading process as:

$$\boxed{F(t) = \mu_F + F'(t)} \tag{6}$$

where $\mu_F = E[F(t)]$ is the mean value of the loading (independent of time since F is stationary) and $F'(t)$ = F(t) - μ_F. As such, it is a random stationary process with zero mean (referred to as the fluctuation or the fluctuating part of F) characterised, in the time domain, by the auto-covariance function $C_{F'F'}(\tau)$. Since $F'(t)$ is zero mean, $C_{F'F'}(\tau)$ coincides with the auto-correlation function $R_{F'F'}(\tau)$ of F'(t) and with the auto-covariance function $C_{FF}(\tau)$ of F(t). In the fundamental case in which F(t) is a normal process, μ_F and $C_{FF}(\tau)$ provide a full probabilistic description of F(t).

Let us consider Eq. (5) and let us apply the transformation $\vartheta = t - \tau$. It follows that:

$$Q(t) = \int_{-\infty}^{\infty} F(t-\vartheta)h(\vartheta)d\vartheta \tag{7}$$

The mean value of Q(t) is given by the relationship:

$$E\left[Q(t)\right] = E\left[\int_{-\infty}^{\infty} F(t-\vartheta)h(\vartheta)d\vartheta\right] = \int_{-\infty}^{\infty} E\left[F(t-\vartheta)\right]h(\vartheta)d\vartheta \Rightarrow$$
$$\mu_Q = E[Q(t)] = \mu_F \int_{-\infty}^{\infty} h(\vartheta)d\vartheta \tag{8}$$

Thus, likewise the mean of the loading, also the mean of the response is independent of time.
Let us consider the complex frequency response function:

$$H(\omega) = \frac{1}{m\omega_0^2}\frac{1}{1 - \frac{\omega^2}{\omega_0^2} + 2i\xi\frac{\omega}{\omega_0}} \tag{9}$$

and let us remember that it is the Fourier transform of the impulse response function:

$$H(\omega) = \int_{-\infty}^{\infty} h(t)e^{-i\omega t}dt \tag{10}$$

It follows that:

$$H(0) = \frac{1}{m\omega_0^2} = \frac{1}{k} = \int_{-\infty}^{\infty} h(t)dt \tag{11}$$

Substituting Eq. (11) into Eq. (8) it results:

$$\boxed{\mu_Q = \frac{\mu_F}{k}} \tag{12}$$

So, <u>the mean value of the response is the deterministic static response to the mean value of the loading</u>.

As a consequence of this result, let us define:

$$\boxed{Q(t) = \mu_Q + Q'(t)} \tag{13}$$

where $\mu_Q = E[Q(t)]$ is the mean value of the response and $Q'(t)$ is the zero mean fluctuation of the response. Substituting Eq. (6) and Eq. (13) into Eq. (2a):

$$\cancel{\ddot{\mu}_Q} + \ddot{Q}'(t) + \cancel{2\xi\omega_0\dot{\mu}_Q} + 2\xi\omega_0\dot{Q}'(t) + \omega_0^2\mu_Q + \omega_0^2 Q'(t) = \frac{1}{m}\mu_F + \frac{1}{m}F'(t)$$

from which, using Eq. (12):

$$\boxed{\ddot{Q}'(t) + 2\xi\omega_0\dot{Q}'(t) + \omega_0^2 Q'(t) = \frac{1}{m}F'(t)} \tag{14}$$

In other words, the fluctuation of the response is the dynamic response to the fluctuation of the loading. This remark points out the opportunity of separating the initial problem into two problems: a static one defined by Eq. (12), and a dynamic one defined by Eq. (14). The static problem applies only if the mean value of F is not null (e.g. in the wind case). The dynamic problem coincides with the initial problem when the mean value of F is null (e.g. in the seismic case). Thanks to Eq. (7):

$$\boxed{Q'(t) = \int_{-\infty}^{\infty} F'(t-\vartheta)h(\vartheta)d\vartheta} \tag{15}$$

Fig. 3 shows some sample functions of the fluctuating loading and corresponding response.

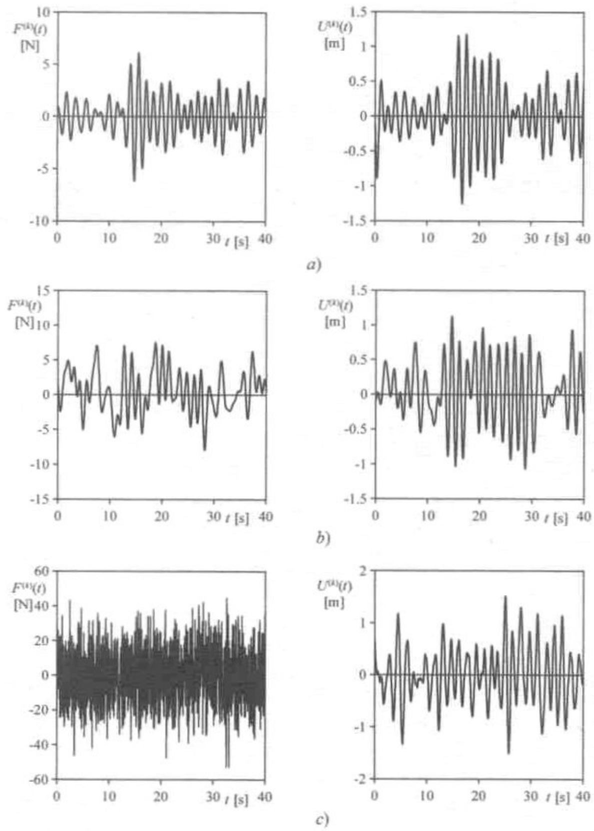


Fig. 3

Auto-covariance function of the response

The auto-covariance function of the response coincides with the auto-correlation function of the fluctuating part of the response. It is given by:

$$\boxed{C_{QQ}(t,t+\tau)=E\left[\{Q(t)-\mu_Q\}\{Q(t+\tau)-\mu_Q\}\right]=E\left[Q'(t)\,Q'(t+\tau)\right]} \tag{16}$$

Applying Eq. (15):

$$C_{QQ}(t,t+\tau)=E\left[\int_{-\infty}^{\infty}F'(t+\tau-\vartheta_1)h(\vartheta_1)d\vartheta_1\int_{-\infty}^{\infty}F'(t-\vartheta_2)h(\vartheta_2)d\vartheta_2\right]=$$
$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}E\left[F'(t-\vartheta_1)F'(t+\tau-\vartheta_2)\right]h(\vartheta_1)h(\vartheta_2)d\vartheta_1\,d\vartheta_2$$

where $E\left[F'(t-\vartheta_1)F'(t+\tau-\vartheta_2)\right]=C_{FF}(\tau-\vartheta_2+\vartheta_1)$. It follows that:

$$\boxed{C_{QQ}(\tau)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}C_{FF}(\tau-\vartheta_2+\vartheta_1)h(\vartheta_1)h(\vartheta_2)d\vartheta_1 d\vartheta_2} \tag{17}$$

Thus, analogously to C_{FF}, also C_{QQ} depends only on the time lag τ. Thus, analogously to the loading process, also the response process is (weakly) stationary. Fig. 4 shows the auto-covariance functions of some loading and response processes.

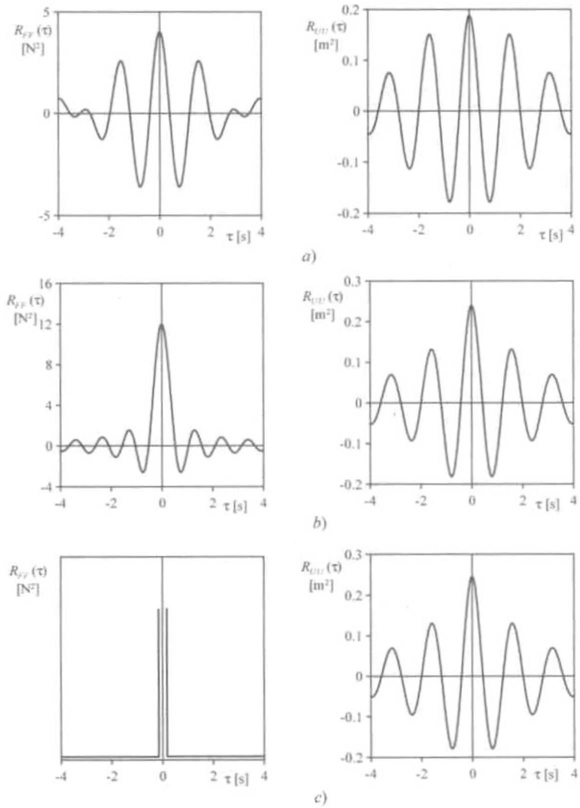


Fig. 4

It is demonstrated that, if the loading process is normal and the structure is linear, also the response process is normal. In this case μ_Q and $C_{QQ}(\tau)$ provide a full probabilistic representation of Q(t).

Power spectral density of the response

Applying the Wiener-Khintchine equations, the power spectral density of the response is given by:

$$\boxed{S_{QQ}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} C_{QQ}(\tau)\, e^{-i\omega\tau} d\tau} \qquad (18)$$

Substituting Eq. (17) into Eq. (18) and multiplying the integrand by $e^{i\omega\vartheta_1}\, e^{-i\omega\vartheta_2}\, e^{-i\omega(\vartheta_1-\vartheta_2)} = 1$:

$$S_{QQ}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} C_{FF}(\tau-\vartheta_2+\vartheta_1)\, h(\vartheta_1)\, h(\vartheta_2)\, e^{-i\omega\tau} e^{i\omega\vartheta_1} e^{-i\omega\vartheta_2} e^{-i\omega(\vartheta_1-\vartheta_2)} d\vartheta_1 d\vartheta_2 d\tau =$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} C_{FF}(\tau-\vartheta_2+\vartheta_1)\, e^{-i\omega(\tau-\vartheta_2+\vartheta_1)} d\tau\right] h(\vartheta_1)\, h(\vartheta_2)\, e^{i\omega\vartheta_1} e^{-i\omega\vartheta_2} d\vartheta_1 d\vartheta_2$$

where $\frac{1}{2\pi}\int_{-\infty}^{\infty} C_{FF}(\tau-\vartheta_2+\vartheta_1)\, e^{-i\omega(\tau-\vartheta_2+\vartheta_1)} d\tau = S_{FF}(\omega)$ is the power spectral density of the loading. Thus:

$$S_{QQ}(\omega) = S_{FF}(\omega)\int_{-\infty}^{\infty} h(\vartheta_1)\, e^{i\omega\vartheta_1} d\vartheta_1 \int_{-\infty}^{\infty} h(\vartheta_2)\, e^{-i\omega\vartheta_2} d\vartheta_2$$

where $\int_{-\infty}^{\infty} h(\vartheta_1)\, e^{i\omega\vartheta_1} d\vartheta_1 = H^*(\omega)$ and $\int_{-\infty}^{\infty} h(\vartheta_2)\, e^{-i\omega\vartheta_2} d\vartheta_2 = H(\omega)$. It follows that:

$$\boxed{S_{QQ}(\omega) = \left|H(\omega)\right|^2 S_{FF}(\omega)} \qquad (19)$$

It is worth noting that, using random dynamics in the frequency domain, the equation that links the power spectral density of the response with the power spectral density of the loading is real (Eq. 19). This does not happen using deterministic dynamics, where the equation that links the Fourier transform of the response with the Fourier transform of the loading is complex:

$$Q(\omega) = H(\omega)\, F(\omega) \qquad (20)$$

The step from the complex Eq. (20) to the real Eq. (19) implies the loss (better the undetermination) of the phases. In the deterministic field, a function expanded by a Fourier integral is a continuous sum of harmonics each characterised by a given circular frequency, amplitude and phase. In the random field the undetermination of the phases (better their random occurrence with a uniform distribution between 0 and 2π) gives rise to infinite functions with the same spectral density. Such functions are the sample functions of the process.

Fig. 5 shows the power spectral densities of some loading processes and the corresponding power spectral densities of the response processes.

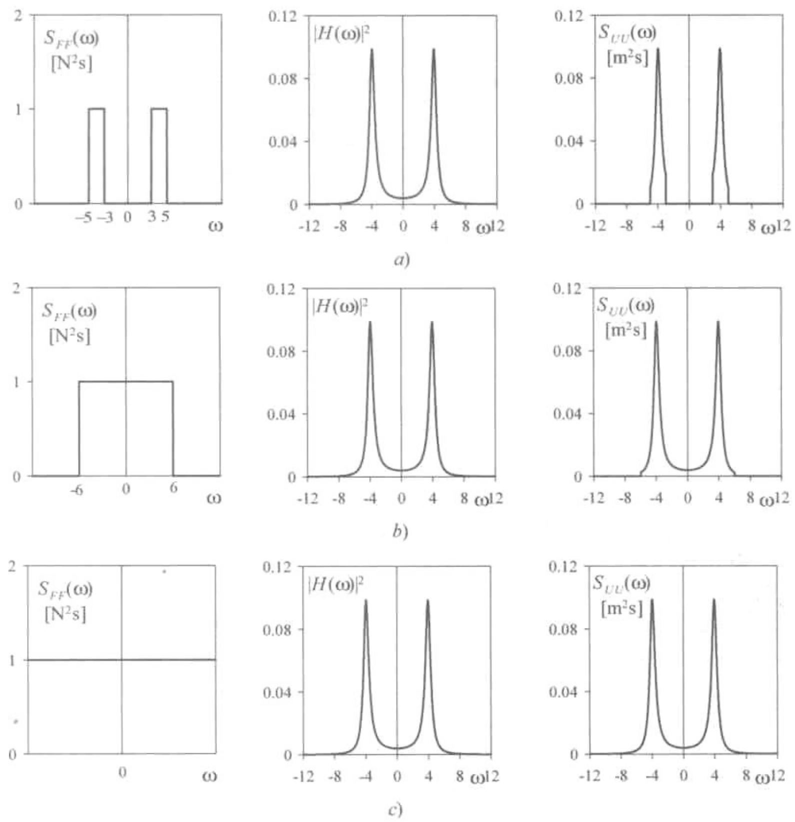


Fig. 5

Variances and spectral moments of the response

The knowledge of the auto-covariance function or the power spectral density of Q(t) allows the derivation of the variance σ_Q^2 through the relationships:

$$\boxed{\sigma_Q^2 = C_{QQ}(0) = E\left[\left\{Q(t) - \mu_Q\right\}^2\right] = \int_{-\infty}^{\infty} S_{QQ}(\omega)\, d\omega} \tag{21}$$

Moreover, due to the properties of the derivation of the stationary processes:

$$S_{Q\dot{Q}}(\omega) = i\omega S_{QQ}(\omega) \tag{22}$$

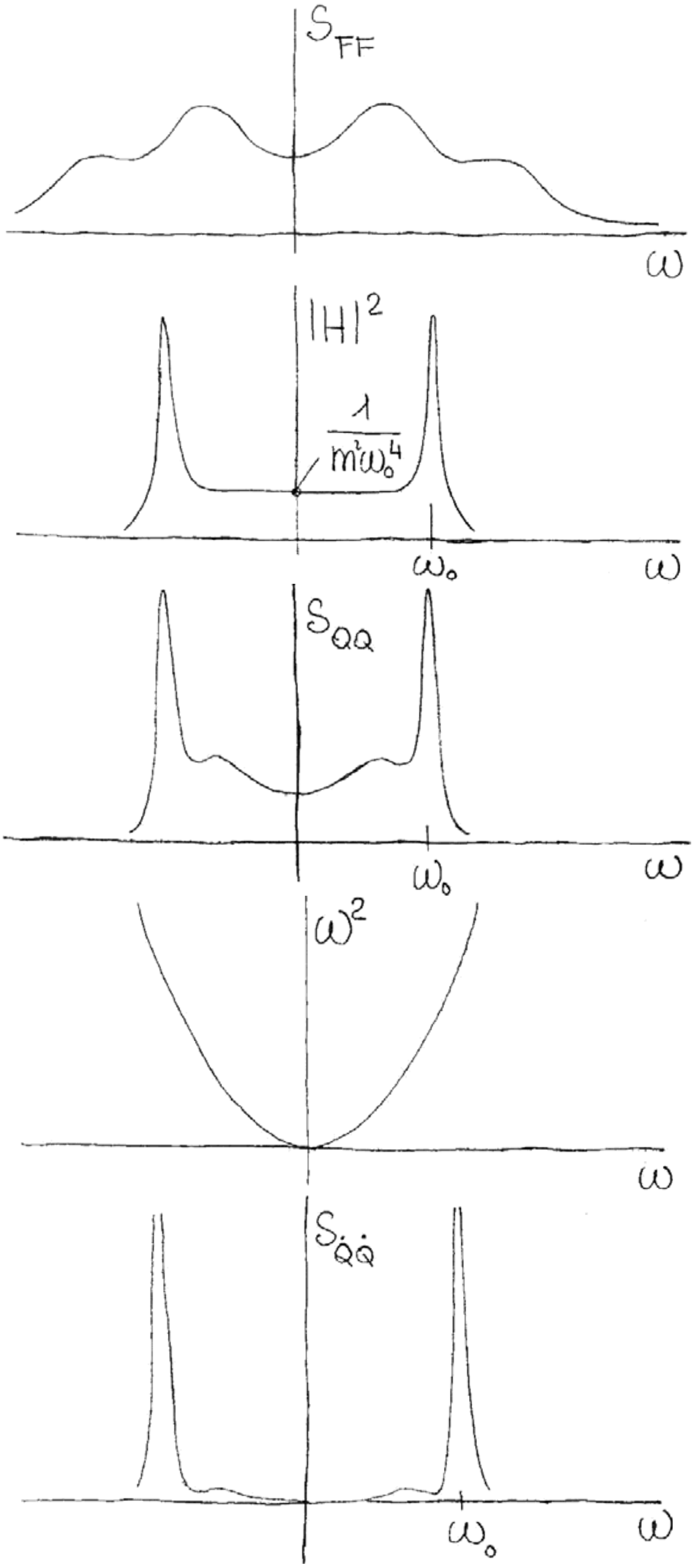
$$S_{\dot{Q}\dot{Q}}(\omega) = \omega^2 S_{QQ}(\omega) \tag{23}$$

Thus:

$$C_{Q\dot{Q}}(0) = E\left[Q(t)\dot{Q}(t)\right] = i\int_{-\infty}^{\infty} \omega S_{QQ}(\omega) d\omega = 0 \tag{24}$$

$$\sigma_{\dot{Q}}^2 = C_{\dot{Q}\dot{Q}}(0) = E\left[\dot{Q}^2(t)\right] = \int_{-\infty}^{\infty} \omega^2 S_{QQ}(\omega) d\omega \tag{25}$$

Fig. 6 summarises the main steps of the frequency domain analysis.



$$\sigma_F^2 = \int_{-\infty}^{\infty} S_{FF}(\omega)\,d\omega$$

$$\left|H(\omega)\right|^2 = \frac{1}{m^2\omega_0^4}\frac{1}{\left(1-\frac{\omega^2}{\omega_0^2}\right)+4\xi\frac{\omega^2}{\omega_0^2}}$$

$$S_{QQ}(\omega) = \left|H(\omega)\right|^2 S_{FF}(\omega)$$

$$\sigma_Q^2 = \int_{-\infty}^{\infty} S_{QQ}(\omega)\,d\omega$$

$$S_{\dot{Q}\dot{Q}}(\omega) = \omega^2 S_{QQ}(\omega)$$

$$\sigma_{\dot{Q}}^2 = \int_{-\infty}^{\infty} S_{\dot{Q}\dot{Q}}(\omega)\,d\omega$$

Fig. 6

Let us remember that the unilateral power spectral density $G_{QQ}(\omega)$ of Q(t) is given by:

$$G_{QQ}(\omega) = 2S_{QQ}(\omega) \quad \text{for} \quad 0 \le \omega < +\infty \tag{26a}$$

$$G_{QQ}(\omega) = 0 \quad \text{for} \quad \omega < 0 \tag{26b}$$

Based on this definition, the first three spectral moments of the response result:

$$\lambda_{Q,0} = \int_0^\infty G_{QQ}(\omega)d\omega = \int_{-\infty}^\infty S_{QQ}(\omega)d\omega = \sigma_Q^2 \tag{27}$$

$$\lambda_{Q,1} = \int_0^\infty \omega G_{QQ}(\omega)d\omega \ne \int_{-\infty}^\infty \omega S_{QQ}(\omega)d\omega = 0 \tag{28}$$

$$\lambda_{Q,2} = \int_0^\infty \omega^2 G_{QQ}(\omega)d\omega = \int_{-\infty}^\infty \omega^2 S_{QQ}(\omega)d\omega = \sigma_Q^2 \tag{29}$$

Distribution of the response

If the response process is stationary and normal, the knowledge of the mean value μ_Q and of the variance σ_Q^2 of Q(t) provides the density function of the first order:

$$p_Q(q) = \frac{1}{\sqrt{2\pi}\sigma_Q} \exp\left\{-\frac{(q-\mu_Q)^2}{2\sigma_Q^2}\right\} \tag{30}$$

The further knowledge of the normalised auto-covariance function $\rho_{QQ}(\tau)$ of Q(t) provides the joint density function of the second order:

$$p_Q(q_1, q_2; \tau) = \frac{1}{2\pi\sigma_Q^2\sqrt{1-\rho_{QQ}^2(\tau)}} \cdot \exp\left\{-\frac{(q_1-\mu_Q)^2 - 2\rho_{QQ}(\tau)(q_1-\mu_Q)(q_2-\mu_Q) + (q_2-\mu_Q)^2}{2\sigma_Q^2\left[1-\rho_{QQ}^2(\tau)\right]}\right\} \tag{31}$$

and of any other order n.

Dynamic response to a white process

Let us consider the case in which the loading process F(t) is a white noise W(t) (Fig. 7):

$$S_{FF}(\omega) = S_0 \tag{32}$$

$$C_{FF}(\tau) = 2\pi S_0 \delta(\tau) \tag{33}$$

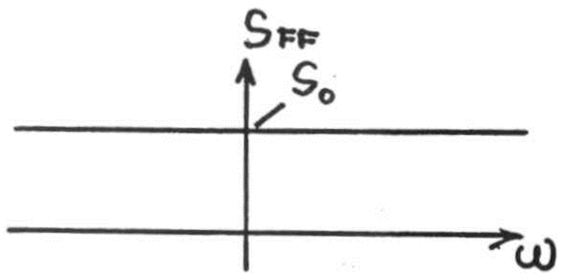


Fig. 7

Applying Eqs. (19) and (23), the power spectral density of the displacement and of its derivative are given by the relationships:

$$S_{QQ}(\omega) = S_0 \left|H(\omega)\right|^2 \tag{34}$$

$$S_{\dot{Q}\dot{Q}}(\omega) = \omega^2 S_{QQ}(\omega) = S_0 \omega^2 \left|H(\omega)\right|^2 \tag{35}$$

from which, applying Eqs. (21) and (25):

$$\sigma_Q^2 = \frac{S_0 \pi}{2m^2 \xi \omega_0^3} \tag{36}$$

$$\sigma_{\dot{Q}}^2 = \frac{S_0 \pi}{2m^2 \xi \omega_0} = \omega_0^2 \sigma_Q^2 \tag{37}$$

It can be demonstrated that, applying Eq. (17) to Eq. (34):

$$C_{QQ}(\tau) = \sigma_Q^2 \exp\left(-\xi\omega_0 |\tau|\right) \left[\cos\left(\omega_0 \sqrt{1-\xi^2}\, |\tau|\right) + \frac{\xi}{\sqrt{1-\xi^2}} \sin\left(\omega_0 \sqrt{1-\xi^2}\, |\tau|\right) \right] \tag{38}$$

$$C_{\dot{Q}\dot{Q}}(\tau) = -\frac{d^2 C_{QQ}(\tau)}{d\tau^2} = \sigma_{\dot{Q}}^2 \exp\left(-\xi\omega_0 |\tau|\right) \left[\cos\left(\omega_0 \sqrt{1-\xi^2}\, |\tau|\right) - \frac{\xi}{\sqrt{1-\xi^2}} \sin\left(\omega_0 \sqrt{1-\xi^2}\, |\tau|\right) \right] \tag{39}$$

It is worth noting that the white process is not realistic from a physical viewpoint since its variance is infinite. Nevertheless, applied as a loading process on a structure, the variance of its response has a finite value. Thus, it constitutes an excellent model for loadings with any uniform harmonic content extended beyond the fundamental circular frequency.

Fig. 8 shows some typical auto-covariance functions and power spectral density functions of Q(t) and of its first derivative with respect to time.

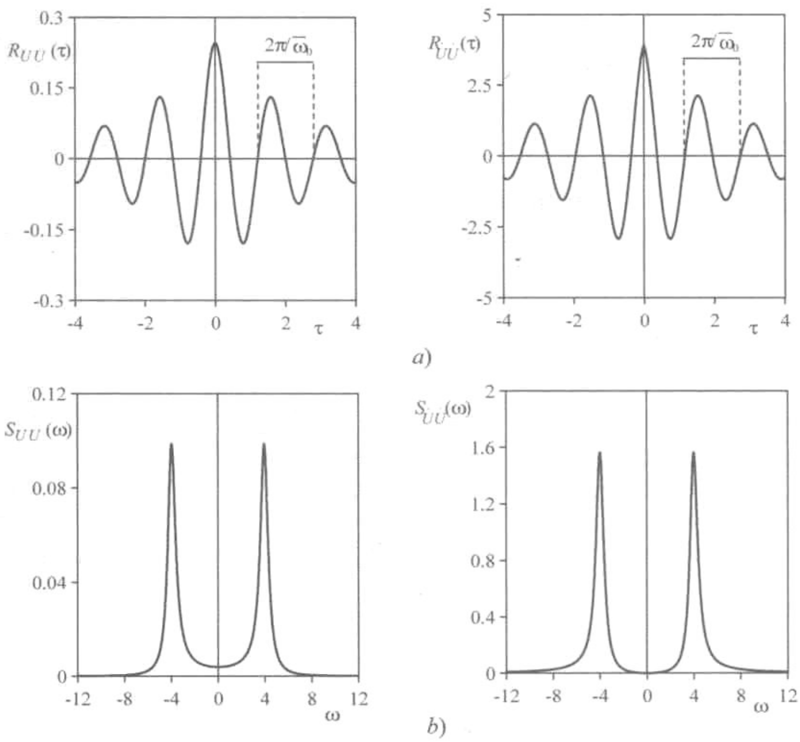


Fig. 8

Applying Eqs. (27)-(29), the spectral moments of the response result:

$$\lambda_{Q,0} = \sigma_Q^2 = \frac{S_0 \pi}{2m^2 \xi \omega_0^3} \tag{40}$$

$$\lambda_{Q,1} = \frac{S_0}{2m^2 \xi \omega_0^2 \sqrt{1-\xi^2}} \left[\pi - 2\mathrm{arctg}\left(\frac{\xi}{\sqrt{1-\xi^2}} \right) \right] \tag{41}$$

$$\lambda_{Q,2} = \sigma_{\dot{Q}}^2 = \frac{S_0 \pi}{2m^2 \xi \omega_0} \tag{42}$$

Consequently, the spectral bandwidth parameter is:

$$q_Q = \sqrt{1 - \frac{\lambda_{Q,1}^2}{\lambda_{Q,0}\lambda_{Q,2}}} = \left\{ 1 - \frac{1}{1-\xi^2} \left[1 - \frac{2}{\pi} \mathrm{arctg}\left(\frac{\xi}{\sqrt{1-\xi^2}} \right) \right]^2 \right\}^{1/2} \tag{43}$$

In the case $\xi \ll 1$, Eq. (43) simplifies and assumes the form:

$$q_Q \simeq 2\sqrt{\frac{\xi}{\pi}} \tag{44}$$