RANDOM DYNAMICS

Multi-Degree-Of-Freedom Systems

Equations of motion

The equation of motion $\mathbf{q}$(t) of a n-D.O.F. system subjected to a deterministic force $\mathbf{f}$(t) and to deterministic initial conditions $\mathbf{q}_0$ and $\dot{\mathbf{q}}_0$ is given by:

$$\mathbf{M}\ddot{\mathbf{q}}(t)+\mathbf{C}\dot{\mathbf{q}}(t)+\mathbf{K}\mathbf{q}(t)=\mathbf{f}(t) \tag{1a}$$

$$\mathbf{q}(0)=\mathbf{q}_0;\dot{\mathbf{q}}(0)=\dot{\mathbf{q}}_0 \tag{1b}$$

Let us assume that $\mathbf{f}$(t) is a generic sample vector of a n-variate loading process $\mathbf{F}$(t). Analogously, the initial conditions $\mathbf{q}_0$ and $\dot{\mathbf{q}}_0$ are random occurrences of a couple of random vectors $\mathbf{Q}_0$ and $\dot{\mathbf{Q}}_0$.
The equation of motion $\mathbf{q}$(t) is the sample vector of the n-variate random response process $\mathbf{Q}$(t) corresponding to $\mathbf{f}$(t).
In this case Eq. (1) is the deterministic relationship that expresses $\mathbf{q}$(t) as a function of $\mathbf{f}$(t), $\mathbf{q}_0$ and $\dot{\mathbf{q}}_0$. Analogously, the relationship that expresses $\mathbf{Q}$(t) as a function of $\mathbf{F}$(t), $\mathbf{Q}_0$ and $\dot{\mathbf{Q}}_0$ has the form:

$$\mathbf{M}\ddot{\mathbf{Q}}(t)+\mathbf{C}\dot{\mathbf{Q}}(t)+\mathbf{K}\mathbf{Q}(t)=\mathbf{F}(t) \tag{2a}$$

$$\mathbf{Q}(0)=\mathbf{Q}_0;\dot{\mathbf{Q}}(0)=\dot{\mathbf{Q}}_0 \tag{2b}$$

Analogously to the S.D.O.F. system, also the n-D.O.F. system retains deterministic properties.
Please also note that assuming the initial conditions as deterministic is equivalent to set:

$$P\left[\mathbf{Q}(t_0)=\mathbf{q}_0\cap\dot{\mathbf{Q}}(t_0)=\dot{\mathbf{q}}_0\right]=1$$

Under the hypothesis of quasi-steady vibrations (i.e. loosing the memory of the initial conditions) the deterministic response of the n-D.O.F. system (Eq. 1) is given by:

$$\mathbf{q}(t)=\int_0^t\mathbf{h}(t-\tau)\mathbf{f}(\tau)d\tau \tag{3}$$

where $\mathbf{h}(\bullet)$ is the impulse response matrix.
Analogously, the quasi-steady response process of a n-D.O.F. system (Eq. 2) excited by the loading process $\mathbf{F}$(t) is given by:

$$\mathbf{Q}(t)=\int_0^t\mathbf{h}(t-\tau)\mathbf{F}(\tau)d\tau \tag{4}$$

Assuming that the loading is a stationary process, and paraphrasing the considerations carried out for the S.D.O.F. system, Eq. (4) can be re-written as:

$$\mathbf{Q}(t)=\int_{-\infty}^{+\infty}\mathbf{h}(t-\tau)\mathbf{F}(\tau)d\tau \tag{5}$$

Mean value and fluctuation of the response

Let us express the loading process as:

$$\mathbf{F}(t) = \boldsymbol{\mu}_{\mathbf{F}} + \mathbf{F}'(t) \tag{6}$$

where $\boldsymbol{\mu}_{\mathbf{F}} = \mathrm{E}\left[\mathbf{F}(t)\right]$ is the mean value of the loading (independent of time since **F** is stationary) and $\mathbf{F}'(t) = \mathbf{F}(t) - \boldsymbol{\mu}_{\mathbf{F}}$. This is a random stationary process with zero mean, characterised by the covariance matrix $\mathbf{C}_{\mathbf{F}'}(\tau)$. Since $\mathbf{F}'(t)$ is a nil mean process, it coincides with the correlation matrix $\mathbf{R}_{\mathbf{F}'}(\tau)$ of $\mathbf{F}'(t)$ and with the covariance matrix $\mathbf{C}_{\mathbf{F}}(\tau)$ of **F**(t). In the fundamental case that **F**(t) is a normal process, $\boldsymbol{\mu}_{\mathbf{F}}$ and $\mathbf{C}_{\mathbf{F}}(\tau)$ provide a full probabilistic description of the process.

Let us consider Eq. (5) and let us apply the transformation $\vartheta = t - \tau$. It follows that:

$$\mathbf{Q}(t) = \int_{-\infty}^{\infty} \mathbf{h}(\vartheta)\mathbf{F}(t-\vartheta)\,d\vartheta \tag{7}$$

The mean value of **Q**(t) is given by the relationship:

$$\begin{aligned}\mathrm{E}\left[\mathbf{Q}(t)\right] &= \mathrm{E}\left[\int_{-\infty}^{\infty} \mathbf{h}(\vartheta)\mathbf{F}(t-\vartheta)\,d\vartheta\right] = \\ &= \int_{-\infty}^{\infty} \mathbf{h}(\vartheta)\mathrm{E}\left[\mathbf{F}(t-\vartheta)\right]d\vartheta \Rightarrow\end{aligned}$$

$$\boldsymbol{\mu}_{\mathbf{Q}} = \int_{-\infty}^{\infty} \mathbf{h}(\vartheta)\,d\vartheta \cdot \boldsymbol{\mu}_{\mathbf{F}} \tag{8}$$

Thus, likewise the mean of the loading, also the mean of the response is independent of time.
Let us consider the complex frequency response matrix of the structure:

$$\mathbf{H}(\omega) = \left(-\omega^2\mathbf{M} + i\omega\mathbf{C} + \mathbf{K}\right)^{-1} \tag{9}$$

and let us remember that it is the Fourier transform of the impulse response matrix:

$$\mathbf{H}(\omega) = \int_{-\infty}^{\infty} \mathbf{h}(t)e^{-i\omega t}dt \tag{10}$$

It follows that:

$$\mathbf{H}(0) = \mathbf{K}^{-1} = \int_{-\infty}^{\infty} \mathbf{h}(\vartheta)\,d\vartheta \tag{11}$$

Substituting Eq. (11) into Eq. (8) it results:

$$\boldsymbol{\mu}_{\mathbf{Q}} = \mathbf{K}^{-1}\boldsymbol{\mu}_{\mathbf{F}} \tag{12}$$

which points out that the mean value of the response is the deterministic static response to the mean value of the random loading process.

Let us remember that, if the structure has classical vibration modes, it results:

$$\mathbf{H}(\omega) = \boldsymbol{\Psi}\mathbf{H}_\mathbf{P}(\omega)\boldsymbol{\Psi}^T \tag{13}$$

where $\boldsymbol{\Psi}$ is the matrix of the eigenvectors and $\mathbf{H}_\mathbf{P}(\omega)$ is the complex frequency response matrix:

$$\mathbf{H}_\mathbf{P}(\omega) = \mathbf{diag}\left\{H_{P_1}(\omega), H_{P_2}(\omega), \ldots H_{P_n}(\omega)\right\} \tag{14}$$

where:

$$H_{P_k}(\omega) = \frac{1}{m_k\omega_k^2}\frac{1}{1-\dfrac{\omega^2}{\omega_k^2}+2i\xi_k\dfrac{\omega}{\omega_k}} \quad (k = 1,..n) \tag{15}$$

is the k-th complex modal frequency response function. Thus:

$$\mathbf{K}^{-1} = \mathbf{H}(0) = \boldsymbol{\Psi}\mathbf{H}_\mathbf{P}(0)\boldsymbol{\Psi}^T \tag{16}$$

where:

$$\mathbf{H}_\mathbf{P}(0) = \mathbf{diag}\left\{\frac{1}{m_1\omega_1^2}, \frac{1}{m_2\omega_2^2}, \ldots \frac{1}{m_n\omega_n^2}\right\} \tag{17}$$

The application of Eqs. (16) and (17) avoids the inversion of the stiffness matrix **K**.

Based on Eq. (12), let us assume:

$$\mathbf{Q}(t) = \boldsymbol{\mu}_\mathbf{Q} + \mathbf{Q}'(t) \tag{18}$$

where $\boldsymbol{\mu}_\mathbf{Q} = E[\mathbf{Q}(t)]$ is the mean value of the response and $\mathbf{Q}'(t)$ is the fluctuating part of the response. Substituting Eq. (18) and Eq. (6) into Eq. (2a) it follows:

$$\cancel{\mathbf{M}\ddot{\boldsymbol{\mu}}_\mathbf{Q}} + \mathbf{M}\ddot{\mathbf{Q}}'(t) + \cancel{\mathbf{C}\dot{\boldsymbol{\mu}}_\mathbf{Q}} + \mathbf{C}\dot{\mathbf{Q}}'(t) + \mathbf{K}\boldsymbol{\mu}_\mathbf{Q} + \mathbf{K}\mathbf{Q}'(t) = \boldsymbol{\mu}_\mathbf{F} + \mathbf{F}'(t)$$

from which, applying Eq. (12):

$$\mathbf{M}\ddot{\mathbf{Q}}'(t) + \mathbf{C}\dot{\mathbf{Q}}'(t) + \mathbf{K}\mathbf{Q}'(t) = \mathbf{F}'(t) \tag{19}$$

Thus, the fluctuating part of the response is the dynamic response of the structure to the fluctuating part of the loading. Also in this case it is apparent the advantage of separating the initial problem into two component problems: a static problem (Eq. 12) and a dynamic problem (Eq. 19). Due to Eq. (7) the solution of the second problem is given by:

$$\mathbf{Q}'(t) = \int_{-\infty}^{\infty}\mathbf{h}(\vartheta)\mathbf{F}'(t-\vartheta)d\vartheta \tag{20}$$

Covariance matrix of the response

The covariance matrix of the response coincides with the correlation matrix of the fluctuating part of the response. It is given by:

$$\mathbf{C_Q}(\tau) = \begin{bmatrix} C_{Q_1Q_1}(\tau) & C_{Q_1Q_2}(\tau) & \cdots & C_{Q_1Q_n}(\tau) \\ C_{Q_2Q_1}(\tau) & C_{Q_2Q_2}(\tau) & \cdots & C_{Q_2Q_n}(\tau) \\ \vdots & \vdots & & \vdots \\ C_{Q_nQ_1}(\tau) & C_{Q_nQ_2}(\tau) & \cdots & C_{Q_nQ_n}(\tau) \end{bmatrix}$$

The on-diagonal terms express the auto-covariance functions of the displacements of each D.O.F.; the off-diagonal terms express the cross-covariance functions of the displacement related to different D.O.F.s. The generic term of the matrix is:

$$C_{Q_iQ_j}(t, t+\tau) = E\left[\left\{Q_i(t) - \mu_{Q_i}\right\}\left\{Q_j(t+\tau) - \mu_{Q_j}\right\}\right] = E\left[Q_i'(t)Q_j'(t+\tau)\right]$$

Thus, the covariance matrix results:

$$\mathbf{C_Q}(\tau) = E\left[\mathbf{Q}'(t)\mathbf{Q}'^T(t+\tau)\right] \tag{21}$$

Applying Eq. (20) it follows:

$$\begin{aligned} \mathbf{C_Q}(\tau) &= E\left[\int_{-\infty}^{\infty} \mathbf{h}(\vartheta_1)\mathbf{F}'(t-\vartheta_1)d\vartheta_1 \int_{-\infty}^{\infty} \mathbf{F}'^T(t+\tau-\vartheta_2)\mathbf{h}^T(\vartheta_2)d\vartheta_2\right] = \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbf{h}(\vartheta_1) E\left[\mathbf{F}'(t-\vartheta_1)\mathbf{F}'^T(t+\tau-\vartheta_2)\right]\mathbf{h}^T(\vartheta_2)d\vartheta_1 d\vartheta_2 \end{aligned}$$

where $E\left[\mathbf{F}'(t-\vartheta_1)\mathbf{F}'^T(t-\vartheta_2)\right] = \mathbf{C_F}(\tau - \vartheta_2 + \vartheta_1)$ is the covariance matrix of the loading:

$$\mathbf{C_F}(\tau) = \begin{bmatrix} C_{F_1F_1}(\tau) & C_{F_1F_2}(\tau) & \cdots & C_{F_1F_n}(\tau) \\ C_{F_2F_1}(\tau) & C_{F_2F_2}(\tau) & \cdots & C_{F_2F_n}(\tau) \\ \vdots & \vdots & & \vdots \\ C_{F_nF_1}(\tau) & C_{F_nF_2}(\tau) & \cdots & C_{F_nF_n}(\tau) \end{bmatrix}$$

Thus:

$$\mathbf{C_Q}(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbf{h}(\vartheta_1)\mathbf{C_F}(\tau - \vartheta_2 + \vartheta_1)\mathbf{h}^T(\vartheta_2)d\vartheta_1 d\vartheta_2 \tag{22}$$

Eq. (22) points out that, likewise the covariance matrix of the loading, also the covariance matrix of the response depends on only the time lag τ. Thus, likewise the loading process, also the response process is (weakly) stationary. Moreover, if the loading process is normal and the structure is linear, also the response is normal. In this case $\boldsymbol{\mu}_\mathbf{Q}$ and $\mathbf{C_Q}(\tau)$ provide a full probabilistic representation of the response.

Finally, let us remember that, if the structure has classical normal modes, then:

$$\mathbf{h}(t) = \boldsymbol{\Psi}\mathbf{h}_\mathbf{P}(t)\boldsymbol{\Psi}^T \tag{23}$$

where $\mathbf{h}_\mathbf{P}(t)$ is the modal impulse response matrix:

$$\mathbf{h}_\mathbf{P}(t) = \mathbf{diag}\left\{h_{P_1}(t), h_{P_2}(t), \ldots h_{P_n}(t)\right\} \tag{24}$$

and $h_{P_k}(t)$ is the k-th modal impulse response function:

$$h_{p_k}(t) = e^{-\xi_k\omega_k t}\frac{1}{m_k\omega_k\sqrt{1-\xi_k^2}}\sin\omega_k\sqrt{1-\xi_k^2}\,t \quad (k=1,..n) \tag{25}$$

Substituting Eq. (23) into Eq. (22):

$$\mathbf{C}_\mathbf{Q}(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\underbrace{\boldsymbol{\Psi}\mathbf{h}_\mathbf{P}(\vartheta_1)\underbrace{\boldsymbol{\Psi}^T\mathbf{C}_\mathbf{F}(\tau-\vartheta_2+\vartheta_1)\boldsymbol{\Psi}}_{\mathbf{C}_{\mathbf{F}_\mathbf{P}}(\tau-\vartheta_2+\vartheta_1)}\mathbf{h}_\mathbf{P}^T(\vartheta_2)}_{\mathbf{C}_\mathbf{P}(\tau)}\boldsymbol{\Psi}^T d\vartheta_1 d\vartheta_2$$

where $\mathbf{C}_{\mathbf{F}_\mathbf{P}}$ is the modal covariance matrix of the loading and $\mathbf{C}_\mathbf{P}$ is the modal covariance matrix of the response.

Power spectral density matrix of the response

The spectral density matrix of the response has the form:

$$\mathbf{S}_\mathbf{Q}(\omega) = \begin{bmatrix} S_{Q_1Q_1}(\omega) & S_{Q_1Q_2}(\omega) & \cdots & S_{Q_1Q_n}(\omega) \\ S_{Q_2Q_1}(\omega) & S_{Q_2Q_2}(\omega) & \cdots & S_{Q_2Q_n}(\omega) \\ \vdots & \vdots & & \vdots \\ S_{Q_nQ_1}(\omega) & S_{Q_nQ_2}(\omega) & \cdots & S_{Q_nQ_n}(\omega) \end{bmatrix}$$

The on-diagonal terms express the power spectral densities of each D.O.F.; the off-diagonal terms express the cross-power spectral densities (usually complex) of the displacements corresponding to different D.O.F.s. Applying the Wiener-Khintchine equations, the generic term of the matrix is:

$$S_{Q_iQ_j}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} C_{Q_iQ_j}(\tau)e^{-i\omega\tau}d\tau$$

Thus, the power spectral density matrix of the response results:

$$\mathbf{S}_\mathbf{Q}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathbf{C}_\mathbf{Q}(\tau)e^{-i\omega\tau}d\tau \tag{26}$$

Substituting Eq. (20) into Eq. (24) and multiplying the integrand by $e^{i\omega\vartheta_1}e^{-i\omega\vartheta_2}e^{-i\omega(\vartheta_1-\vartheta_2)}=1$:

$$\begin{aligned}\mathbf{S_Q}(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathbf{h}(\vartheta_1)\mathbf{C_F}(\tau-\vartheta_2+\vartheta_1)\mathbf{h}^{\mathrm{T}}(\vartheta_2)e^{-i\omega\tau}e^{i\omega\vartheta_1}e^{-i\omega\vartheta_2}e^{-i\omega(\vartheta_1-\vartheta_2)}d\vartheta_1 d\vartheta_2 d\tau = \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathbf{h}(\vartheta_1)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathbf{C_F}(\tau-\vartheta_2+\vartheta_1)e^{-i\omega(\tau-\vartheta_2+\vartheta_1)}d\tau\right]\mathbf{h}^{\mathrm{T}}(\vartheta_2)e^{i\omega\vartheta_1}e^{-i\omega\vartheta_2}d\vartheta_1 d\vartheta_2\end{aligned}$$

$\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathbf{C_F}(\tau-\vartheta_2+\vartheta_1)e^{-i\omega(\tau-\vartheta_2+\vartheta_1)}d\tau = \mathbf{S_F}(\omega)$ being the power spectral density matrix of the loading:

$$\mathbf{S_F}(\omega)=\begin{bmatrix} S_{F_1F_1}(\omega) & S_{F_1F_2}(\omega) & \cdots & S_{F_1F_n}(\omega) \\ S_{F_2F_1}(\omega) & S_{F_2F_2}(\omega) & \cdots & S_{F_2F_n}(\omega) \\ \vdots & \vdots & & \vdots \\ S_{F_nF_1}(\omega) & S_{F_nF_2}(\omega) & \cdots & S_{F_nF_n}(\omega)\end{bmatrix}$$

Thus it follows:

$$\mathbf{S_Q}(\omega)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathbf{h}(\vartheta_1)\mathbf{S_F}(\omega)\mathbf{h}^{\mathrm{T}}(\vartheta_2)e^{i\omega\vartheta_1}e^{-i\omega\vartheta_2}e^{-i\omega(\vartheta_1-\vartheta_2)}d\vartheta_1 d\vartheta_2$$

where $\int_{-\infty}^{\infty}\mathbf{h}(\vartheta_1)e^{i\omega\vartheta_1}d\vartheta_1 = \mathbf{H}^*(\omega)$; $\int_{-\infty}^{\infty}\mathbf{h}^{\mathrm{T}}(\vartheta_2)e^{-i\omega\vartheta_2}d\vartheta_2 = \mathbf{H}^{\mathrm{T}}(\omega)$.

Therefore:

$$\mathbf{S_Q}(\omega)=\mathbf{H}^*(\omega)\mathbf{S_F}(\omega)\mathbf{H}^{\mathrm{T}}(\omega) \tag{27}$$

Let us remember that, if the structure has classical normal modes, then $\mathbf{H}(\omega)=\mathbf{\Psi}\mathbf{H_P}(\omega)\mathbf{\Psi}^{\mathrm{T}}$ (Eqs. 13-15). Substituting this expression into Eq. (27):

$$\mathbf{S_Q}(\omega)=\mathbf{\Psi}\underbrace{\mathbf{H_P^*}\underbrace{\mathbf{\Psi}^{\mathrm{T}}\mathbf{S_F}(\omega)\mathbf{\Psi}}_{\mathbf{S_{F_P}}(\omega)}\mathbf{H_P}(\omega)}_{\mathbf{S_P}(\omega)}\mathbf{\Psi}^{\mathrm{T}} \Rightarrow$$

where $\mathbf{S_{F_P}}(\omega)$ is the power spectral density matrix of the modal loading and $\mathbf{S_P}(\omega)$ is the power spectral density matrix of the modal response:

$$\mathbf{S_Q}(\omega)=\mathbf{\Psi}\mathbf{S_P}(\omega)\mathbf{\Psi}^{\mathrm{T}} \tag{28}$$

$$\mathbf{S_P}(\omega)=\mathbf{H_P^*}(\omega)\mathbf{S_{F_P}}(\omega)\mathbf{H_P}(\omega) \tag{29}$$

$$\mathbf{S_{F_P}}(\omega)=\mathbf{\Psi}^{\mathrm{T}}\mathbf{S_F}(\omega)\mathbf{\Psi} \tag{30}$$

Fig. 1 shows an auto-covariance function and the corresponding power spectral density function of the displacement. It also shows the cross-covariance function and the corresponding cross-power spectral density function of the displacement and its derivative.

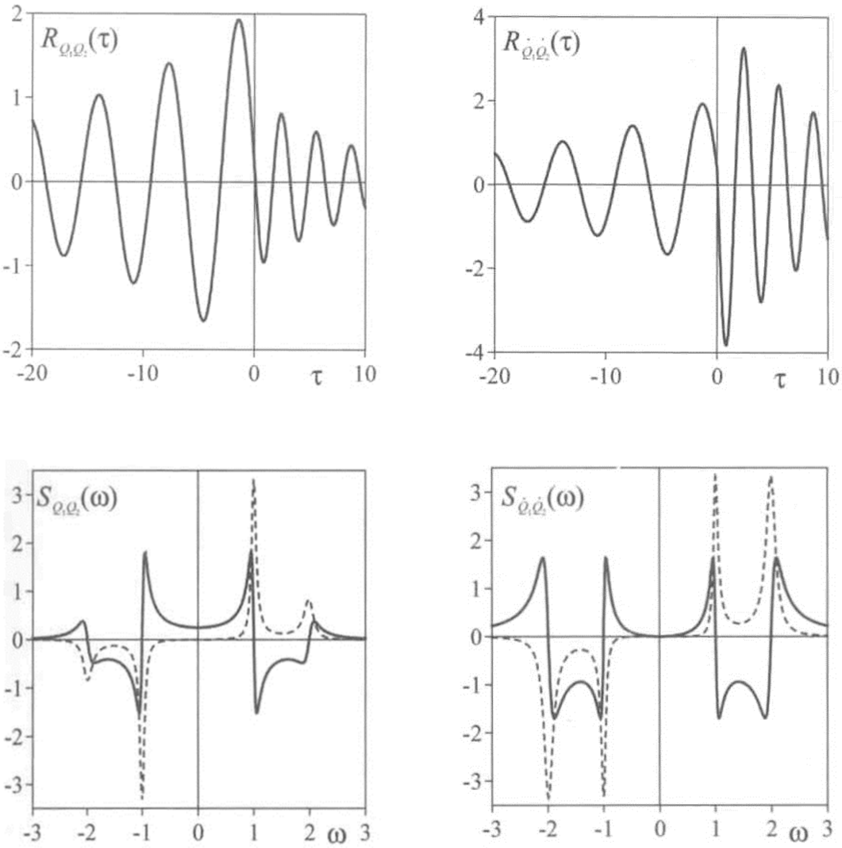


Fig. 1

Covariance matrix of the response at τ = 0

The knowledge of the covariance matrix and of the power spectral density matrix of **Q**(t) allows one to obtain the covariance matrix of the response at τ = 0:

$$\boldsymbol{\Sigma}_{\mathbf{Q}} = \mathbf{C}_{\mathbf{Q}}(0) = \int_{-\infty}^{\infty} \mathbf{S}_{\mathbf{Q}}(\omega)\,d\omega \tag{31}$$

The on-diagonal terms of $\boldsymbol{\Sigma}_{\mathbf{Q}}$ are the variances of the displacements of each D.O.F.; the off-diagonal terms are the covariances (for τ = 0) of the displacements related to different D.O.F.s:

$$\boldsymbol{\Sigma}_{\mathbf{Q}} = \begin{bmatrix} \sigma^2_{Q_1} & C_{Q_1Q_2}(0) & \cdots & C_{Q_1Q_n}(0) \\ C_{Q_2Q_1}(0) & \sigma^2_{Q_2} & \cdots & C_{Q_2Q_n}(0) \\ \vdots & \vdots & & \vdots \\ C_{Q_nQ_1}(0) & C_{Q_nQ_2}(0) & \cdots & \sigma^2_{Q_n} \end{bmatrix}$$

It is immediate to verify that, if the structure has classical normal modes, then:

$$\boldsymbol{\Sigma}_{\mathbf{Q}} = \boldsymbol{\Psi}\boldsymbol{\Sigma}_{\mathbf{P}}\boldsymbol{\Psi}^{T} \tag{32}$$

where $\boldsymbol{\Sigma}_{\mathbf{P}}$ is the covariance matrix of the modal response for τ = 0:

$$\boldsymbol{\Sigma}_{\mathbf{P}} = \begin{bmatrix} \sigma_{P_1}^2 & C_{P_1P_2}(0) & \cdots & C_{P_1P_n}(0) \\ C_{P_2P_1}(0) & \sigma_{P_2}^2 & \cdots & C_{P_2P_n}(0) \\ \vdots & \vdots & & \vdots \\ C_{P_nP_1}(0) & C_{P_nP_2}(0) & \cdots & \sigma_{P_n}^2 \end{bmatrix}$$

The on-diagonal terms of $\boldsymbol{\Sigma}_{\mathbf{P}}$ express the variances of the modal displacements associated with each mode of vibration; the off-diagonal terms are the covariances (for τ = 0) of the modal displacements associated to different modes of vibration.

Distribution of the response

If the n-variate response process is normal, the mean vector $\boldsymbol{\mu}_{\mathbf{Q}}$ and the covariance matrix $\mathbf{C}_{\mathbf{Q}}(\tau)$ provide a full information to derive the joint density functions of any order n.
In particular, the joint density function of order n at a fixed time t is given by:

$$p_{\mathbf{Q}}(\mathbf{q};0) = \frac{1}{(2\pi)^{n/2}\left|\Sigma_{\mathbf{Q}}\right|^{1/2}} \exp\left\{-\frac{1}{2}\left(\mathbf{q}-\boldsymbol{\mu}_{\mathbf{Q}}\right)^T \boldsymbol{\Sigma}_{\mathbf{Q}}^{-1}\left(\mathbf{q}-\boldsymbol{\mu}_{\mathbf{Q}}\right)\right\}$$

Correlation of the principal coordinates

If the natural frequencies are well separated and the normal modes are low damped, it can be shown that the principal coordinates of the response are, with good approximation, un-correlated. Thus:

$$\mathbf{S}_{\mathbf{P}}(\omega) \simeq \begin{bmatrix} S_{P_1P_1}(\omega) & 0 & \cdots & 0 \\ 0 & S_{P_2P_2}(\omega) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & S_{P_nP_n}(\omega) \end{bmatrix} \tag{34}$$

$$\boldsymbol{\Sigma}_{\mathbf{P}} \simeq \begin{bmatrix} \sigma_{P_1}^2 & 0 & \cdots & 0 \\ 0 & \sigma_{P_2}^2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \sigma_{P_n}^2 \end{bmatrix} \tag{35}$$

In other words, not only the principal transformation decouples the equations of motion:

$$\mathbf{M}\ddot{\mathbf{Q}}(t) + \mathbf{C}\dot{\mathbf{Q}}(t) + \mathbf{K}\mathbf{Q}(t) = \mathbf{F}(t) \tag{36}$$

$$\mathbf{Q}(t) = \boldsymbol{\Psi}\mathbf{P}(t) \tag{37}$$

$$\ddot{P}_k(t) + 2\xi_k\omega_k\dot{P}_k(t) + \omega_k^2 P_k(t) = \frac{1}{m_k}\psi_k^T \mathbf{F}(t) \quad (k = 1, 2, ..n) \tag{38}$$

but the principal coordinates are also identically uncorrelated. Thus, their statistical properties can be derived, separately, using the principles of the random dynamics of the S.D.O.F. systems:

$$S_{P_kP_k}(\omega) = \left|H_{P_k}(\omega)\right|^2 S_{F_{P_k}F_{P_k}}(\omega) \tag{39}$$

$$S_{F_{P_k}F_{P_k}}(\omega) = \boldsymbol{\psi}_k^T \mathbf{S}_\mathbf{F}(\omega)\boldsymbol{\psi}_k \tag{40}$$

$$\sigma_{P_k}^2 = \int_{-\infty}^{\infty} S_{P_kP_k}(\omega)\,d\omega \tag{41}$$

Substituting Eqs. (34) and (35) into Eqs. (27) and (31) it follows:

$$S_{Q_iQ_j}(\omega) \cong \sum_1^n {}_k\psi_{ki}\psi_{kj} S_{P_kP_k}(\omega) \tag{42}$$

$$C_{Q_iQ_j}(0) \cong \sum_1^n {}_k\psi_{kj}\psi_{kj}\sigma_{P_k}^2 \tag{43}$$

In particular, since Eq. (39) is real, also $S_{Q_iQ_j}(\omega)$ is real. Thus, the un-correlation of the principal coordinates makes real the cross-power spectral densities of the response.

Example. Consider a 2 D.O.F. system whose motion is described by Eq. (36), being:

$$\mathbf{M} = m\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \mathbf{K} = k\begin{bmatrix} 1+\varepsilon & -\varepsilon \\ -\varepsilon & 1+\varepsilon \end{bmatrix}; \mathbf{C} = c\begin{bmatrix} 1+\varepsilon & -\varepsilon \\ -\varepsilon & 1+\varepsilon \end{bmatrix}$$
$$\boldsymbol{\mu}_\mathbf{F} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}; \mathbf{S}_\mathbf{F}(\omega) = S_0\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

The eigenvalues and the eigenvectors are given by:

$$\left(\mathbf{K} - \omega^2\mathbf{M}\right)\psi = \mathbf{0}$$

whose solution involves:

$$\det\left(\mathbf{K} - \omega^2\mathbf{M}\right) = 0 \Rightarrow$$

$$\det\begin{bmatrix} k(1+\varepsilon)-\omega^2 m & -k\varepsilon \\ -k\varepsilon & k(1+\varepsilon)-\omega^2 m \end{bmatrix} = 0 \Rightarrow \omega_1 = \sqrt{k/m}\ ;\ \omega_2 = \sqrt{(1+2\varepsilon)k/m}$$

$$\omega_0 = \sqrt{k/m} \Rightarrow \omega_1 = \omega_0\ ;\ \omega_2 = \omega_0\sqrt{(1+2\varepsilon)} \Rightarrow$$

$$\mathbf{\Omega} = \omega_0^2\begin{bmatrix} 1 & 0 \\ 0 & 1+2\varepsilon \end{bmatrix};\quad \boldsymbol{\psi}_1 = \lambda_1\begin{Bmatrix} 1 \\ 1 \end{Bmatrix};\ \boldsymbol{\psi}_2 = \lambda_2\begin{Bmatrix} 1 \\ -1 \end{Bmatrix} \Rightarrow \mathbf{\Psi} = \begin{bmatrix} \lambda_1 & \lambda_2 \\ \lambda_1 & -\lambda_2 \end{bmatrix}$$

$$\mathbf{L} = \mathbf{\Psi}^T\mathbf{M}\mathbf{\Psi} = \mathbf{I} \Rightarrow \mathbf{\Psi} = \frac{1}{\sqrt{2m}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Let us apply the principal transformation law (Eq. 37). It decouples the equations of motion as::

$$\mathbf{Q}(t) = \mathbf{\Psi}\mathbf{P}(t) \Rightarrow$$
$$\mathbf{M}\mathbf{\Psi}\ddot{\mathbf{P}}(t) + \mathbf{C}\mathbf{\Psi}\dot{\mathbf{P}}(t) + \mathbf{K}\mathbf{\Psi}\mathbf{P}(t) = \mathbf{F}(t) \Rightarrow$$
$$\ddot{\mathbf{P}}(t) + \mathbf{\Gamma}\dot{\mathbf{P}}(t) + \mathbf{\Omega}\mathbf{P}(t) = \mathbf{\Psi}^T\mathbf{F}(t)$$

where $\mathbf{\Gamma}$ is diagonal since $\mathbf{C} = (c/k)\,\mathbf{K}$. In particular:

$$\mathbf{\Gamma} = \mathbf{\Psi}^T\mathbf{C}\mathbf{\Psi} = \frac{c}{m}\begin{bmatrix} 1 & 0 \\ 0 & 1+2\varepsilon \end{bmatrix} = \begin{bmatrix} 2\xi_1\omega_1 & 0 \\ 0 & 2\xi_2\omega_2 \end{bmatrix} \Rightarrow$$

$$\xi_1 = \frac{1}{2\omega_1}\frac{c}{m} = \frac{1}{2\omega_0}\frac{c}{m}$$

$$\xi_2 = \frac{1}{2\omega_2}\frac{c}{m}(1+2\varepsilon) = \frac{1}{2\omega_0\sqrt{1+2\varepsilon}}\frac{c}{m}(1+2\varepsilon) = \frac{1}{2\omega_0}\frac{c}{m}\sqrt{1+2\varepsilon}$$

$$\xi_0 = \frac{1}{2\omega_0}\frac{c}{m} = \frac{c}{2\sqrt{km}} \Rightarrow \xi_1 = \xi_0\ ;\ \xi_2 = \xi_0\sqrt{1+2\varepsilon}$$

The power spectral density matrix of the modal forces results (Eq. 28):

$$\mathbf{S}_{\mathbf{F_P}}(\omega) = \mathbf{\Psi}^T\mathbf{S}_{\mathbf{F}}(\omega)\mathbf{\Psi} \Rightarrow$$
$$\mathbf{S}_{\mathbf{F_P}}(\omega) = \frac{S_0}{2m}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

The power spectral density matrix of the principal coordinates results (Eq. 29):

$$\mathbf{S}_{\mathbf{P}}(\omega) = \mathbf{H}_{\mathbf{P}}^*(\omega)\mathbf{S}_{\mathbf{F_P}}(\omega)\mathbf{H}_{\mathbf{P}}(\omega) \Rightarrow$$
$$\mathbf{S}_{\mathbf{P}}(\omega) = \frac{S_0}{2m}\begin{bmatrix} |H_{P_1}(\omega)|^2 & H_{P_1}^*(\omega)H_{P_2}(\omega) \\ H_{P_1}(\omega)H_{P_2}^*(\omega) & |H_{P_2}(\omega)|^2 \end{bmatrix}$$

The covariance matrix of the modal response for $\tau = 0$ has the form:

$$\mathbf{\Sigma}_{\mathbf{P}} = \int_{-\infty}^{\infty}\mathbf{S}_{\mathbf{P}}(\omega)\,d\omega = \begin{bmatrix} \sigma_{P_1}^2 & C_{P_1P_2}(0) \\ C_{P_2P_1}(0) & \sigma_{P_2}^2 \end{bmatrix}$$

Assuming that the principal coordinates associated with different vibration modes are un-correlated:

$$\mathbf{S_P}(\omega) \simeq \frac{S_0}{2m}\begin{bmatrix} \left|H_{P_1}(\omega)\right|^2 & 0 \\ 0 & \left|H_{P_2}(\omega)\right|^2 \end{bmatrix}$$

$$\mathbf{\Sigma_P} \simeq \begin{bmatrix} \sigma_{P_1}^2 & 0 \\ 0 & \sigma_{P_2}^2 \end{bmatrix}$$

where $\sigma_{P_i}^2 = \int_{-\infty}^{\infty} S_{P_iP_i}(\omega)\,d\omega = \frac{S_0}{2m}\int_{-\infty}^{\infty}\left|H_{P_i}(\omega)\right|^2 d\omega = \frac{\pi S_0}{4m\xi_i\omega_i^3}$

Finally: $\mathbf{\Sigma_Q} = \mathbf{\Psi}\mathbf{\Sigma_P}\mathbf{\Psi}^T$

Using the approximate solution neglecting the correlation between different principal coordinates:

$$\sigma_{Q_1}^2 = \frac{\pi S_0}{4kc}$$
$$\sigma_{Q_2}^2 = \frac{\pi S_0}{4kc}$$

Using the rigorous solution:

$$\sigma_{Q_1}^2 = \frac{\pi S_0}{4kc}\left[1 + \frac{1}{(1+2\varepsilon)^2} + \frac{2\left(c^2/km\right)}{\varepsilon^2/(1+\varepsilon) + (1+2\varepsilon)\left(c^2/km\right)}\right]$$
$$\sigma_{Q_2}^2 = \frac{\pi S_0}{4kc}\left[1 + \frac{1}{(1+2\varepsilon)^2} - \frac{2\left(c^2/km\right)}{\varepsilon^2/(1+\varepsilon) + (1+2\varepsilon)\left(c^2/km\right)}\right]$$

Fig. 2 shows the percent errors $\eta_i = \left[\left(\sigma_{Q_i}^2\right)_{rig} - \left(\sigma_{Q_i}^2\right)_{ap}\right] / \left(\sigma_{Q_i}^2\right)_{rig} \times 100\%$, with i =1,2.

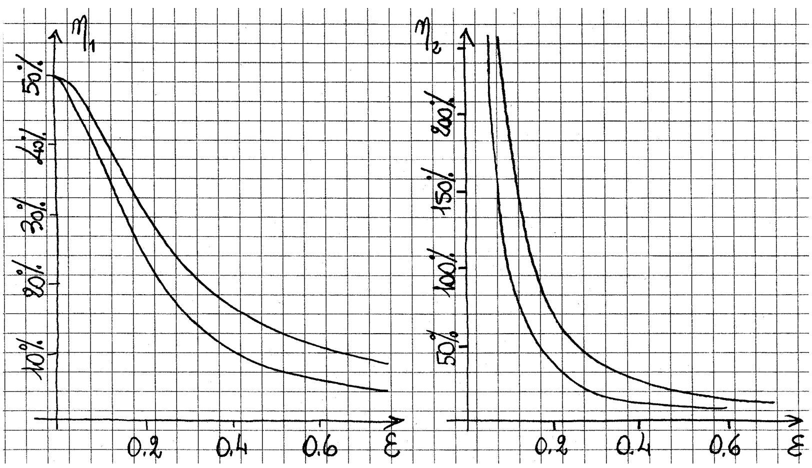
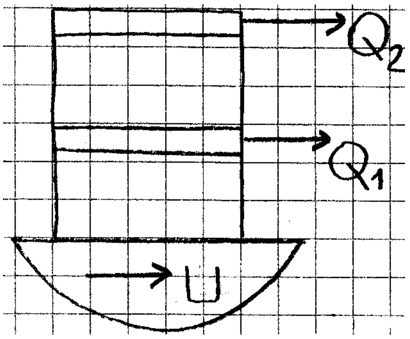


Fig. 2

Example. Consider the seismic response of a r.c. two-storey shear-type building.
The seismic acceleration is dealt with as stationary and schematised by a white noise (T = 10 s).



$$\mathbf{M}\ddot{\mathbf{Q}}(t)+\mathbf{C}\dot{\mathbf{Q}}(t)+\mathbf{K}\mathbf{Q}(t)=\mathbf{M}\mathbf{r}\ddot{U}(t)$$

$$\mathbf{Q}(t)=\begin{Bmatrix} Q_1(t) \\ Q_2(t) \end{Bmatrix};\ \mathbf{r}=\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$\mathbf{M}=\begin{bmatrix} 271.200 & 0 \\ 0 & 146.325 \end{bmatrix}(\mathrm{kg});\ \mathbf{K}=\begin{bmatrix} 1.694\times10^8 & -0.758\times10^8 \\ -0.758\times10^8 & 0.758\times10^8 \end{bmatrix}(\mathrm{N/m})$$

$$\mathbf{\Omega}=\begin{bmatrix} 187.063 & 0 \\ 0 & 956.056 \end{bmatrix};\ \mathbf{\Psi}=\begin{bmatrix} 1.260 & -1.448 \\ 1.972 & 1.716 \end{bmatrix}\times10^{-3}\ (\mathbf{L}=\mathbf{I})$$

$$\omega_1=13.677\,\mathrm{rad/s};\ \omega_2=30.920\,\mathrm{rad/s}$$

Principal transformation: $\mathbf{Q}(t)=\mathbf{\Psi}\mathbf{P}(t)\Rightarrow$

$$\ddot{P}_1(t)+2\xi_1\omega_1\dot{P}_1(t)+\omega_1^2 P_1(t)=-g_1\ddot{U}(t)$$

$$\ddot{P}_2(t)+2\xi_2\omega_2 P_2(t)+\omega_2^2 P_2(t)=-g_2\ddot{U}(t)$$

$$\xi_1=\xi_2=0.05;\ g_1=630.265,\ g_2=141.604$$

$$S_{P_kP_k}(\omega)=\left|H_{P_k}(\omega)\right|^2 g_k^2 S_{\ddot{U}\ddot{U}}(\omega)$$

$$H_{P_k}(\omega)=\frac{1}{\omega^2+2i\xi_k\omega\omega_k+\omega_k^2}$$

$$S_{\ddot{U}\ddot{U}}(\omega)=S_0;\ S_0=0.0217$$

$$\sigma_{P_k}^2=\int_{-\infty}^{\infty}S_{P_kP_k}(\omega)d\omega;\ \sigma_{\dot{P}_k}^2=\int_{-\infty}^{\infty}\omega^2 S_{P_kP_k}(\omega)d\omega$$

$$\sigma_{P_i}^2=\frac{g_k^2 S_0\pi}{2\xi_k\omega_k^3};\ \sigma_{\dot{P}_{ki}}^2=\frac{g_k^2 S_0\pi}{2\xi_k\omega_k}$$

$$\sigma_{P_1}^2=105.848,\quad \sigma_{P_2}^2=0.462$$

$$\sigma_{\dot{P}_1}^2=19800.280,\quad \sigma_{\dot{P}_2}^2=442.105$$

$$\sigma_{P_1}=10.288,\quad \sigma_{\dot{P}_1}=140.713,\quad \nu_{P_1}=2.177\ \mathrm{Hz}$$

$$\sigma_{P_2}=0.680,\quad \sigma_{\dot{P}_2}=21.026,\quad \nu_{P_2}=4.921\ \mathrm{Hz}$$

$$\nu_{P_1}=\frac{1}{2\pi}\frac{\sigma_{\dot{P}_1}}{\sigma_{P_1}}=2.177\ \mathrm{Hz};\ \nu_{P_2}=\frac{1}{2\pi}\frac{\sigma_{\dot{P}_2}}{\sigma_{P_2}}=4.921\ \mathrm{Hz}$$

$$g_{P_1} = \sqrt{2\ell n\left(2\nu_{P_1}T\right)} + \frac{0.5772}{\sqrt{2\ell n\left(2\nu_{P_1}T\right)}} = 2.950$$

$$g_{P_2} = \sqrt{2\ell n\left(2\nu_{P_2}T\right)} + \frac{0.5772}{\sqrt{2\ell n\left(2\nu_{P_2}T\right)}} = 3.220$$

$$\mu_{\hat{P}_1} = g_{P_1}\sigma_{P_1} = 2.950 \times 10.288 = 30.350$$

$$\mu_{\hat{P}_2} = g_{P_2}\sigma_{P_2} = 3.220 \times 0.680 = 2.190$$

$$\mathbf{\Sigma_P} \cong \begin{bmatrix} \sigma_{P_1}^2 & 0 \\ 0 & \sigma_{P_2}^2 \end{bmatrix} \simeq \begin{bmatrix} 105.848 & 0 \\ 0 & 0.462 \end{bmatrix}$$

$$\mathbf{\Sigma_{\dot{P}}} \cong \begin{bmatrix} \sigma_{\dot{P}_1}^2 & 0 \\ 0 & \sigma_{\dot{P}_2}^2 \end{bmatrix} = \begin{bmatrix} 19800.280 & 0 \\ 0 & 442.105 \end{bmatrix}$$

$$\mathbf{\Sigma_Q} = \mathbf{\Psi\Sigma_P\Psi}^T = \begin{bmatrix} 1.260 & -1.448 \\ 1.972 & 1.716 \end{bmatrix}\begin{bmatrix} 105.848 & 0 \\ 0 & 0.462 \end{bmatrix}\begin{bmatrix} 1.260 & 1.972 \\ -1.448 & 1.716 \end{bmatrix} \times 10^{-6} =$$

$$= \begin{bmatrix} 1.69 \times 10^{-4} & 2.62 \times 10^{-4} \\ 2.62 \times 10^{-4} & 4.13 \times 10^{-4} \end{bmatrix} \Rightarrow$$

$$\sigma_{Q_1} = 0.013\text{ m}, \quad \sigma_{Q_2} = 0.020\text{ m}, \quad \rho_{Q_1Q_2} = 0.99$$

$$\mathbf{\Sigma_{\dot{Q}}} = \mathbf{\Psi\Sigma_{\dot{P}}\Psi}^T = \begin{bmatrix} 1.260 & -1.448 \\ 1.972 & 1.716 \end{bmatrix}\begin{bmatrix} 19800.28 & 0 \\ 0 & 442.105 \end{bmatrix}\begin{bmatrix} 1.260 & 1.972 \\ -1.448 & 1.716 \end{bmatrix} \times 10^{-6} =$$

$$= \begin{bmatrix} 0.0324 & 0.0481 \\ 0.0481 & 0.0783 \end{bmatrix} \Rightarrow$$

$$\sigma_{\dot{Q}_1} = 0.18\text{ m/s}, \quad \sigma_{\dot{Q}_2} = 0.28\text{ m/s}, \quad \rho_{\dot{Q}_1\dot{Q}_2} = 0.95$$

$$\nu_{Q_1} = \frac{1}{2\pi}\frac{\sigma_{\dot{Q}_1}}{\sigma_{Q_1}} = 2.204\text{ Hz}\,; \nu_{Q_2} = \frac{1}{2\pi}\frac{\sigma_{\dot{Q}_2}}{\sigma_{Q_2}} = 2.228\text{ Hz}$$

$$g_{Q_1} = \sqrt{2\ell n\left(2\nu_{Q_1}T\right)} + \frac{0.5772}{\sqrt{2\ell n\left(2\nu_{Q_1}T\right)}} = 2.961$$

$$g_{Q_2} = \sqrt{2\ell n\left(2\nu_{Q_2}T\right)} + \frac{0.5772}{\sqrt{2\ell n\left(2\nu_{Q_2}T\right)}} = 2.965$$

$$\mu_{\hat{Q}_1} = g_{Q_1}\sigma_{Q_1} = 2.961 \times 0.013 = 0.038\text{ m}$$

$$\mu_{\hat{Q}_2} = g_{Q_2}\sigma_{Q_2} = 2.965 \times 0.020 = 0.059\text{ m}$$

Neglecting the contribution of the second mode of vibration:

$$Q_1(t) = \Psi_{11}P_1(t) \Rightarrow \sigma_{Q_1} = \Psi_{11}\sigma_{P_1} = 0.013\text{ m}$$

$$Q_2(t) = \Psi_{12}P_1(t) \Rightarrow \sigma_{Q_2} = \Psi_{12}\sigma_{P_1} = 0.020\text{ m}$$

The bending moment at the base of a column of the first storey is given by:

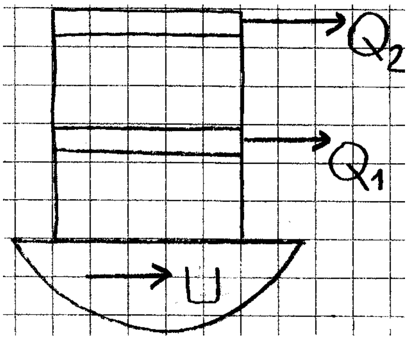
$$M_I(t) = \frac{6EJ}{h^2} Q_1(t) \Rightarrow$$
$$\sigma_{M_I} = \frac{6EJ}{h^2}\sigma_{Q_1} = \frac{6\times 0.3\times 10^{11}\times 5.208\times 10^{-3}}{6^2}\times 0.013 = 0.338\times 10^6 \text{ Nm}$$

The bending moment at the base of a column of the second storey is given by:

$$M_{II}(t) = \frac{6EJ}{h^2}\left[Q_2(t) - Q_1(t)\right] \Rightarrow$$
$$\sigma^2_{M_{II}} = \left(\frac{6EJ}{h^2}\right)^2 \left(\sigma^2_{Q_2} + \sigma^2_{Q_1} - 2C_{Q_1Q_2}(0)\right) \Rightarrow$$
$$\sigma_{M_{II}} = \frac{6EJ}{h^2}\sqrt{\sigma^2_{Q_2} + \sigma^2_{Q_1} - 2\rho_{Q_1Q_2}(0)\sigma_{Q_1}\sigma_{Q_2}} \Rightarrow$$
$$\sigma_{M_{II}} \simeq \frac{6EJ}{h^2}\left(\sigma_{Q_2} - \sigma_{Q_1}\right) = \frac{6\times 0.3\times 10^{11}\times 2.133\times 10^{-3}}{4.5^2}\times (0.020 - 0.013) = 0.136\times 10^6 \text{ Nm}$$

The approximation is good since $\rho_{Q_1Q_2}(0) = 0.99 \simeq 1.00$.

Example. Consider the wind-excited response of a r.c. two-storey shear-type building.



$$\mathbf{M}\ddot{\mathbf{Q}}(t)+\mathbf{C}\dot{\mathbf{Q}}(t)+\mathbf{K}\mathbf{Q}(t)=\mathbf{F}(t)$$

$$\mathbf{Q}(t)=\begin{Bmatrix} Q_1(t) \\ Q_2(t) \end{Bmatrix};\ \mathbf{F}(t)=\begin{Bmatrix} F_1(t) \\ F_2(t) \end{Bmatrix}$$

The wind velocity is defined by the relationships:

$$\mathbf{V}(t)=\mu_V+\mathbf{V}'(t)$$

$$\mu_{V_1}=\mu_{V_2}=\mu_V=35\ \mathrm{m/s}$$

$$S_{V_1V_1}(\omega)=S_{V_2V_2}(\omega)=S_{VV}(\omega)=\frac{\sigma_V^2}{4\pi}\frac{1.093L_V/\mu_V}{\left(1+1.640|\omega|L_V/\mu_V\right)^{5/3}}$$

$$\sigma_{V_1}=\sigma_{V_2}=\sigma_V=5\ \mathrm{m/s}\ ;\quad L_{V_1}=L_{V_2}=L_V=100\ \mathrm{m}$$

$$\gamma_{V_1V_2}(\omega)=\gamma_{VV}(\omega)=\exp\left\{-\frac{|\omega|C_z\Delta z}{2\pi\mu_V}\right\}$$

$$C_z=10\ ;\quad \Delta z=4.5\ \mathrm{m}$$

$$\boldsymbol{\mu}_{\mathbf{V}}=\begin{Bmatrix} \mu_{V_1} \\ \mu_{V_2} \end{Bmatrix}=\begin{Bmatrix} 35 \\ 35 \end{Bmatrix}\mathrm{m/s}$$

$$\mathbf{S}_{\mathbf{V}}(\omega)=\begin{bmatrix} S_{V_1V_1}(\omega) & S_{V_1V_2}(\omega) \\ S_{V_2V_1}(\omega) & S_{V_2V_2}(\omega) \end{bmatrix}=\begin{bmatrix} S_{V_1V_1}(\omega) & \sqrt{S_{V_1V_1}(\omega)S_{V_2V_2}(\omega)}\gamma_{V_1V_2}(\omega) \\ \sqrt{S_{V_1V_1}(\omega)S_{V_2V_2}(\omega)}\gamma_{V_2V_1}(\omega) & S_{V_2V_2}(\omega) \end{bmatrix}\Rightarrow$$

$$\mathbf{S}_{\mathbf{V}}(\omega)=S_{VV}(\omega)\begin{bmatrix} 1 & \gamma_{VV}(\omega) \\ \gamma_{VV}(\omega) & 1 \end{bmatrix}$$

The aerodynamic wind actions are defined by the relationships:

$$\mathbf{F}(t)=\mu_F+\mathbf{F}'(t)$$

$$\mu_{F_k}=\frac{1}{2}\rho\mu_{V_k}^2A_kC_{D_k}\ ;\quad F_k'(t)=\rho\mu_{V_k}A_kC_{D_k}V_k'(t)$$

ρ = 1.25 kg/m^3; A_1 = 5×6 = 30 m^2; A_2 = 2.5×6 = 15 m^2; $C_{D_1}=C_{D_2}=1$

$$\boldsymbol{\mu}_{\mathbf{F}}=\begin{Bmatrix} \mu_{F_1} \\ \mu_{F_2} \end{Bmatrix}=\begin{Bmatrix} 22969 \\ 11484 \end{Bmatrix}\mathrm{N}$$

$$\mathbf{F}'(t)=\begin{Bmatrix} F_1'(t) \\ F_2'(t) \end{Bmatrix}=\mathbf{A}\mathbf{V}'(t)\ ;\quad \mathbf{A}=\begin{bmatrix} \rho\mu_{V_1}A_1C_{D_1} & 0 \\ 0 & \rho\mu_{V_2}A_2C_{D_2} \end{bmatrix}=\begin{bmatrix} 1312 & 0 \\ 0 & 656 \end{bmatrix}\mathrm{kg/s}$$

$$\mathbf{S_F}(\omega)=\mathbf{A}\mathbf{S_V}(\omega)\mathbf{A}^T=S_{VV}(\omega)\begin{bmatrix}A_1 & 0\\ 0 & A_2\end{bmatrix}\begin{bmatrix}1 & \gamma_{VV}(\omega)\\ \gamma_{VV}(\omega) & 1\end{bmatrix}\begin{bmatrix}A_1 & 0\\ 0 & A_2\end{bmatrix}\Rightarrow$$

$$\mathbf{S_F}(\omega)=S_{VV}(\omega)\begin{bmatrix}A_1^2 & A_1A_2\gamma_{VV}(\omega)\\ A_1A_2\gamma_{VV}(\omega) & A_2^2\end{bmatrix}$$

A_1 = 1312 kg/s; A_2 = 656 kg/s

The dynamic properties of the structure are defined as:

$$\mathbf{M}=\begin{bmatrix}271.200 & 0\\ 0 & 146.325\end{bmatrix}(\mathrm{kg})\,;\,\mathbf{K}=\begin{bmatrix}1.694\times10^8 & -0.758\times10^8\\ -0.758\times10^8 & 0.758\times10^8\end{bmatrix}(\mathrm{N/m})$$

$$\mathbf{\Omega}=\begin{bmatrix}187.063 & 0\\ 0 & 956.056\end{bmatrix};\,\mathbf{\Psi}=\begin{bmatrix}1.260 & -1.448\\ 1.972 & 1.716\end{bmatrix}\times10^{-3}\,(\mathbf{L}=\mathbf{I})$$

$$\omega_1=13.677\,\mathrm{rad/s}\,;\,\omega_2=30.920\,\mathrm{rad/s}$$

$$\xi_1=\xi_2=0.02$$

Using the principal transformation law:

$$\mathbf{Q}(t)=\mathbf{\Psi}\mathbf{P}(t)\simeq\psi_1P_1(t)$$

$$\ddot{P}_1(t)+2\xi_1\omega_1\dot{P}_1(t)+\omega_1^2P_1(t)=\psi_1^T\mathbf{F}(t)$$

$$S_{P_1P_1}(\omega)=\left|H_{P_1}(\omega)\right|^2S_{F_{P_1}F_{P_1}}(\omega)$$

$$H_{P_1}(\omega)=\frac{1}{-\omega^2+2i\xi_1\omega\omega_1+\omega_1^2}$$

$$S_{F_{P_1}F_{P_1}}(\omega)=\psi_1^T\mathbf{S_F}(\omega)\psi_1\Rightarrow$$

$$S_{F_{P_1}F_{P_1}}(\omega)=S_{VV}(\omega)\{1.260\ \ 1.972\}\begin{bmatrix}A_1^2 & A_1A_2\gamma_{VV}(\omega)\\ A_1A_2\gamma_{VV}(\omega) & A_2^2\end{bmatrix}\begin{Bmatrix}1.260\\ 1.972\end{Bmatrix}\times10^{-6}$$

$$\sigma_{P_1}^2=\int_{-\infty}^{\infty}S_{P_1P_1}(\omega)d\omega\,;\,\sigma_{\dot{P}_1}^2=\int_{-\infty}^{\infty}\omega^2S_{P_1P_1}(\omega)d\omega$$

$$\sigma_{P_1}^2=0.0017,\ \ \sigma_{\dot{P}_1}^2=0.1575$$

$$\sigma_{P_1}=0.0413,\ \ \ \sigma_{\dot{P}_1}=0.396,\ \ \ \nu_{P_1}=1.528\,\mathrm{Hz}\ \ ,\ \ \ T=600\ \mathrm{s}$$

$$g_{P_1}=\sqrt{2\ell n\left(\nu_{P_1}T\right)}+\frac{0.5772}{\sqrt{2\ell n\left(\nu_{P_1}T\right)}}=3.850$$

$$\mu_{Q_1}=0.000368\,\mathrm{m},\ \ \ \mu_{Q_2}=0.000514\,\mathrm{m}$$

$$\sigma_{Q_1}=0.0413\times1.260\times10^{-3}=5.209\times10^{-5}\ \mathrm{m},\ \ \ \sigma_{Q_2}=0.0413\times1.972\times10^{-3}=8.153\times10^{-5}\ \mathrm{m}$$

$$\mu_{\hat{Q}_1}=\mu_{Q_1}+g_{Q_1}\sigma_{Q_1}=0.000368+3.850\times5.209\times10^{-5}=0.000569\,\mathrm{m}$$

$$\mu_{\hat{Q}_2}=\mu_{Q_2}+g_{Q_2}\sigma_{Q_2}=0.000514+3.850\times8.153\times10^{-5}=0.000830\,\mathrm{m}$$