MONTE CARLO SIMULATION

The knowledge of a set of the sample functions of a random process allows to derive its power spectral density. On the other hand, the knowledge of a suitable model of the power spectral density of a random process allows to derive artificially its sample functions. This operation is known as simulation of a random process and falls into the broad family of the Monte Carlo methods.

Monte Carlo methods to simulate a random process may be classified into two main families: 1) the methods based on the superposition of harmonic waves with random phase angles (random phase method); 2) ARMA methods based on the filtering of uncorrelated white noises (Auto-Regressive and/or Mobile-Average methods). Both these methods may be applied to simulate stationary and non-stationary random processes, as well as normal and non-normal random processes. This section is aimed at providing a synthetic description of the application of the random phase method to random stationary normal processes with zero mean.

Mono-variate processes

Let $X(t)$ be a random stationary normal process with zero mean. $S_{XX}(\omega)$ is its power spectral density. Using the random phase method, a generic sample function of $X(t)$ is given by:

$$\boxed{x(t) = 2\sum_{1}^{N}{}_{j}\sqrt{S_{XX}(\omega_j)\Delta\omega_j}\,\sin(\omega_j t + \varphi_j)} \tag{1}$$

where $\Delta\omega_j\,(j=1,..N)$ is the amplitude of the frequency steps into which the harmonic content of the process is sub-divided (with $\omega \geq 0$); $\omega_j\,(j=1,..N)$ is the central value of each step (Fig. 1a); φ_j is the j-th occurrence of the random phase Φ uniformly distributed between 0 and 2π (Fig. 1b):

$$\boxed{p_\Phi(\varphi) = \begin{cases} 1/2\pi & \text{for } 0 < \varphi < 2\pi \\ 0 & \text{elsewhere} \end{cases}} \tag{2}$$

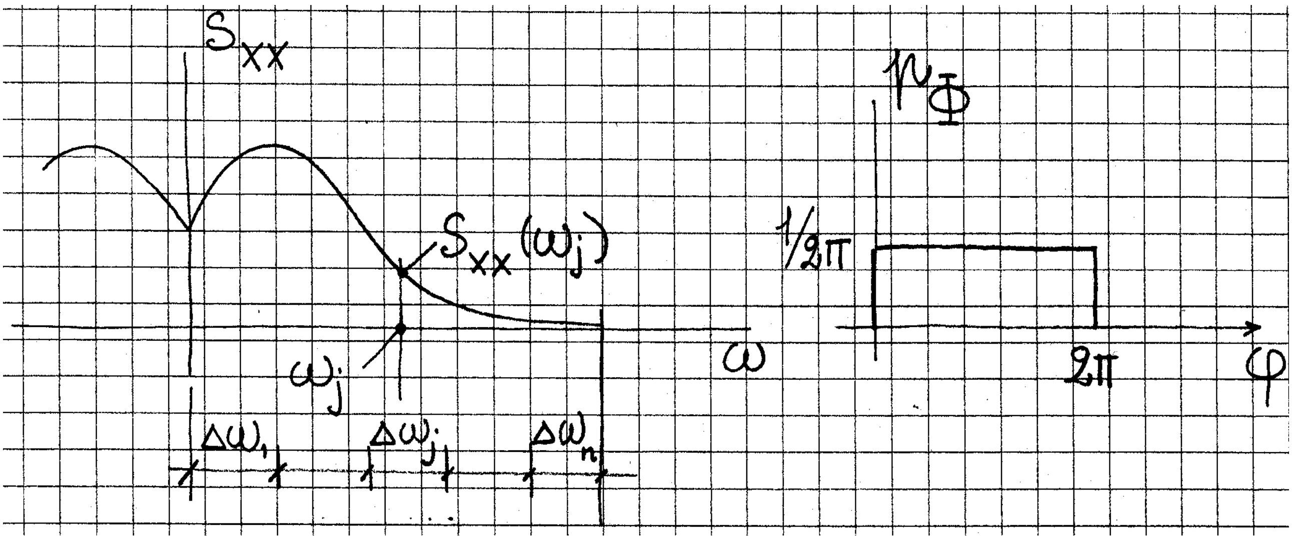

Fig. 1

To interpret Eq. (1), let us consider the j-th term of the sum:

$$x_j(t) = 2\sqrt{S_{XX}(\omega_j)\Delta\omega_j}\,\sin(\omega_j t + \varphi_j) \tag{3}$$

and let us evaluate its variance (the variance of a harmonic function with unit amplitude is 1/2). Thus:

$$\sigma^2_{x_j} = 4S_{XX}(\omega_j)\Delta\omega_j \frac{1}{2} = 2S_{XX}(\omega_j)\Delta\omega_j$$

Since the harmonics that constitute the sample function defined by Eq. (1) is uncorrelated with each other (having different circular frequencies), the variance of $X(t)$ is the sum of the variances of its components. Thus (Fig. 2):

$$\sigma^2_x = \sum_1^N {}_j\sigma^2_{x_j} = 2\sum_1^N {}_jS_{XX}(\omega_j)\Delta\omega_j \simeq \sigma^2_X$$

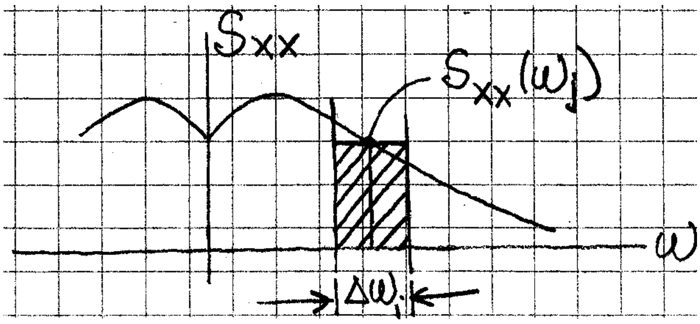

Fig. 2.

Moreover, the amplitude of each harmonic defines the power content and the distribution of the sample function and of the process. Fig. 3 shows the simulation of 3 sample functions of the wind velocity studied in the previous section. The functions are normalised in order to have $\sigma_V = 1$.

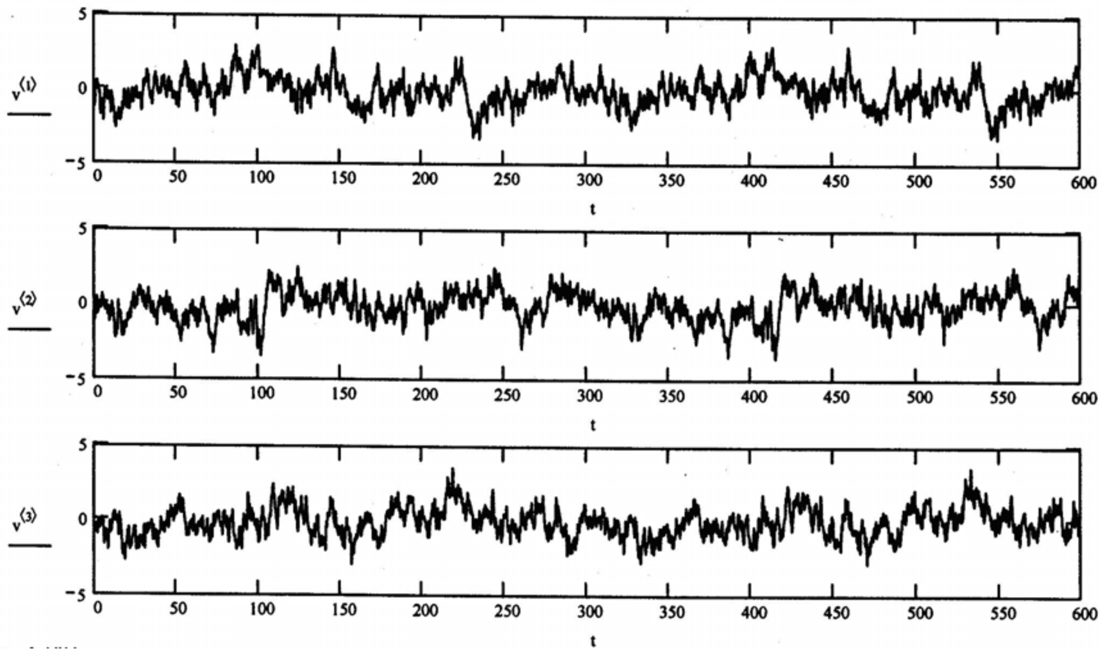

Fig. 3

Multi-variate processes

Let $\mathbf{X}(t)$ be a n-variate random stationary normal process with zero mean. Let $\mathbf{S}_X(\omega)$ be its power spectral density matrix. The random phase method allows to simulate any number of sample vectors $\mathbf{x}(t)$ of $\mathbf{X}(t)$. Fig. 4 shows the basic concepts of the simulation of a 3-variate process.

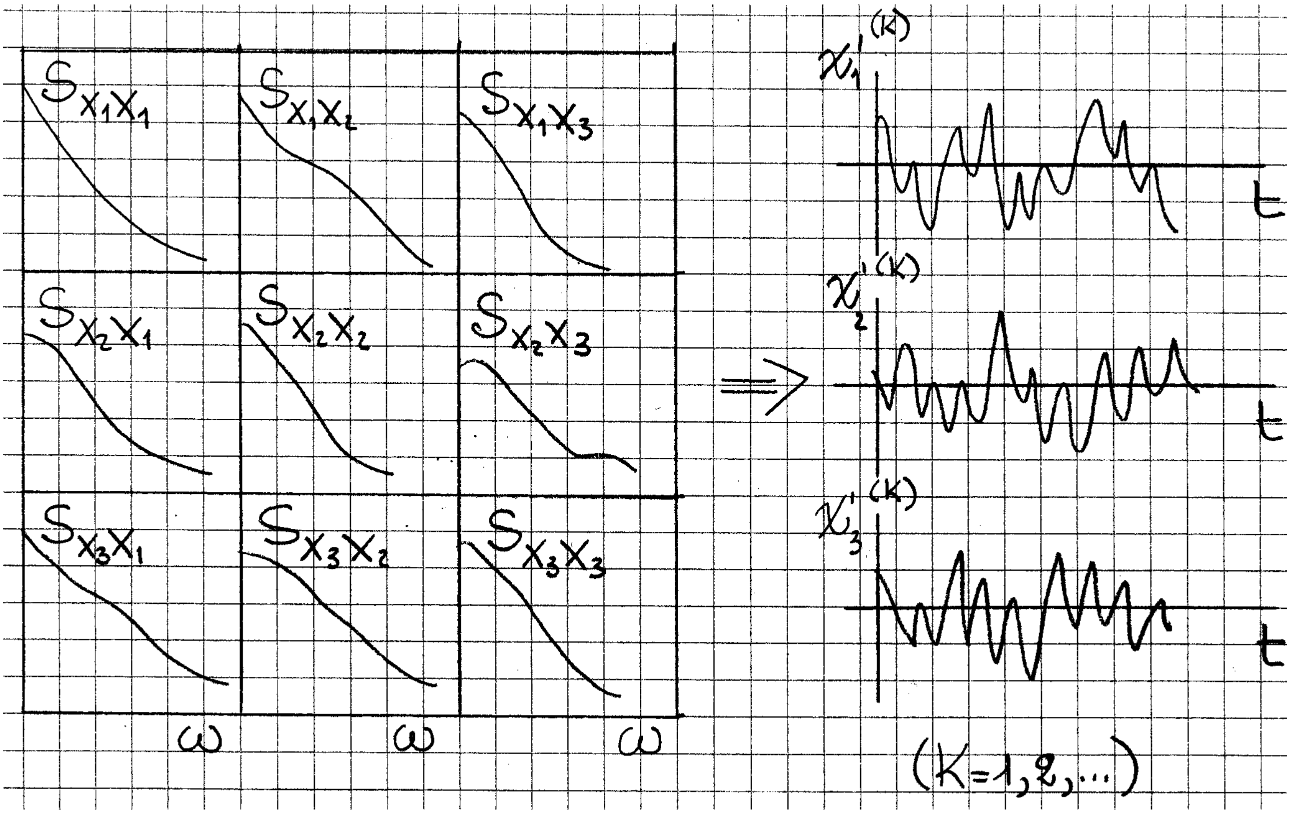

Fig. 4

It is possible to demonstrate that the i-th component of the sample vector $\mathbf{x}(t)$ of $\mathbf{X}(t)$ is given by:

$$\boxed{x_i(t) = 2\sum_{1}^{N}{}_j \sum_{1}^{n}{}_k D_{ik}(\omega_j)\sqrt{\Delta\omega_j}\,\sin(\omega_j t + \varphi_{jk}) \quad (i = 1, 2, ..n)} \tag{4}$$

where $\Delta\omega_j \; (j = 1,..N)$ is the amplitude of the frequency steps into which the harmonic content of the process is sub-divided (with $\omega \geq 0$); $\omega_j \; (j = 1,..N)$ is the central value of each frequncy step (Fig. 1a); φ_{jk} is the j,k-th occurrence of the random phase Φ uniformly distributed between 0 and 2π.

Eq. (4) may be re-written in the following matrix form:

$$\boxed{\mathbf{x}(t) = 2\sum_{1}^{N}{}_j \mathbf{D}(\omega_j)\sqrt{\Delta\omega_j}\,\sin(\omega_j \mathbf{1} t + \boldsymbol{\varphi}_j)} \tag{5}$$

where $\mathbf{1} = \{1\;1\dots1\}^T$ is a vecton of n unit components, $\boldsymbol{\varphi}_j = \{\varphi_{j1}\;\varphi_{j2}\;...\;\varphi_{jn}\}^T$, $\mathbf{D}$ is a matrix provided by the relationship:

$$\boxed{\mathbf{D}(\omega)\mathbf{D}^{T}(\omega)=\mathbf{S}_{X}(\omega)} \tag{6}$$

Eq. (6) is referred to as matrix decomposition. There are infinite possible matrices $\mathbf{D}(\omega)$ that satisfy Eq. (6) and several methods to determine such matrices. The most well-known methods are referred to as the Cholesky and spectral decompositions.

Fig. 5 shows the simulation of 3 correlated time-histories of the wind velocity.

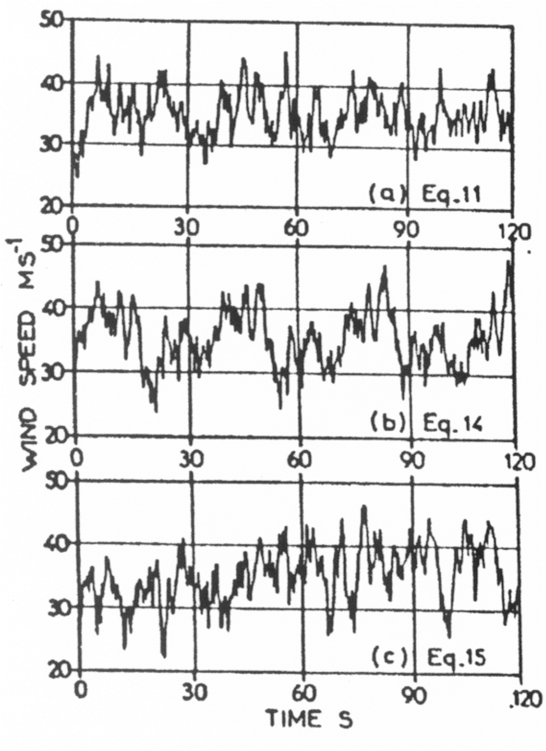

Fig. 5

Cholesky decomposition

A square matrix **A** of order n that is real, symmetric, semi-positive definite can be decomposed as:

$$\mathbf{A}=\mathbf{L}\mathbf{L}^{T} \tag{7}$$

$$\mathbf{A}=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix};\mathbf{L}=\begin{bmatrix} \ell_{11} & 0 & \cdots & 0 \\ \ell_{21} & \ell_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ \ell_{n1} & \ell_{n2} & \cdots & \ell_{nn} \end{bmatrix} \tag{8}$$

where **L** is a lower triangular matrix whose terms are given by the recursive formulae:

$$\ell_{ii}=\left[a_{ii}-\sum_{1}^{i-1}{}_{j}\ell_{ij}^{2}\right]^{1/2} \quad (i=1,2,..n) \tag{9}$$

$$\ell_{ki}=\frac{1}{\ell_{ii}}\left[a_{ik}-\sum_{1}^{i-1}{}_{j}\ell_{ij}\ell_{kj}\right] \quad (k=i+1,i+2,..n;i=1,2,..n) \tag{10}$$

Example

$$\mathbf{A} = \begin{bmatrix} 4 & -1 \\ -1 & 2 \end{bmatrix}$$

From Eq. (9): $i = 1 \Rightarrow \ell_{11} = \sqrt{a_{11}} = \sqrt{4} = 2$

From Eq. (10): $k = 2, i = 1 \Rightarrow \ell_{21} = \frac{1}{\ell_{11}}\left[a_{12}\right] = \frac{1}{2}(-1) = -\frac{1}{2}$

From Eq. (9): (a) $i = 2 \Rightarrow \ell_{22} = \left[a_{22} - \left(\ell_{21}^2\right)\right]^{1/2} = \sqrt{1 - \frac{1}{4}} = \frac{\sqrt{3}}{2}$

Therefore:

$$\mathbf{L} = \begin{bmatrix} 2 & 0 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \Rightarrow$$

$$\mathbf{LL}^{\mathrm{T}} = \begin{bmatrix} 2 & 0 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 2 & -\frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ -1 & 1 \end{bmatrix} = \mathbf{A}$$

Spectral decomposition

Let us consider again Eq. (7) and let us evaluate the eigenvalues and the eigenvectors of **A**:

$$(\mathbf{A} - \lambda\mathbf{I})\psi = \mathbf{0} \tag{11}$$

$\boldsymbol{\Psi} = [\psi_1\ \psi_2\ ...\psi_n]$ is the matrix of the orthonormal eigenvectors; $\boldsymbol{\Lambda} = \mathbf{diag}\{\lambda_1, \lambda_2, ..\lambda_n\}$ is the matrix of the eigenvalues:

$$\boldsymbol{\Psi}^{\mathrm{T}}\boldsymbol{\Psi} = \mathbf{I} \quad ; \quad \boldsymbol{\Psi}^{\mathrm{T}}\mathbf{A}\boldsymbol{\Psi} = \boldsymbol{\Lambda} \tag{12}$$

Using Eq. (12):

$$\underbrace{\boldsymbol{\Psi}\boldsymbol{\Psi}^{\mathrm{T}}}_{\mathbf{I}} \mathrm{A} \underbrace{\boldsymbol{\Psi}\boldsymbol{\Psi}^{\mathrm{T}}}_{\mathbf{I}} = \boldsymbol{\Psi}\boldsymbol{\Lambda}\boldsymbol{\Psi}^{\mathrm{T}} \Rightarrow$$

$$\mathrm{A} \underbrace{\boldsymbol{\Psi}\sqrt{\boldsymbol{\Lambda}}}_{\mathbf{L}} \underbrace{\sqrt{\boldsymbol{\Lambda}}\boldsymbol{\Psi}^{\mathrm{T}}}_{\mathbf{L}^{\mathrm{T}}} = \mathbf{LL}^{\mathrm{T}} \Rightarrow$$

$$\mathbf{L} = \boldsymbol{\Psi}\sqrt{\boldsymbol{\Lambda}} \tag{13}$$

Example

$$\mathbf{A}=\begin{bmatrix}4 & -1\\ -1 & 1\end{bmatrix}$$

$$\begin{bmatrix}4-\lambda & -1\\ -1 & 1-\lambda\end{bmatrix}\begin{Bmatrix}\psi_1\\ \psi_2\end{Bmatrix}=\begin{Bmatrix}0\\ 0\end{Bmatrix}$$

The characteristic equation results:

$$(4-\lambda)(1-\lambda)-1=0\Rightarrow\lambda^2-5\lambda+3=0\Rightarrow\lambda=\frac{5\mp\sqrt{13}}{2}$$

$$\begin{cases}\left(4-\dfrac{5\mp\sqrt{13}}{2}\right)\psi_1-\psi_2=0\\ -\psi_1+\left(1-\dfrac{5\mp\sqrt{13}}{2}\right)\psi_2=0\end{cases}$$

$$\bar{\mathbf{\Psi}}=\begin{bmatrix}\dfrac{\sqrt{13}-3}{2} & \dfrac{-\sqrt{13}-3}{2}\\ 1 & 1\end{bmatrix}$$ (matrix of the orthogonal eigenvectors)

$$\bar{\mathbf{\Psi}}^{\mathrm{T}}\bar{\mathbf{\Psi}}=\begin{bmatrix}\dfrac{\sqrt{13}-3}{2} & 1\\ \dfrac{-\sqrt{13}-3}{2} & 1\end{bmatrix}\begin{bmatrix}\dfrac{\sqrt{13}-3}{2} & \dfrac{-\sqrt{13}-3}{2}\\ 1 & 1\end{bmatrix}=\begin{bmatrix}1.09 & 0\\ 0 & 11.91\end{bmatrix}\Rightarrow$$

$$\psi_1=\bar{\mathbf{\Psi}}_1^{\mathrm{T}}/\sqrt{1.09}\ ;\ \psi_2=\bar{\mathbf{\Psi}}_2^{\mathrm{T}}/\sqrt{11.91}\Rightarrow$$

$$\mathbf{\Psi}=\begin{bmatrix}0.29 & -0.96\\ 0.96 & 0.29\end{bmatrix}$$ (matrix of the orthonormal eigenvectors)

$$\mathbf{L}=\mathbf{\Psi}\sqrt{\mathbf{\Lambda}}=\begin{bmatrix}0.29 & -0.96\\ 0.96 & 0.29\end{bmatrix}\begin{bmatrix}\sqrt{\dfrac{5-\sqrt{13}}{2}} & 0\\ 0 & \sqrt{\dfrac{5-\sqrt{13}}{2}}\end{bmatrix}=\begin{bmatrix}0.24 & -1.99\\ 0.80 & 0.60\end{bmatrix}$$

$$\mathbf{L}\mathbf{L}^{\mathrm{T}}=\begin{bmatrix}0.24 & -1.99\\ 0.80 & 0.60\end{bmatrix}\begin{bmatrix}0.24 & 0.80\\ -1.99 & 0.60\end{bmatrix}=\begin{bmatrix}4 & -1\\ -1 & 1\end{bmatrix}=\mathbf{A}$$