VECTOR OF RANDOM PROCESSES

Definitions

Let us consider an experiment whose result is represented by a vector of n random processes $\mathbf{X}(t)=\{X_1(t)\,X_2(t)..X_n(t)\}^T$ (for example the components of the seismic motion at the base of n piers of a viaduct, the wind velocities registered by n anemometers, the dynamic response of a n-D.O.F. system). The vector **X**(t) is also called a n-variate random process.

Let us consider the values $x_i^{(j)}(t_i)$ (i = 1,2,..n; j = 1,2,..) assumed by the sample functions $x_i^{(j)}(t)$ of $X_i(t)$ for $t=t_i$. The set of these values constitutes n random variables $X_i=X(t_i)$ (i = 1,2,..n). They are characterised by the joint density function of order n $p_{X_1X_2..X_n}(x_1,t_1;x_2,t_2;..x_n,t_n)$ = $p_{\mathbf{X}}(\mathbf{x},\mathbf{t})$, where $\mathbf{x}=\{x_1\,x_2\,..x_n\}^T$ is the vector of the state variables and $\mathbf{t}=\{t_1\,t_2\,..t_n\}^T$ is the vector of the times in correspondence of which the random variables are extracted.
If **X**(t) is a n-variate normal random process, its complete probabilistic definition calls for the knowledge of only the joint density functions of the second order of all the possible couples of random variables $X_i(t_i),X_j(t_j)\,(i,j=1,2,...n)$: $p_{X_iX_j}(x_i,t_i;x_j,t_j)$, $\forall i,j=1..n$ and $\forall t_i,t_j\in\mathbf{R}$.

Stationary processes

If **X**(t) is a (weakly) stationary random process, the joint desity functions of the second order of all the possible couples of random variables $X_i(t_i),X_j(t_j)\,(i,j=1,2,...n)$ do not depend on the instants t_i, t_j but only on the time lag $\tau = t_j - t_i$: $p_{X_iX_j}(x_i,t_i;x_j,t_j)$ = $p_{X_iX_j}(x_i,x_j;\tau)$, $\forall i,j=1..n$ and $\forall\tau\in\mathbf{R}$.

Let us define as mean vector $\boldsymbol{\mu}_{\mathbf{X}}$ of the stationary process $\mathbf{X}(t)$, a vector whose components $\mu_{X_i}(t)$ are the means of the processes $X_i(t)\,(i=1,..n)$:

$$\boldsymbol{\mu}_{\mathbf{X}}=E[\mathbf{X}(t)]=\{\mu_{X_1}\,\mu_{X_2}\,..\mu_{X_n}\}^T \tag{44}$$

Let us define as correlation matrix and covariance matrix of **X**(t), respectively, the matrices whose i,j-th terms are the functions $R_{X_iX_j}(\tau)$ and $C_{X_iX_j}(\tau)$:

$$\mathbf{R}_{\mathbf{X}}(\tau)=E\left[\mathbf{X}(t)\mathbf{X}^T(t+\tau)\right]=\begin{bmatrix} R_{X_1X_1}(\tau) & R_{X_1X_2}(\tau) & .. & R_{X_1X_n}(\tau) \\ R_{X_2X_1}(\tau) & R_{X_2X_2}(\tau) & .. & R_{X_2X_n}(\tau) \\ \vdots & \vdots & \vdots & \vdots \\ R_{X_nX_1}(\tau) & R_{X_nX_2}(\tau) & .. & R_{X_nX_n}(\tau) \end{bmatrix} \tag{45}$$

$$\mathbf{C}_{\mathbf{X}}(\tau)=E\left[\{\mathbf{X}(t)-\boldsymbol{\mu}_{\mathbf{X}}\}\{\mathbf{X}(t+\tau)-\boldsymbol{\mu}_{\mathbf{X}}\}^T\right]=\begin{bmatrix} C_{X_1X_1}(\tau) & C_{X_1X_2}(\tau) & .. & C_{X_1X_n}(\tau) \\ C_{X_2X_1}(\tau) & C_{X_2X_2}(\tau) & .. & C_{X_2X_n}(\tau) \\ \vdots & \vdots & \vdots & \vdots \\ C_{X_nX_1}(\tau) & C_{X_nX_2}(\tau) & .. & C_{X_nX_n}(\tau) \end{bmatrix} \tag{46}$$

The on-diagonal terms are, respectively, the auto-correlation functions and the auto-covariance functions of each process; the off-diagonal terms are, respectively, the cross-correlation functions and the cross-covariance functions of all the possible couples of the processes.
Eqs. (45) and (46) involve:

$$\mathbf{R_X}(\tau) = \mathbf{C_X}(\tau) + \boldsymbol{\mu}_\mathbf{X}\boldsymbol{\mu}_\mathbf{X}^\mathbf{T} \tag{47}$$

Due to Eqs. (16) and (21):

$$\mathbf{R_X}(\tau) = \mathbf{R}_\mathbf{X}^\mathbf{T}(-\tau) \tag{48}$$

$$\mathbf{C_X}(\tau) = \mathbf{C}_\mathbf{X}^\mathbf{T}(-\tau) \tag{49}$$

Thus, the matrices $\mathbf{R_X}$ and $\mathbf{C_X}$ are symmetric for $\tau = 0$:

$$\mathbf{R_X}(0) = \mathbf{R}_\mathbf{X}^\mathbf{T}(0) \tag{50}$$

$$\mathbf{C_X}(0) = \mathbf{C}_\mathbf{X}^\mathbf{T}(0) \tag{51}$$

It can be shown that $\mathbf{R_X}$ and $\mathbf{C_X}$ are semi-positive definite for any value of τ ($f_\mathbf{R} = \mathbf{u}^\mathrm{T}\mathbf{R_X}\mathbf{u} \geq 0$, $f_\mathbf{C} = \mathbf{v}^\mathrm{T}\mathbf{C_X}\mathbf{v} \geq 0$ for $\forall \mathbf{u}, \mathbf{v} \neq \mathbf{0}$).

Let us define as power spectral density matrix of **X**(t) the matrix whose i,j-th term is the function $S_{X_iX_j}(\omega)$:

$$\mathbf{S_X}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathbf{C_X}(\tau)e^{-i\omega\tau}d\tau = \begin{bmatrix} S_{X_1X_1}(\omega) & S_{X_1X_2}(\omega) & .. & S_{X_1X_n}(\omega) \\ S_{X_2X_1}(\omega) & S_{X_2X_2}(\omega) & .. & S_{X_2X_n}(\omega) \\ \vdots & \vdots & \vdots & \vdots \\ S_{X_nX_1}(\omega) & S_{X_nX_2}(\omega) & .. & S_{X_nX_n}(\omega) \end{bmatrix} \tag{52}$$

$$\mathbf{C_X}(\tau) = \int_{-\infty}^{\infty}\mathbf{S_X}(\omega)e^{i\omega\tau}d\omega = \begin{bmatrix} C_{X_1X_1}(\tau) & C_{X_1X_2}(\tau) & .. & C_{X_1X_n}(\tau) \\ C_{X_2X_1}(\tau) & C_{X_2X_2}(\tau) & .. & C_{X_2X_n}(\tau) \\ \vdots & \vdots & \vdots & \vdots \\ C_{X_nX_1}(\tau) & C_{X_nX_2}(\tau) & .. & C_{X_nX_n}(\tau) \end{bmatrix} \tag{53}$$

The on-diagonal terms of the spectral matrix are the power spectral densities of each process; the off-diagonal terms are the cross-power spectral densities of all the possible couples of the processes.

Due to Eq. (37) $\mathbf{S_X}(\omega)$ is a Hermitian matrix. It can be proved that it is also semi-positive definite for any value of ω. From Eq. (53) it results:

$$\boldsymbol{\Sigma} = \mathbf{C}_{\mathbf{X}}(0) = \int_{-\infty}^{\infty} \mathbf{S}_{\mathbf{X}}(\omega)\,d\omega = \begin{bmatrix} \varphi^2_{X_1} & C_{X_1X_2}(0) & .. & C_{X_1X_n}(0) \\ C_{X_2X_1}(0) & \varphi^2_{X_2} & .. & C_{X_2X_n}(0) \\ \vdots & \vdots & \vdots & \vdots \\ C_{X_nX_1}(0) & C_{X_nX_2}(0) & .. & \varphi^2_{X_n} \end{bmatrix} \tag{54}$$

The on-diagonal terms are the mean square values of each random process; the off-diagonal terms are the cross-covariance functions (i.e. the cross-correlation functions for a zero mean process) of all the possible couples of processes, at the time lag $\tau = 0$.

n-variate normal process

A random stationary process **X**(t) has a normal n-variate distribution if all the couples of the random processes that constitute **X**(t) have a joint density function of order 2 given by the Eq. (24).

A particular linear transformation

Let **Y**(t) be a n-variate process linked with the n-variate process **X**(t) by the relationship:

$$Y(t) = \mathbf{A}\mathbf{X}(t)$$

It can be shown that:

$$\boldsymbol{\mu}_{\mathbf{Y}} = \mathbf{A}\boldsymbol{\mu}_{\mathbf{X}}$$

$$\mathbf{C}_{\mathbf{Y}}(\tau) = \mathbf{A}\mathbf{C}_{\mathbf{X}}(\tau)\mathbf{A}^{T}$$

$$\mathbf{R}_{\mathbf{Y}}(\tau) = \mathbf{A}\mathbf{R}_{\mathbf{X}}(\tau)\mathbf{A}^{T}$$

$$\mathbf{S}_{\mathbf{Y}}(\omega) = \mathbf{A}\mathbf{S}_{\mathbf{X}}(\omega)\mathbf{A}^{T}$$