RANDOM PROCESSES

Definitions

Let us consider an experiment whose outcome is a random function of time (the seismic motion, the time-history of a wind velocity, the equation of motion of a S.D.O.F. system). Each time-history representing one outcome of the experiment is indicated by $x^{(j)}(t)$ $(j=1,2,...)$ and is called sample function. The set of all the possible sample functions (Fig. 1) associated with the same physical phenomenon and registered in the same conditions is indicated by X(t) and is called random process or stochastic process or random function.

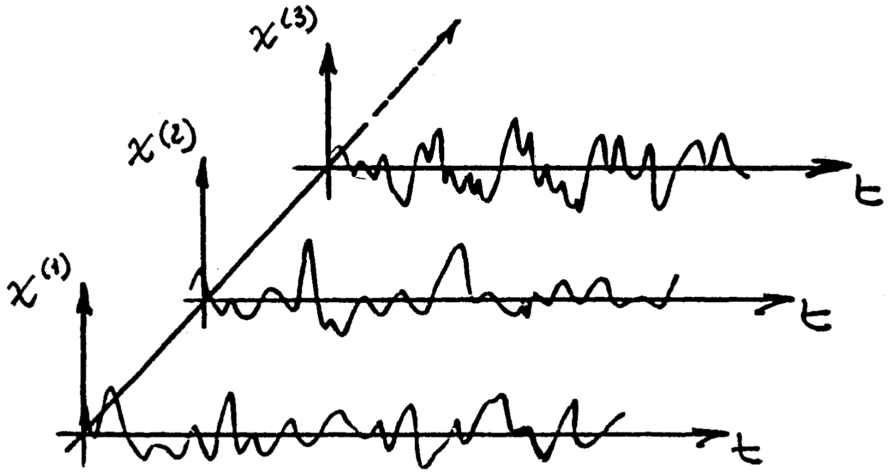

Fig. 1

Let us consider the random process X(t), and let us examine the values $x^{(j)}(t_1)$ $(j=1,2,...)$ that each sample function assumes at time $t = t_1$. The set of these values constitutes the random variable $X_1 = X(t_1)$ and is characterised by the density function $p_X(x_1,t_1)$ (written with this notation to remember the extraction of X at time t_1) (Fig. 2).

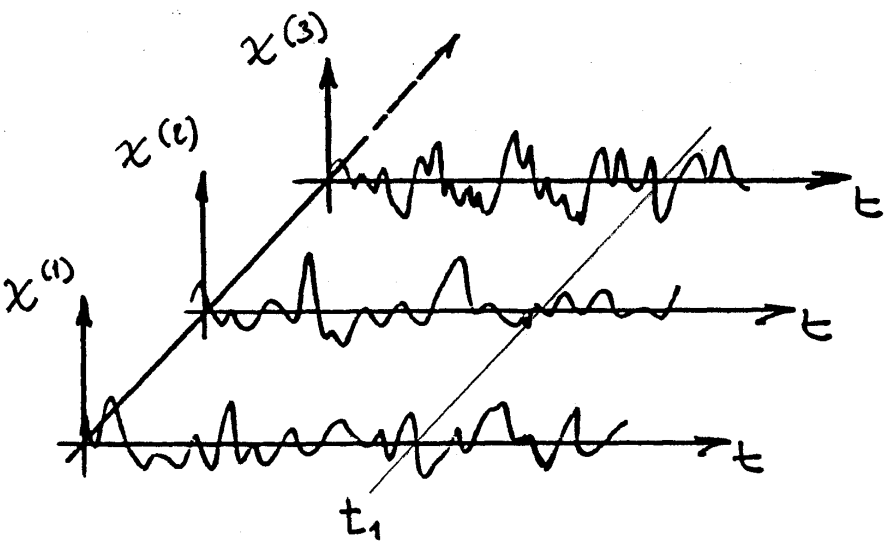

Fig. 2

Assuming t_1 as variable, $X(t)$ can be interpreted as a random variable depending on time.

Let $X_1(t_1), X_2(t_2), .. X_n(t_n)$ be a family (or a vector) of n random variables extracted from $X(t)$ at times $t_1, t_2, .. t_n$. It is characterised by the joint density function of order n: $p_{X_1 X_2 .. X_n}(x_1, x_2, .. x_n) = p_X(x_1, t_1; x_2, t_2; .. x_n, t_n)$ (Fig. 3).

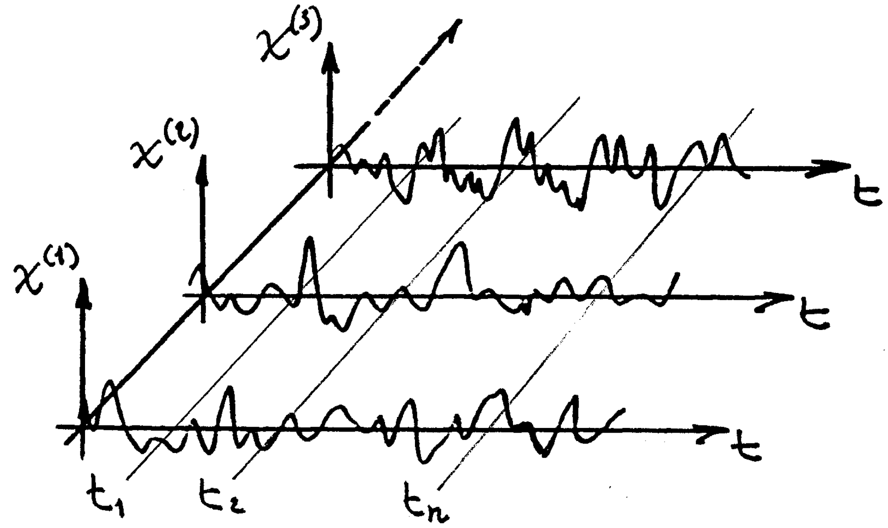

Fig. 3

Since the process $X(t)$ is a family of ∞ random variables, it constitutes an infinite-variate random vector. Thus, it is characterised by a joint density function of order ∞: $p_X(x_1, t_1; x_2, t_2; ...)$.

If the process is normal, the knowledge of the joint density functions of order 2, $p_X(x_1, t_1; x_2, t_2)$, for any value of t_1 and t_2 (Fig. 4), allows to derive the joint density function for any order n.

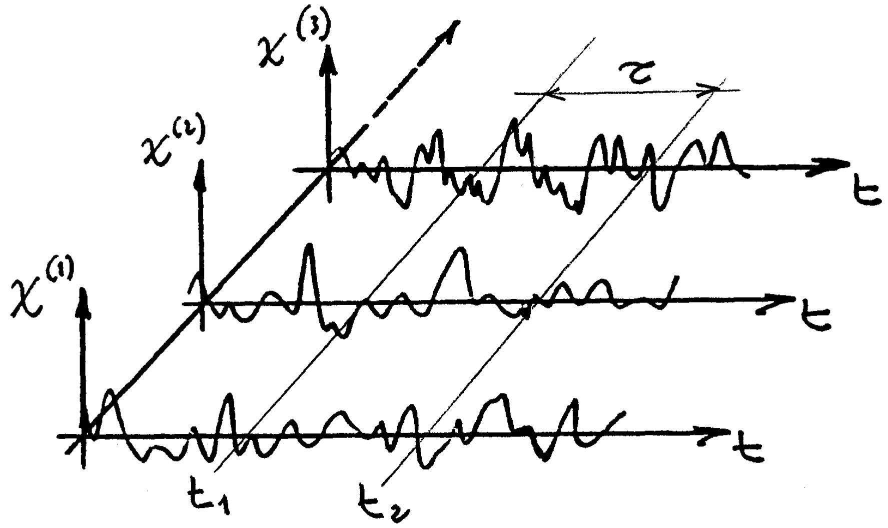

Fig. 4

Statistical averages of the first order

Let us consider Fig. 2 and the random variable $X_1 = X(t_1)$. It is described by the density function of the first order $p_X(x_1, t_1)$. The statistical averages of the first order include all the moments of the random variable X_1 that can be derived from $p_X(x_1, t_1)$.

The mean value of the random process is defined as:

$$\boxed{\mu_X(t_1) = E\left[X(t_1)\right] = \int_{-\infty}^{\infty} x_1 p_X(x_1, t_1)\, dx_1} \tag{1}$$

The mean square value of the random process is defined as:

$$\boxed{\varphi_X^2(t_1) = E\left[X^2(t_1)\right] = \int_{-\infty}^{\infty} x_1^2 p_X(x_1, t_1)\, dx_1} \tag{2}$$

The variance of the random process is defined as:

$$\boxed{\sigma_x^2(t_1) = E\left[\left\{X(t_1) - \mu_X(t_1)\right\}^2\right] = \int_{-\infty}^{\infty} \left[x_1 - \mu_X(t_1)\right]^2 p_X(x_1, t_1)\, dx_1} \tag{3}$$

Expanding Eq. (3) it follows:

$$\sigma_x^2(t_1) = \int_{-\infty}^{\infty} x_1^2 p_X(x_1, t_1)\, dx_1 - 2\mu_X(t_1) \int_{-\infty}^{\infty} x_1^2 p_X(x_1, t_1)\, dx_1 +$$
$$+\mu_x^2(t_1) \int_{-\infty}^{\infty} p_X(x_1, t_1)\, dx_1 = \varphi_X^2(t_1) - 2\mu_X(t_1) + \mu_x^2(t_1) \cdot 1 \Rightarrow$$

$$\boxed{\sigma_x^2(t_1) = \varphi_X^2(t_1) - \mu_x^2(t_1)} \tag{4}$$

All the statistical averages of the first order are (deterministic) functions of the generic time t_1.

Statistical averages of the second order

Let us consider the process $X(t)$, and let us examine the values $x^{(j)}(t_1)$ and $x^{(j)}(t_2)$ $(j=1,2,...)$ assumed by each sample function at times t_1 and t_2 (Fig. 4). The set of these values constitutes a couple of random variables $X_1 = X(t_1)$ and $X_2 = X(t_2)$ characterised by the joint density function of the second order $p_X(x_1,t_1;x_2,t_2)$. The statistical averages of the second order include all the joint moments of X_1 e X_2; they can be derived from $p_X(x_1,t_1;x_2,t_2)$.

The auto-correlation function of the process is defined as:

$$\boxed{R_{XX}(t_1,t_2) = E\left[X(t_1)X(t_2)\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 p_X(x_1,t_1;x_2,t_2)\,dx_1 dx_2} \tag{5}$$

The auto-covariance function of the process is defined as:

$$\boxed{\begin{aligned} C_{XX}(t_1,t_2) &= E\left[\{X(t_1)-\mu_X(t_1)\}\{X(t_2)-\mu_X(t_2)\}\right] = \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[x_1-\mu_X(t_1)\right]\left[x_2-\mu_X(t_2)\right]p_X(x_1,t_1;x_2,t_2)\,dx_1\,dx_2 \end{aligned}} \tag{6}$$

It follows that:

$$\boxed{C_{XX}(t_1,t_2) = R_{XX}(t_1,t_2) - \mu_X(t_1)\mu_X(t_2)} \tag{7}$$

The normalised auto-covariance function of the process is defined as:

$$\boxed{\rho_{XX}(t_1,t_2) = \frac{C_{XX}(t_1,t_2)}{\sigma_X(t_1)\sigma_X(t_2)}} \tag{8}$$

The prefix "auto" indicates that the random variables $X(t_1)$ and $X(t_2)$ are extracted from the same random process $X(t)$.

From Eqs. (5), (6), (8) the following properties derive:

$$\begin{aligned} R_{XX}(t_1,t_2) &= R_{XX}(t_2,t_1) \\ C_{XX}(t_1,t_2) &= C_{XX}(t_2,t_1) \\ \rho_{XX}(t_1,t_2) &= \rho_{XX}(t_2,t_1) \end{aligned} \tag{9}$$

Moreover:

$$\begin{aligned} R_{XX}(t_1,t_1) &= \varphi_X^2(t_1) \\ C_{XX}(t_1,t_1) &= \sigma_x^2(t_1) \\ \rho_{XX}(t_1,t_1) &= 1 \end{aligned} \tag{10}$$

Stationary processes

A random process is defined as strongly stationary when its joint density functions of any order n are independent of any translation τ of the origin of the axis of time:

$$p_X(x_1, t_1) = p_X(x_1, t_1 + \tau) \tag{11a}$$

$$p_X(x_1, t_1; x_2, t_2) = p_X(x_1, t_1 + \tau; x_2, t_2 + \tau) \tag{11b}$$

…… ……

$$p_X(x_1, t_1; x_2, t_2; ..x_n, t_n) = p_X(x_1, t_1 + \tau; x_2, t_2 + \tau; ..x_n, t_n + \tau) \tag{11n}$$

Setting $\tau = -t_1$, it is immediate to verify that Eq. (11) involves the following properties:

(a) the density function of the first order is independent of time t_1;

(b) the joint density function of the second order depends on only the time interval $(t_2 - t_1)$;

....

(n) the joint density function of order n depends on the n–1 time intervals $(t_2 - t_1)$, $(t_3 - t_1)$, .. $(t_n - t_1)$.

A random process $X(t)$ is defined as weakly stationary when only the Eqs. (11a, b) are satisfied. Considering that normal processes are characterised by only the joint density functions of the second order, the weakly stationary processes will be called below "simply" stationary processes.

It is immediate to demonstrate that, in the class of the (weakly) stationary processes, the statistical averages of the first order are independent of time. From Eqs. (1, 2, 3):

$$\boxed{\mu_X(t_1) = \mu_X}$$

$$\boxed{\varphi_X^2(t_1) = \varphi_X^2}$$

$$\boxed{\sigma_X^2(t_1) = \sigma_X^2}$$

Analogously, again in the class of (weakly) stationary processes, the statistical averages of the second order depend only on the time interval $\tau = t_2 - t_1$:

$$\boxed{R_{XX}(t_1, t_2) = R_{XX}(\tau)}$$

$$\boxed{C_{XX}(t_1, t_2) = C_{XX}(\tau)}$$

$$\boxed{\rho_{XX}(t_1, t_2) = \rho_{XX}(\tau)}$$

The interval τ is referred to as the time lag.

The auto-correlation function of a stationary process:

$$R_{XX}(r) = E[X(t)X(t+r)] \tag{12}$$

has several noteworthy properties (Fig. 5):

1) Setting $\tau = 0$ in Eq. (1):

$$\boxed{R_{XX}(0) = E\left[X^2(t)\right] = \varphi_X^2} \tag{13}$$

2) $R_{XX}(\tau) = C_{XX}(\tau) + \mu_X^2 = \rho_{XX}(\tau)\sigma_X^2 + \mu_X^2$. Thus, since $|\rho_{XX}(\tau)| \leq 1$:

$$\mu_X^2 - \sigma_X^2 < R_{XX}(\tau) < \mu_X^2 + \sigma_X^2 = \varphi_X^2 \quad \forall\tau \Rightarrow$$
$$\boxed{|R_{XX}(\tau)| < R_{XX}(0) \ \ \forall\tau} \tag{14}$$

3) For $|\tau|$ tending to infinite, the couple of random variables $X(t)$, $X(t+\tau)$ tends to become not correlated $(\rho_{XX} = 0) \Rightarrow$

$$\boxed{\lim_{|\tau|\to\infty} R_{XX}(\tau) = \mu_X^2} \tag{15}$$

4) Setting $\bar{t} = t + \tau$, Eq. (12) becomes $R_{XX}(\tau) = E\left[X(\bar{t}-\tau)X(\bar{t})\right]$. The comparison with Eq. (12) shows that $R_{XX}(\tau)$ is a symmetric function with respect to τ:

$$\boxed{R_{XX}(\tau) = R_{XX}(-\tau)} \tag{16}$$

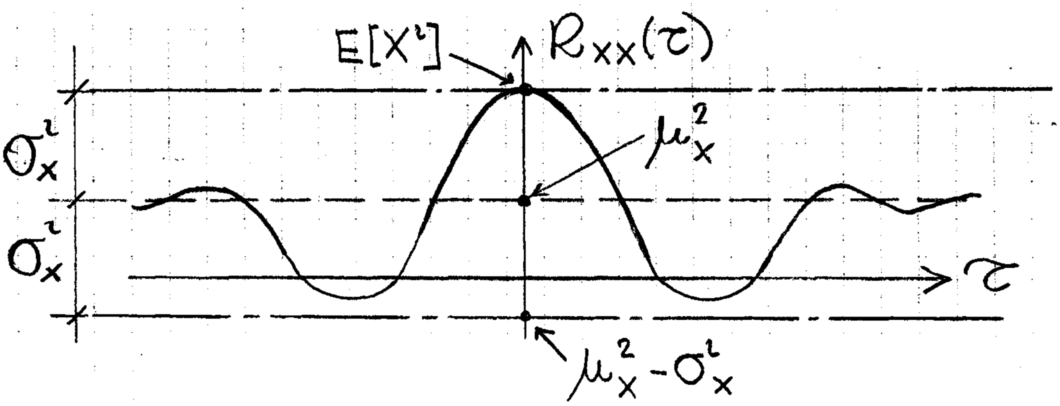

Fig. 5

The auto-covariance function of a stationary process:

$$C_{XX}(\tau) = E\left[\{X(t) - \mu_X\}\{X(t+\tau) - \mu_X\}\right] \tag{17}$$

has properties analogous to those of the auto-correlation function (Fig. 6). In particular:

$$\boxed{C_{XX}(0) = \sigma_X^2} \tag{18}$$

$$-\sigma_X^2 < C_{XX}(\tau) < \sigma_X^2 \quad \forall\tau \Rightarrow$$

$$\boxed{\left|C_{XX}(\tau)\right| < \sigma_X^2 \quad \forall\tau} \tag{19}$$

$$\boxed{\lim_{|\tau|\to\infty} C_{XX}(\tau) = 0} \tag{20}$$

$$\boxed{C_{XX}(\tau) = C_{XX}(-\tau)} \tag{21}$$

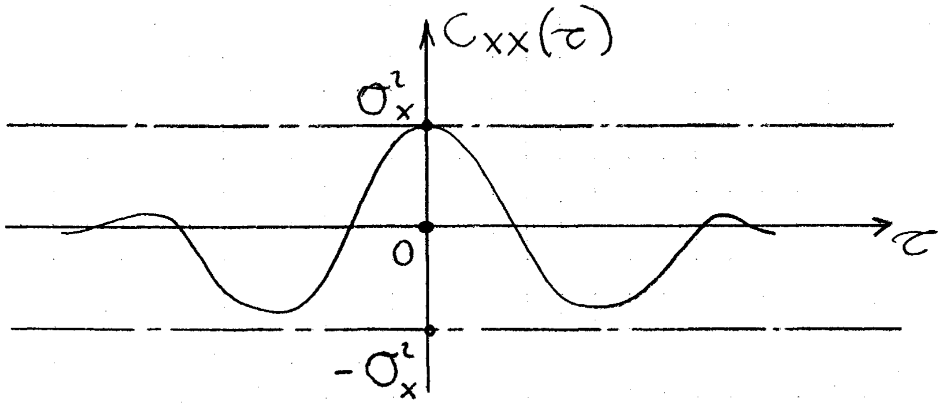

Fig. 6

A necessary condition to define a process as rigorously stationary is that it has no beginning and no end; in other words, each sample function shall be defined for any time belonging to **R**.

In the reality the hypothesis of stationarity is widely and reasonably applied when the nonstationary effects associated with the beginning of the process have a short duration in comparison with the length of the process itself. Based on this remark, the probabilistic concepts of stationarity and non-stationarity are clearly linked with the deterministic concepts of transient and quasi-steady regime.

In structural dynamics the hypothesis of stationarity is frequently used when the fundamental period of oscillation is much shorter than the duration T of the exciting force. It follows that the stationarity hypothesis is normally used to study wind actions (T ~ 600 – 3600 s). The same hypothesis is questionable (and even more often unreliable) for seismic actions (T = 15 – 30s).

Normal random process

The normal random process has a fundamental role in structural dynamics. For instance it provides an excellent representation for wind velocity and seismic motion. A stationary random process is defined as normal or Gaussian if the joint density function of the second order of the random variables $X(t_1)$ and $X(t_2)$ is normal for any t_1 and t_2 along **R** (Fig. 7). In this case it is also normal the joint density function of any order n of any n-variate random vector extracted from the random process at n arbitrary instants.

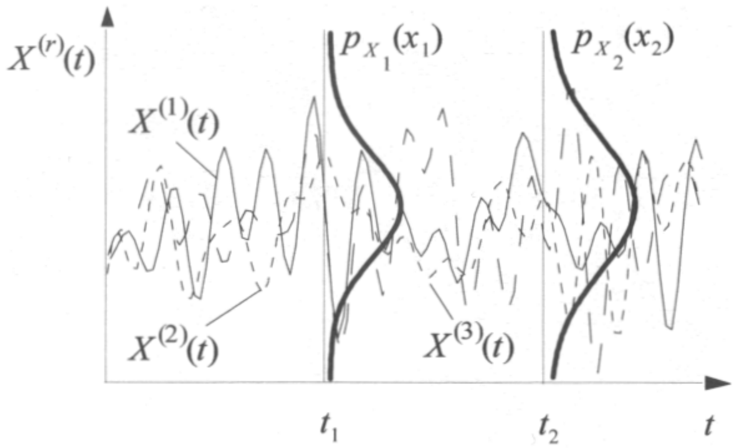

Fig. 7

Therefore:

$$p_X(x) = \frac{1}{\sigma_X\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2\right\} \tag{22}$$

$$p_X(x_1,x_2;\tau) = \frac{1}{2\pi\sigma_X^2\sqrt{1-\rho_{XX}^2(\tau)}} \cdot$$
$$\cdot\exp\left\{-\frac{(x_1-\mu_X)^2-2\rho_{XX}(\tau)(x_1-\mu_X)(x_2-\mu_X)+(x_2-\mu_X)^2}{2\sigma_X^2\left[1-\rho_{XX}^2(\tau)\right]}\right\} \tag{23}$$

$$p_{X_1X_2..X_n}(x_1,x_2,...x_n;\tau) = \frac{1}{(2\pi)^{n/2}\left|C_{XX}(\tau)\right|^{1/2}} \cdot$$
$$\cdot\exp\left\{-\frac{1}{2\left|C_{XX}(\tau)\right|}\sum_1^n {}_j\sum_1^n {}_k \left|C_{XX}(\tau)\right|_{jk}(x_j-\mu_X)(x_k-\mu_X)\right\} \tag{24}$$

Temporal averages of a sample function

All the quantities and the functions defined above have been deduced through statistical averages carried out on the whole of the sample functions of the process; this operation involves the knowledge of the density functions of the process.

Analogous quantities may be defined with reference to each sample function x(t) of the process, calculating suitable averages in the time domain. These averages are called temporal averages.

The following treatment deals with stationary processes and their sample functions. It also presumes that the sample functions x(t) are defined on an unlimited temporal interval T $\left(-\infty < t < +\infty\right)$.

The (temporal) mean of a sample function is defined as:

$$\boxed{\overline{x} = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x\left(t\right)dt} \tag{25}$$

The mean square value of a sample function is defined as:

$$\boxed{\overline{x^2} = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x^{2}\left(t\right)dt} \tag{26}$$

The variance of a sample function is defined as:

$$\boxed{\overline{x^{2}-\overline{x}^{2}} = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2}\left[\,x\left(\,t\right)-\overline{x}\right]^{2}\,dt = \overline{x}^{2}-\overline{x}^{2}} \tag{27}$$

The auto-correlation function of a sample function is defined as:

$$\boxed{r_{xx}\left(\tau\right) = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x\left(t\right)x\left(t+\tau\right)dt} \tag{28}$$

The auto-covariance function of a sample function is defined as:

$$\boxed{c_{xx}\left(\tau\right) = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2}\left[x\left(t\right)-\overline{x}\right]\left[x\left(t+\tau\right)-\overline{x}\right]dt = r_{xx}\left(\tau\right)-\overline{x}^{\,2}} \tag{29}$$

The normalised auto-covariance function is defined as:

$$\boxed{p_{xx}\left(\tau\right) = \frac{c_{xx}\left(\tau\right)}{\overline{x^{2}-\overline{x}^{2}}}} \tag{30}$$

The functions $r_{xx}\left(\tau\right)$ and $c_{xx}\left(\tau\right)$ have analogous properties to the functions $R_{XX}\left(\tau\right)$ and $C_{XX}\left(\tau\right)$.

Temporal averages of a process

The sets of the means, mean square values and variances of each sample function of a random stationary process constitute random variables referred to as, respectively, the temporal mean, mean square value and variance of a stationary process. These random variables are defined as:

$$\bar{X} = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} X(t)\,dt \tag{31}$$

$$\overline{X^2} = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} X^2(t)\,dt \tag{32}$$

$$\overline{X^2 - \bar{X}^2} = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} \left[X(t) - \bar{X}\right]^2 dt = \bar{X}^2 - \bar{X}^2 \tag{33}$$

The quantities $\bar{x}$ (Eq. 25), $\overline{x^2}$ (Eq. 26) and $\overline{x^2 - \bar{x}^2}$ (Eq. 27) associated with each sample function are occurrences of these random variables.

The sets of the auto-correlation, auto-covariance and normalised auto-covariance functions of each sample function of a random stationary process constitute random processes referred to as, respectively, the temporal auto-correlation, auto-covariance and normalised auto-covariance function of the stationary process (as functions of τ). These random processes are defined as:

$$\mathcal{R}_{XX}(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} X(t)X(t+\tau)\,dt \tag{34}$$

$$\mathcal{C}_{XX}(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} \left[X(t) - \bar{X}\right]\left[X(t+\tau) - \bar{X}\right] dt = \mathcal{R}_{XX}(\tau) - \bar{X}^2 \tag{35}$$

$$\mathcal{P}_{XX}(\tau) = \frac{\mathcal{C}_{XX}(\tau)}{\overline{\bar{X}^2 - \bar{X}^2}} \tag{36}$$

The functions $r_{xx}(\tau)$ (Eq. 27), $c_{xx}(\tau)$ (Eq. 28) and $p_{xx}(\tau)$ (Eq. 29) associated with each sample function are sample functions themselves of the new random process.

It is possible to demonstrate that the statistical averages of the process identify with the statistical averages of the corresponding temporal averages, i.e.:

$$\boxed{\mu_X = E\left[\bar{X}\right]} \tag{37}$$

$$\boxed{\varphi_X^2 = E\left[\overline{X^2}\right]} \tag{38}$$

$$\boxed{\sigma_X^2 = E\left[\overline{X^2 - \bar{X}^2}\right]} \tag{39}$$

$$\boxed{R_{XX}(\tau) = E\left[\mathcal{R}_{XX}(\tau)\right]} \tag{40}$$

$$\boxed{C_{XX}(\tau) = E\left[\mathcal{C}_{XX}(\tau)\right]} \tag{41}$$

$$\boxed{\rho_{XX}(\tau) = E\left[\mathcal{P}_{XX}(\tau)\right]} \tag{42}$$

For instance, considering Eq. (37), it results:

$$E\left[\overline{X}\right] = E\left[\lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} X(t)\,dt\right] =$$
$$= \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} E\left[X(t)\right]dt = \mu_X \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} dt = \mu_X$$

This treatment has great importance especially with reference to the numeric analysis of the sample functions of a random process deduced, for instance, through measurements or simulations.

Due to the definition of statistical average, the above equations are rigorously valid if the number of the available sample functions tends to infinite; this situation does not occur in the real cases and, even more, the number of the available sample functions is often very limited. In these cases the calculation of the statistical averages is critical and their evaluations is more appropriate by averaging the available temporal averages.

Ergodic processes

The ergodic processes constitute a sub-class of the stationary processes. A stationary process $X(t)$ is defined as ergodic when all its statistical properties can be determined from only one sample function $x(t)$ of the process. Since all the statistical properties can be interpreted as statistical averages of temporal averages, a process can be defined as ergodic when its statistical averages coincide with the temporal averages:

$$\mu_X = \overline{X} \tag{43}$$

$$\varphi_X^2 = \overline{X^2} \tag{44}$$

$$\sigma_X^2 = \overline{X^2 - \overline{X}^2} \tag{45}$$

$$R_{XX}(\tau) = \mathcal{R}_{XX}(\tau) \tag{46}$$

$$C_{XX}(\tau) = \mathcal{C}_{XX}(\tau) \tag{47}$$

$$\rho_{XX}(\tau) = \mathcal{P}_{XX}(\tau) \tag{48}$$

Power spectral density

Let us consider a stationary random process. The power spectral density, or more simply the power spectrum $S_{XX}(\omega)$ of the random process X(t), is defined as:

$$\boxed{S_{XX}(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}C_{XX}(\tau)e^{-i\omega\tau}d\tau} \tag{49}$$

Unless the factor $1/2\pi$, it coincides with the Fourier transform of the auto-covariance function $C_{XX}(\tau)$ of X(t). The auto-covariance function is the inverse Fourier transform (unless the factor 2π) of the power spectral density $S_{XX}(\omega)$ of $X(t)$:

$$\boxed{C_{XX}(\tau)=\int_{-\infty}^{\infty}S_{XX}(\omega)e^{i\omega\tau}d\omega} \tag{50}$$

Eqs. (49) and (50) are referred to as the Wiener-Khintchine equations.
$S_{XX}(\omega)$ exists if $C_{XX}(\tau)$ is absolutely integrable:

$$\int_{-\infty}^{\infty}\left|C_{XX}(\tau)\right|d\tau<\infty$$

If the random process is zero mean, the auto-covariance function $C_{XX}(\tau)$ coincides with the auto-correlation function $R_{XX}(\tau)$. So, the above equations hold also replacing $C_{XX}(\tau)$ by $R_{XX}(\tau)$.

The power spectral density has several noteworthy properties:

1) Applying the Euler's formula to Eq. (49):

$$S_{XX}(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}C_{XX}(\tau)\cos(\omega\tau)d\tau-\frac{i}{2\pi}\int_{-\infty}^{\infty}C_{XX}(\tau)\sin(\omega\tau)d\tau$$

Moreover, remembering that $C_{XX}(\tau)$ is a real symmetric function (of τ), then:

$$S_{XX}(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}C_{XX}(\tau)\cos(\omega\tau)d\tau$$

Thus, $S_{XX}(\omega)$ is a real symmetric function of ω:

$$\boxed{S_{XX}(\omega)=S_{XX}(-\omega)} \tag{51}$$

Moreover, it is a non-negative function:

$$\boxed{S_{XX}(\omega)\geq 0 \quad \forall\omega} \tag{52}$$

2) Setting in Eq. (50) $\tau = 0$:

$$C_{XX}(0) = \int_{-\infty}^{\infty} S_{XX}(\omega)\, d\omega$$

Thus, being $C_{XX}(0) = \sigma_X^2$:

$$\boxed{\sigma_X^2 = \int_{-\infty}^{\infty} S_{XX}(\omega)\, d\omega} \tag{53}$$

This means that the variance of the process is the area under the power spectral density.

3) The elementary area $2S_{XX}(\omega)d\omega$ is the contribution to σ_X^2 given by the harmonic components of the process with circular frequency in the interval $(\omega, \omega + d\omega)$ (Fig. 8). Thus, the power spectral density describes the power or the harmonic content of the process.

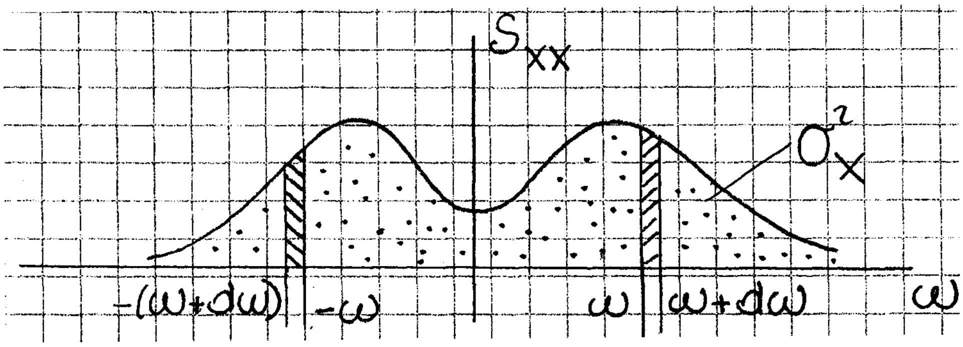

Fig. 8

Power spectral density of a sample function

Let us consider a sample function $x(t)$ of a stationary random process and let us assume, for sake of simplicity, that the temporal mean $\overline{x}$ of $x(t)$ is null (Fig. 9).

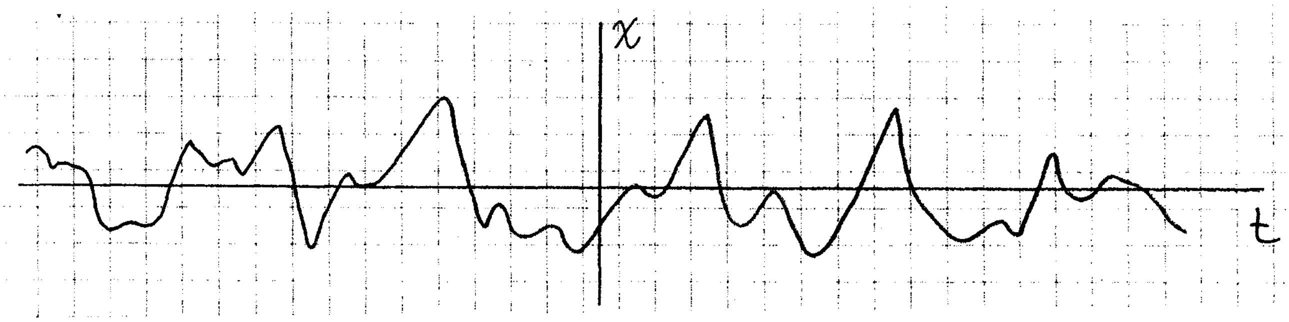

Fig. 9

Such a function cannot be expanded in a Fourier series since, in general, it is not periodic. In addition, it cannot be expressed through a Fourier integral: since the stationarity excludes that x(t) tends to zero for $|t|$ tending to infinite, it is obvious that x(t) is not absolutely integrable.

Thus the paradox occurs that, in the most simple case of a sample function belonging to a stationary process, the fundamental tools of the harmonic calculus do not apply. This shortcoming can be overcome by means of two alternative approaches:

1. using more powerful mathematical tools as the generalised Fourier transforms and the Fourier-Stieltjes integrals;
2. developing the treatment in a “limit” form using the classical tools previously described.

With this second aim, let us consider a new function $x_T(t)$ defined as (Fig. 10):

$$\begin{aligned} & x_T(t) = x(t) \quad \text{for } t \in (-T/2, T/2) \\ & x_T(t) = 0 \quad \text{elsewhere} \end{aligned} \tag{54}$$

In other words x_T identifies with x in $(-T/2, T/2)$, being null outside this interval.

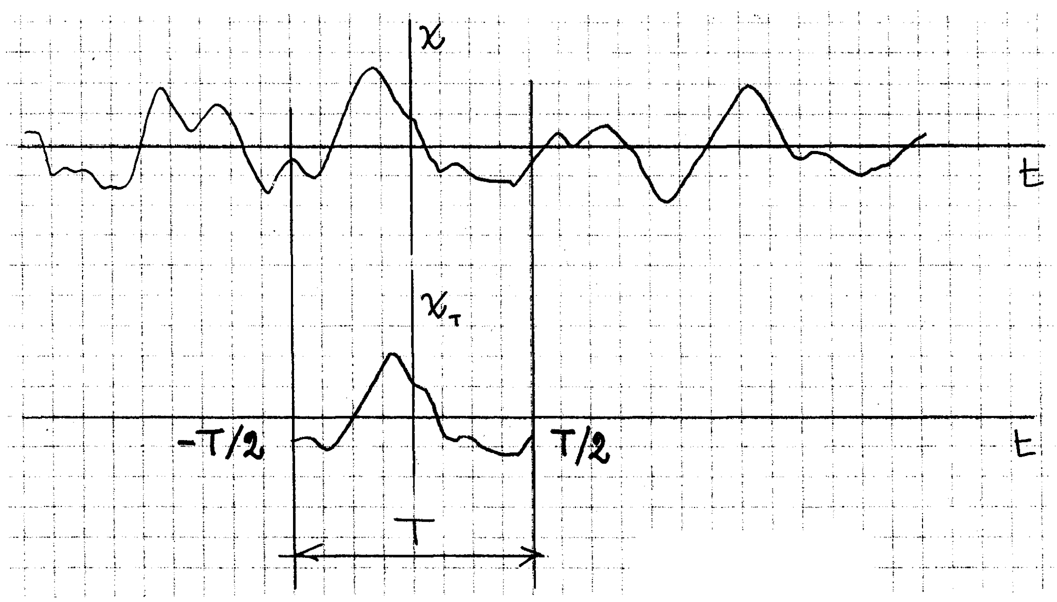

Fig. 10

Eq. (54) may be expressed through the Fourier integral:

$$x_T(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{x}_T(\omega)\, e^{i\omega t}\, d\omega \tag{55}$$

$$\tilde{x}_T(\omega) = \int_{-\infty}^{\infty} x_T(t)\, e^{-i\omega t}\, dt = \int_{-T/2}^{T/2} x(t)\, e^{-i\omega t}\, dt \tag{56}$$

The energy of $x_T(t)$ is defined as:

$$e_{xT} = \int_{-\infty}^{\infty} x_T^2(t)dt = \int_{-T/2}^{T/2} x^2(t)dt \tag{57}$$

It is finite when T is finite. It tends to infinite when T tends to infinite.
The power of $x_T(t)$ is its energy per unit time. It is given by the relationship:

$$p_{xT} = \frac{e_{xT}}{T} = \frac{1}{T}\int_{-T/2}^{T/2} x^2(t)dt \tag{58}$$

It is finite also when T tends to infinite.

Let us consider again Eq. (55), and let us multiply both its members by $x_T(t)$; then, let us execute the integral over t between -∞ and +∞. It results:

$$\int_{-\infty}^{\infty} x^2(t)\,dt = \int_{-\infty}^{\infty}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{x}(\omega)\,e^{-i\omega t}\,d\omega\right]x(t)\,dt =$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{x}(\omega)\left[x(t)\,e^{i\omega t}\,dt\right]d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{x}(\omega)\tilde{x}^*(\omega)\,d\omega \Rightarrow$$

$$\int_{-\infty}^{\infty} x^2(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\tilde{x}(\omega)|^2\, d\omega \tag{59}$$

Eq. (59) is referred to as the Parseval theorem and represents the basic integral transformation tool from the time domain to the frequency domain and viceversa. From this theorem it follows that the power of $x_T(t)$ is given by:

$$p_{xT}(t) = \frac{1}{2\pi T}\int_{-\infty}^{\infty} |\tilde{x}_T(\omega)|^2 d\omega \tag{60}$$

where $|\tilde{x}_T(\omega)|^2 d\omega / 2\pi T$ is the contribution to the total power given by the harmonic components of $x_T(t)$ within $(\omega, \omega + d\omega)$. Thus, the limit $|\tilde{x}_T(\omega)|^2 d\omega / 2\pi T$ for $T \rightarrow \infty$ is the contribution to the total power given by the harmonic components of $x(t)$ within $(\omega, \omega + d\omega)$. Applying this concept, the power spectral density function of the sample function $x(t)$ is defined as:

$$s_{xx}(\omega) = \lim_{T\to\infty} \frac{\left|\tilde{x}_T(\omega)\right|^2}{2\pi T} \tag{61}$$

Eq. (61) has several noteworty properties:

1) The power spectral density is a real, not negative, symmetric function of ω:

$$s_{xx}(\omega) = s_{xx}(-\omega) \tag{62}$$

2) The variance of $x(t)$ is given by:

$$\sigma_x^2 = \int_{-\infty}^{\infty} s_{xx}(\omega)\, d\omega \tag{63}$$

Demonstration: Remembering that $\overline{x} = 0$, then $\overline{x}^2 = \sigma_x^2$. Thus:

$$\begin{aligned}
\sigma_x^2 &= \overline{x}^2 = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x^2(\,t\,)\ dt = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x_T^2(\,t\,)\ dt = \\
&= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x^2(t) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{x}_T(\omega)\ e^{i\omega t}\ d\omega \right] dt = \\
&= \lim_{T\to\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{x}_T(\omega) \left[\int_{-T/2}^{T/2} x^2(t)\, e^{i\omega t}\ dt \right] d\omega = \\
&= \lim_{T\to\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{x}_T(\omega)\, \tilde{x}_T^*(\omega)\, d\omega \ \int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{\left|\tilde{x}_T(\omega)\right|^2}{2\pi T}\, d\omega \Rightarrow (63)
\end{aligned}$$

Thanks to Eq. (63), the elementary area $2 s_{XX}(\omega)\, d\omega$ is the contribution to σ_x^2 given by the harmonic components with circular natural frequency within the interval $(\omega, \omega + d\omega)$ (Fig. 11).

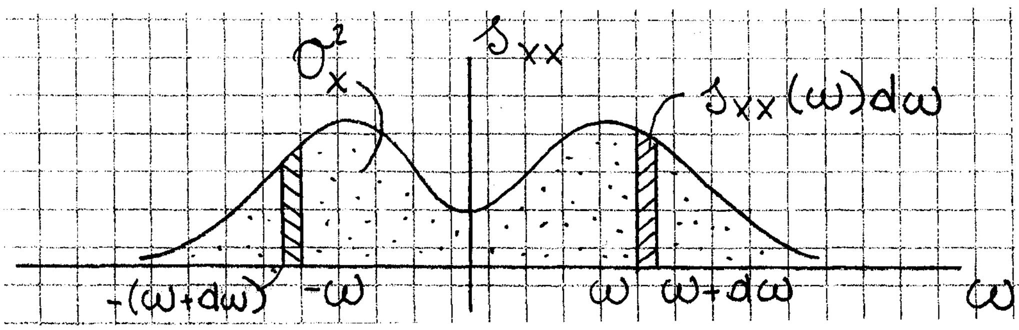

Fig. 11

3) The power spectral density $s_{xx}(\omega)$ is the Fourier transform (unless the factor 1/2π) of the auto-covariance function $c_{xx}(\tau)$ of $x(t)$. Thus $c_{xx}(\tau)$ is the inverse Fourier transform (unless the factor 2π) of the power spectral density $s_{xx}(\omega)$ of $x(t)$:

$$s_{xx}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} c_{xx}(\tau)\, e^{-i\omega\tau}\, d\tau \tag{64}$$

$$c_{xx}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} s_{xx}(\omega) e^{i\omega\tau} d\omega \tag{65}$$

Analogously to Eqs. (49), (50), also the Eqs. (64), (65) are referred to as the Wiener-Khintchine equations.

Dem: $c_{xx}(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x(t)x(t+\tau)dt =$

$$= \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x_T(t)x_T(t+\tau)dt \Rightarrow$$

$$\int_{-\infty}^{\infty} c_{xx}(\tau)e^{-i\omega\tau}d\tau = \int_{-\infty}^{\infty}\left[\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} x_T(t)x_T(t+\tau)dt\right] e^{-i\omega\tau}d\tau =$$

$$= \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} x_T(t)e^{i\omega t}dt\int_{-\infty}^{\infty} x_T(t+\tau)e^{-i\omega(t+\tau)}d\tau =$$

$$= \lim_{T\to\infty}\frac{1}{T}\tilde{x}_T^*(\omega)\int_{-T/2}^{T/2} x_T(\eta)e^{-i\omega\eta}d\eta =$$

$$= \lim_{T\to\infty}\frac{1}{T}\tilde{x}_T^*(\omega)\tilde{x}_T(\omega) = \lim_{T\to\infty}\frac{\left|\tilde{x}(\omega)\right|^2}{T} = 2\pi s_{xx}(\omega) \Rightarrow (64)$$

Power spectral density of a process

The set of the temporal auto-covariance functions $c_{xx}(\tau)$ of each sample function $x(t)$ of $X(t)$ constitutes the process $\mathcal{C}_{XX}(\tau)$. Analogously, the set of the power spectral densities $s_{xx}(\omega)$ of each sample function $x(t)$ constitutes the process $\mathsf{S}_{XX}(\omega)$. From Eqs. (64) and (65), it derives:

$$\mathsf{S}_{XX}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{C}_{XX}(\tau)e^{-i\omega\tau}d\tau \tag{66}$$

$$\mathcal{C}_{XX}(\tau) = \int_{-\infty}^{\infty} \mathsf{S}_{XX}(\omega)e^{i\omega\tau}d\omega \tag{67}$$

Since $C_{XX}(\tau) = E\left[\mathcal{C}_{XX}(\tau)\right]$, then:

$$\boxed{S_{XX}(\omega) = E\left[\mathsf{S}_{XX}(\omega)\right]} \tag{68}$$

In other words, the power spetral density of the process is the statistical average of the power spectral densities of each sample function of the process.

Derivation of stationary processes

The derivation of the stationary processes needs much deeper considerations than those developed below. At an indicative level, let $X^{(n)}(t)$ be the n-th derivative of X(t) with respect to time t:

$$X^{(n)}(t) = \frac{d^n}{dt^n} X(t) \tag{69}$$

If X(t) is a stationary zero mean process, it can be demonstrated that:

$$\frac{d}{d\tau} C_{XX}(\tau) = C_{X\dot{X}}(\tau) = -C_{\dot{X}X}(\tau) \tag{70}$$

$$\frac{d^2}{d\tau^2} C_{XX}(\tau) = C_{X\ddot{X}}(\tau) = -C_{\dot{X}\dot{X}}(\tau) \tag{71}$$

Applying several times the Wiener-Khinchine equations, it follows:

$$C_{XX}(\tau) = \int_{-\infty}^{\infty} S_{XX}(\omega) e^{i\omega\tau} d\omega \tag{72}$$

$$\frac{d}{d\tau} C_{XX}(\tau) = C_{X\dot{X}}(\tau) = i\int_{-\infty}^{\infty} \omega S_{XX}(\omega) e^{i\omega\tau} d\omega \tag{73}$$

$$\frac{d^2}{d\tau^2} C_{XX}(\tau) = -C_{\dot{X}\dot{X}}(\tau) = -\int_{-\infty}^{\infty} \omega^2 S_{XX}(\omega) e^{i\omega\tau} d\omega \tag{74}$$

and, moreover:

$$S_{X\dot{X}}(\omega) = i\omega S_{XX}(\omega) \tag{75}$$

$$S_{\dot{X}\dot{X}}(\omega) = \omega^2 S_{XX}(\omega) \tag{76}$$

From Eqs. (72)-(76) it derives:

$$C_{XX}(0) = E\left[X^2(t)\right] = \sigma_X^2 = \int_{-\infty}^{\infty} S_{XX}(\omega) d\omega \tag{77}$$

$$C_{X\dot{X}}(0) = E\left[X(t)\dot{X}(t)\right] = i\int_{-\infty}^{\infty} \omega S_{XX}(\omega) d\omega = 0 \tag{78}$$

$$C_{\dot{X}\dot{X}}(0) = E\left[\dot{X}^2(t)\right] = \sigma_{\dot{X}}^2 = \int_{-\infty}^{\infty} \omega^2 S_{XX}(\omega) d\omega \tag{79}$$

Finally, it can be demonstrated that:

$$\boxed{S_{X^{(n)}X^{(n)}}(\omega) = \omega^{2n} S_{XX}(\omega)} \tag{80}$$

and then:

$$\boxed{\sigma^2_{X^{(n)}} = \int_{-\infty}^{\infty} S_{X^{(n)}X^{(n)}}(\omega)d\omega = \int_{-\infty}^{\infty} \omega^{2n} S_{XX}(\omega)d\omega} \tag{81}$$

Spectral moments

The unilateral power spectral density (or the unilateral power spectrum) $G_{XX}(\omega)$ is the following function (Fig. 12):

$$G_{XX}(\omega) = 2S_{XX}(\omega) \quad \text{per} \quad 0 \le \omega < +\infty \tag{82a}$$
$$G_{XX}(\omega) = 0 \quad \text{per} \quad \omega < 0 \tag{82b}$$

It is a real non negative function, defined for $\omega \ge 0$, which has the following property:

$$\sigma_X^2 = \int_0^{\infty} G_{XX}(\omega)d\omega \tag{83}$$

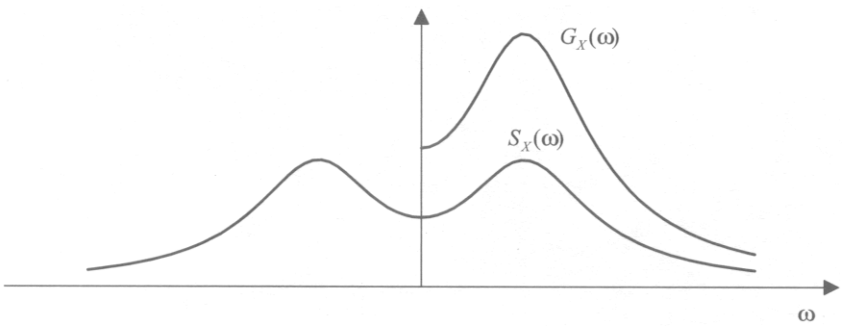

Fig. 12

Let us define as the spectral moments (or the Vanmarcke moments) the following quantities:

$$\lambda_{X,i} = \int_0^{\infty} \omega^i G_{XX}(\omega)d\omega \quad i = 0,1,2,... \tag{84}$$

In particular, the first three spectral moments have the form:

$$\lambda_{X,0} = \int_0^{\infty} G_{XX}(\omega)d\omega = \int_{-\infty}^{\infty} S_{XX}(\omega)d\omega = \sigma_X^2 \tag{85}$$

$$\lambda_{X,1} = \int_0^{\infty} \omega G_{XX}(\omega)d\omega \ne \int_{-\infty}^{\infty} \omega S_{XX}(\omega)d\omega = 0 \tag{86}$$

$$\lambda_{X,2} = \int_0^{\infty} \omega^2 G_{XX}(\omega)d\omega = \int_{-\infty}^{\infty} \omega^2 S_{XX}(\omega)d\omega = \sigma_{\dot{X}}^2 \tag{87}$$

The position $\omega_{X,1}$ of the barycentre of the area under $G_{XX}(\omega)$ is given by the relationship (Fig. 13):

$$\omega_{X,1} = \frac{\int_0^{\infty} \omega G_{XX}(\omega)d\omega}{\int_0^{\infty} G_{XX}(\omega)d\omega} = \frac{\lambda_{X,1}}{\lambda_{X,0}} \tag{88}$$

The radius of gyration of the area under $G_{XX}(\omega)$ is given by:

$$\omega_{X,2} = \left(\frac{\int_0^\infty \omega^2 G_{XX}(\omega)d\omega}{\int_0^\infty G_{XX}(\omega)d\omega} \right)^{1/2} = \sqrt{\frac{\lambda_{X,2}}{\lambda_{X,0}}} \tag{89}$$

It will be shown later that the quantity $\nu_X = \omega_{X,2}/2\pi = (\sigma_{\dot{X}}/\sigma_X)/2\pi$, called the expected frequency of the process X(t), has a fundamental role in random dynamics.

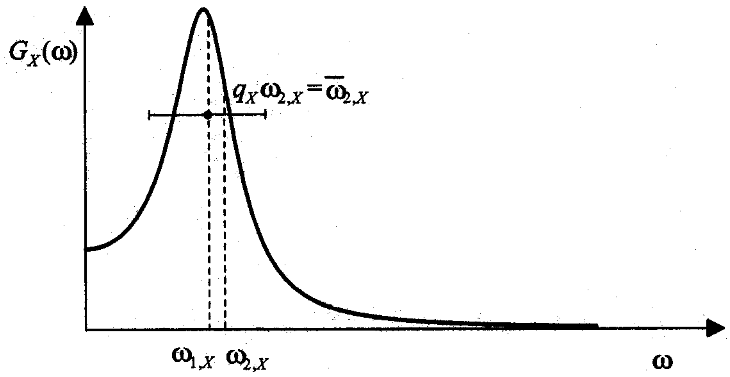

Fig. 13

The radius of gyration of the area under $G_{XX}(\omega)$ with respect to its barycentre, $\bar{\omega}_{X,2}$, provides a measure of the dispersion of the area around the barycentre (Fig. 13). Thus it offers an estimate of the amplitude of the spectral bandwidth containing the harmonic or power content of the process. It is defined as:

$$\bar{\omega}_{X,2} = \sqrt{\frac{1}{\lambda_{X,0}}\left(\lambda_{X,2} - \frac{\lambda_{X,1}^2}{\lambda_{X,0}}\right)} = q_X\omega_{X,2} \tag{90}$$

where:

$$q_X = \sqrt{1 - \frac{\lambda_{X,1}^2}{\lambda_{X,0}\lambda_{X,2}}} \tag{91}$$

is a non-dimensional quantity between 0 and 1, called the spectral bandwidth parameter. A small value of q_X is typical of a process with a harmonic content in a small frequency band. A large value of q_X is typical of a process with a harmonic content distributed over a large frequency band. The two limit cases $q_X = 0$ and $q_X = 1$ correspond, respectively, to $G_{XX}(\omega) = \lambda_{X,0}\delta(\omega-\omega_{X,1})$ and to $G_{XX}(\omega) = G_0 = 2S_0$ = constant.

Particular random processes

Four random processes characterised by particular properties are considered below: the sinusoidal, narrow band, broad band and white noise processes. They have increasingly wide spectral bandwidth.

Sinusoidal random process

A zero mean stationary random process is defined as sinusoidal (Fig. 14) if any sample function is given by the relationship:

$$x^{(j)}(t) = A\sin(\omega_0 t + \theta^{(j)}) \tag{92}$$

where the phase angle $\theta^{(j)}$ is the j-th occurrence of a random variable Θ uniformly distributed over the interval $[0, 2\pi]$:

$$p_\Theta(\theta) = \frac{1}{2\pi} \quad (0 \le \theta \le 2\pi) \tag{93}$$

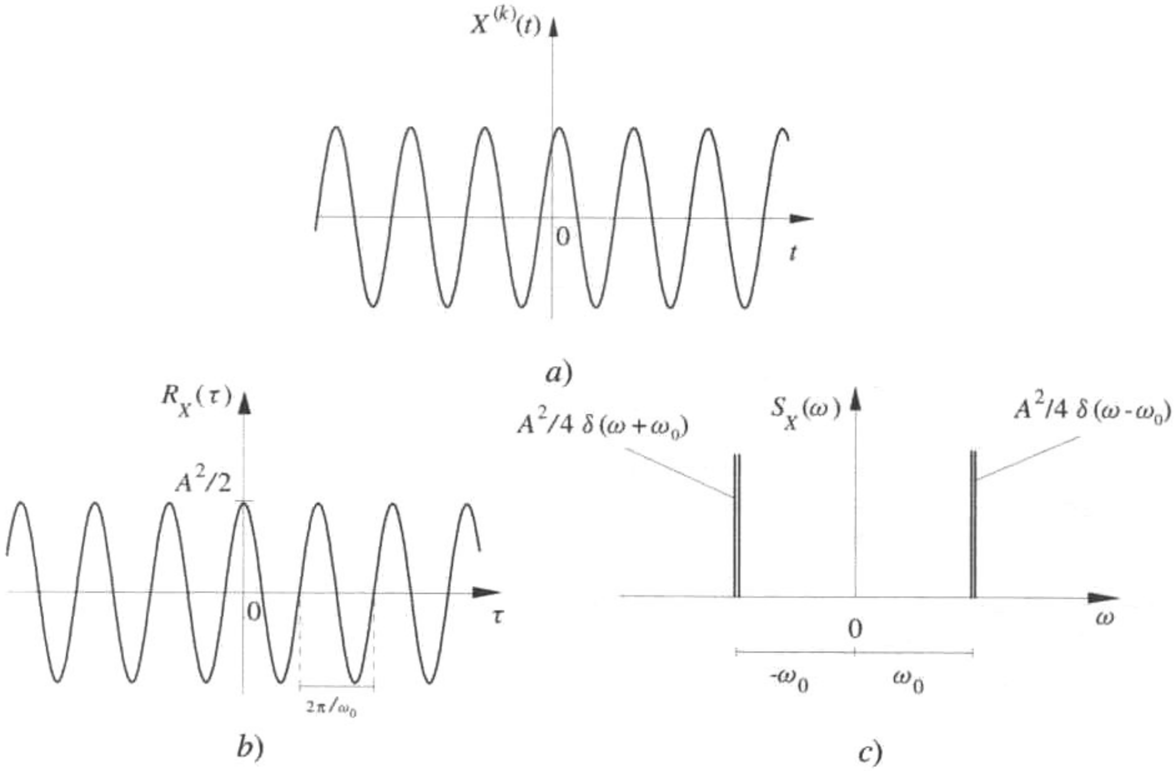

Fig. 14

The auto-covariance function coincides with the auto-correlation function and is given by:

$$C_{XX}(\tau) = E\left[X(t)X(t+\tau)\right] = A^2\int_0^{2\pi}\sin(\omega_0 t+\theta)\sin\left[\omega_0(t+\tau)+\theta\right]p(\theta)d\theta =$$

$$= \frac{A^2}{2\pi}\int_0^{2\pi}\sin(\omega_0 t+\theta)\sin\left[\omega_0(t+\tau)+\theta\right]d\theta \Rightarrow$$

$$C_{XX}(\tau) = \frac{A^2}{2}\cos(\omega_0 t) \tag{94}$$

Thus, the power spectral density results:

$$S_{XX}(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}C_{XX}(\tau)e^{-i\omega\tau}d\tau=\frac{A^2}{4\pi}\int_{-\infty}^{\infty}\cos(\omega_0\tau)e^{-i\omega\tau}d\tau\Rightarrow$$

$$S_{XX}(\omega)=\frac{A^2}{4}\left[\delta(\omega+\omega_0)+\delta(\omega-\omega_0)\right] \tag{95}$$

It follows that $\sigma_X^2=A^2/2$. Moreover, $\lambda_{X,i}=\omega_0^i A^2/2$. Finally, $q_X=0$.

Narrow band process

A stationary random process is defined as narrow band if its power spectral density is different from zero only in a limited frequency range with amplitude $B=|\omega_2-\omega_1|$, where $B/\omega_0\approx 0$, ω_0 being the mean value of B: $\omega_0=\pm(\omega_1+\omega_2)/2$. A narrow band process is defined as ideal (Fig. 15) if its power spectral density is given by:

$$S_{XX}(\omega)=S_0 \quad \text{for} \quad \omega_1\le|\omega|\le\omega_2 \tag{96a}$$

$$S_{XX}(\omega)=0 \quad \text{elsewhere} \tag{96b}$$

Thus: $\sigma_X^2=2S_0(\omega_2-\omega_1)$.

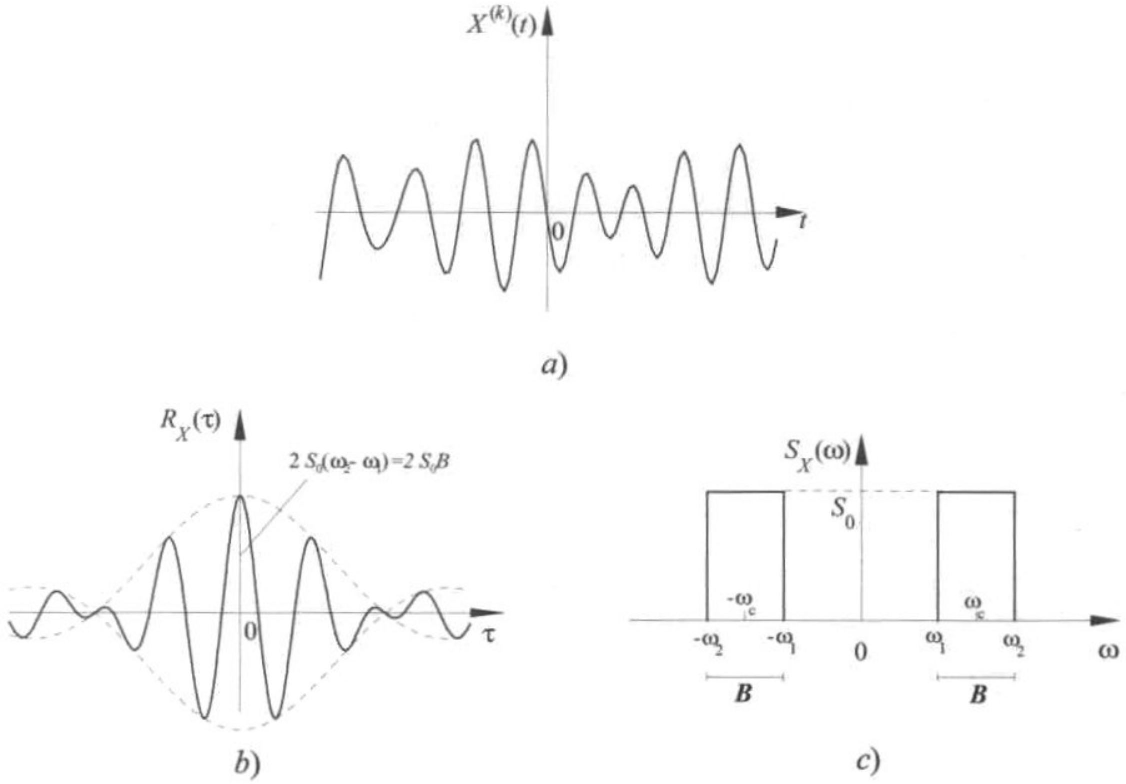

Fig. 15

The auto-covariance function is given by:

$$C_{XX}(\tau)=\int_{-\infty}^{\infty}S_{XX}(\omega)e^{i\omega\tau}d\omega=2\int_{-\infty}^{\infty}S_0\cos(\omega\tau)\,d\omega\Rightarrow$$

$$C_{XX}(\tau) = \frac{2S_0}{\tau}\left[\sin(\omega_2\tau) - \sin(\omega_1\tau)\right] = 2S_0B\left\{\frac{\sin\left(\frac{B}{2}\tau\right)}{\frac{B}{2}\tau}\right\}\cos(\omega_0\tau) \tag{97}$$

Moreover:

$$\lambda_{X,0} = 2S_0B \quad ; \quad \lambda_{X,1} = 2S_0B\omega_0 \quad ; \quad \lambda_{X,2} = 2S_0B\left(\omega_0^2 + \frac{B^2}{12}\right) \quad ; \quad q_X = \frac{B}{\sqrt{12\omega_0^2 + B^2}} \cong \frac{B}{\omega_0\sqrt{12}}$$

The sample functions of the narrow band random process are characterised by a harmonic content concentrated around the central circular frequency of the harmonic band. For B tending to 0 the narrow band process tends to the sinusoidal process.

Broad band process

A random stationary process is defined as a broad band process if the power spectral density is different from zero in a wide frequency band. A broad band process is defined as ideal (Fig. 16) if its power spectral density is given by the relationship:

$$S_{XX}(\omega) = S_0 \quad \text{for} \quad |\omega| \leq B \tag{98a}$$

$$S_{XX}(\omega) = 0 \quad \text{elsewhere} \tag{98b}$$

Thus: $\sigma_X^2 = 2S_0B$.

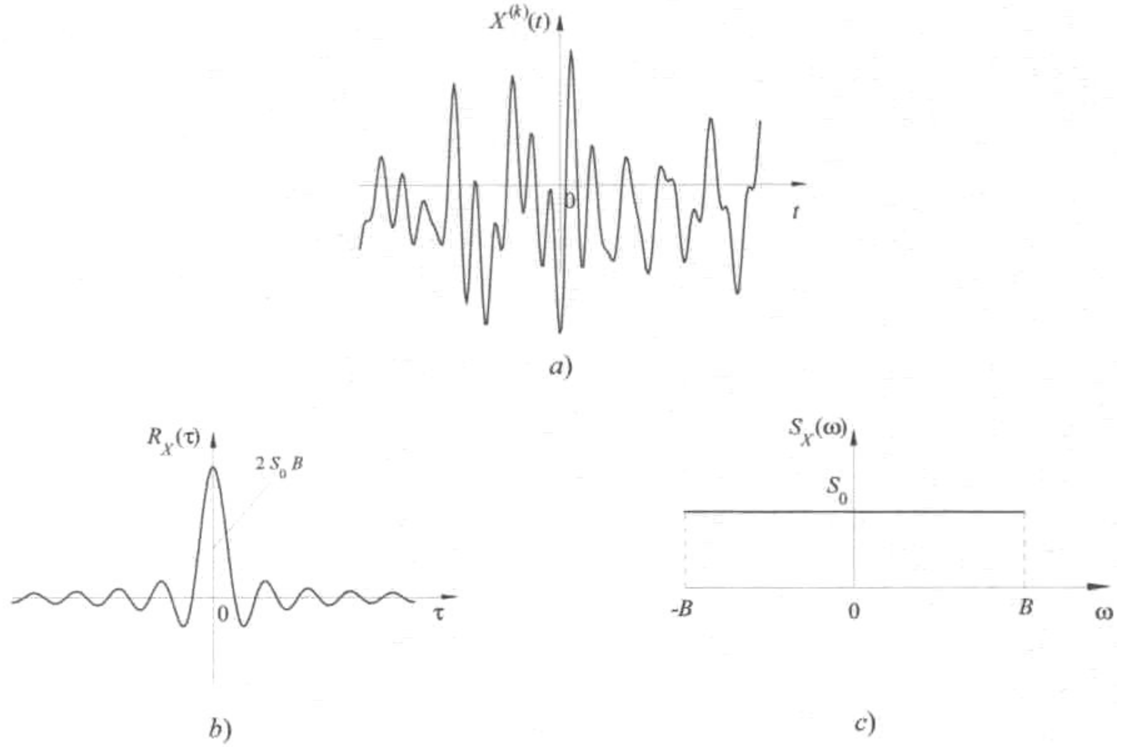

Fig. 16

The auto-covariance function is given by:

$$C_{XX}(\tau)=\int_{-\infty}^{\infty} S_{XX}(\omega)e^{i\omega\tau}d\omega=2\int_0^B S_0\cos(\omega\tau)\,d\omega \Rightarrow$$

$$C_{XX}(\tau)=2S_0B\left\{\frac{\sin(B\tau)}{B\tau}\right\} \tag{99}$$

Moreover:

$$\lambda_{X,0}=2S_0B \quad ; \quad \lambda_{X,1}=S_0B^2 \quad ; \quad \lambda_{X,2}=\frac{2S_0B^3}{3} \quad ; \quad q_X=\frac{1}{2}$$

The sample functions of the process have irregular shape due to the width of the harmonic content.

White random process
A stationary random process is defined as a white (noise) process (Fig. 17) if its power spectral density is constant over the whole frequency range. It is generally indicated by the symbol W(t):

$$S_{XX}(\omega)=S_0 \quad \text{for} \quad \forall\omega \tag{100}$$

Thus $\sigma_X^2=\infty$; therefore the white process is physically not realizable. However, its importance in structural dynamics is fundamental. Some of its properties will be discussed in the next sections.

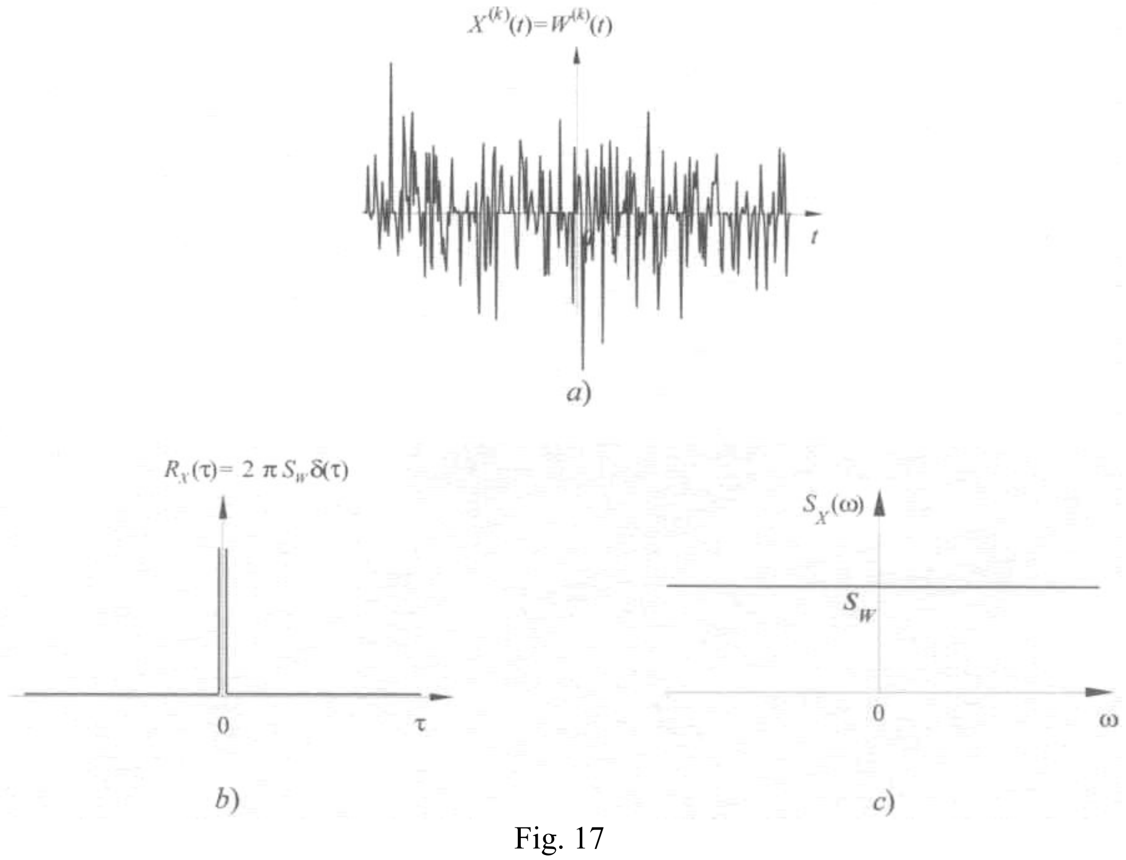

Fig. 17

The auto-covariance function is given by:

$$C_{XX}(\tau) = \int_{-\infty}^{\infty} S_{XX}(\omega) e^{i\omega\tau} d\omega = S_0 \int_{-\infty}^{\infty} e^{i\omega\tau} d\omega \Rightarrow$$

$$C_{XX}(\tau) = 2\pi S_0 \delta(\tau) \tag{101}$$

Thanks to this expression the white process is also referred to as the delta-correlated process. Moreover: $\lambda_{X,i} = \infty$ for $\forall i$, $q_X = 1$. The sample functions of the white process have maximum irregularity due to the infinite amplitude of the spectral band.

A particular linear transformation

Let Y(t) be a stationary random process proportional to the stationary random process X(t):

$$\boxed{Y(t) = aX(t)} \tag{102}$$

It results:

$$\mu_Y = E\left[Y(t)\right] = E\left[aX(t)\right] = aE\left[X(t)\right] \Rightarrow$$

$$\mu_Y = a\mu_X \tag{103}$$

$$C_{YY}(\tau) = E\left[Y(t)Y(t+\tau)\right] = E\left[aX(t) \cdot aX(t+\tau)\right] = a^2 E\left[X(t)X(t+\tau)\right] \Rightarrow$$

$$C_{YY}(\tau) = a^2 C_{XX}(\tau) \tag{104}$$

$$S_{XX}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} C_{YY}(\tau)\, e^{-i\omega\tau}\, d\tau = \frac{1}{2\pi}\int_{-\infty}^{\infty} a^2 C_{XX}(\tau) e^{-i\omega\tau} d\tau = a^2 \frac{1}{2\pi}\int_{-\infty}^{\infty} C_{XX}(\tau)\, e^{-i\omega\tau}\, d\tau \Rightarrow$$

$$\boxed{S_{YY}(\omega) = a^2 S_{XX}(\omega)} \tag{105}$$