COUPLE OF RANDOM PROCESSES

Definitions

Let us consider an experiment whose result is represented by a couple of random processes X(t) and Y(t) (for example the components of the seismic motion at the base of 2 piers of a viaduct, the wind velocities registered by 2 anemometers, the dynamic response of a 2-D.O.F. system). The couple of the processes X(t) and Y(t) is also called a 2-variate random process.

Let us consider the values $x^{(j)}(t_1)$ and $y^{(j)}(t_2)$ $(j = 1, 2, ...)$ assumed by the sample functions $x^{(j)}(t)$ and $y^{(j)}(t)$ of X(t) and Y(t) for $t = t_1$ and $t = t_2$ (Fig. 1). The set of these values constitutes the couple of random variables $X_1 = X(t_1)$ and $Y_2 = Y(t_2)$. They are characterised by the joint density function of the second order $p_{XY}(x_1, t_1; y_2, t_2)$. From this it is immediate to derive the marginal density functions of the first order of $X(t_1)$ and $Y(t_2)$:

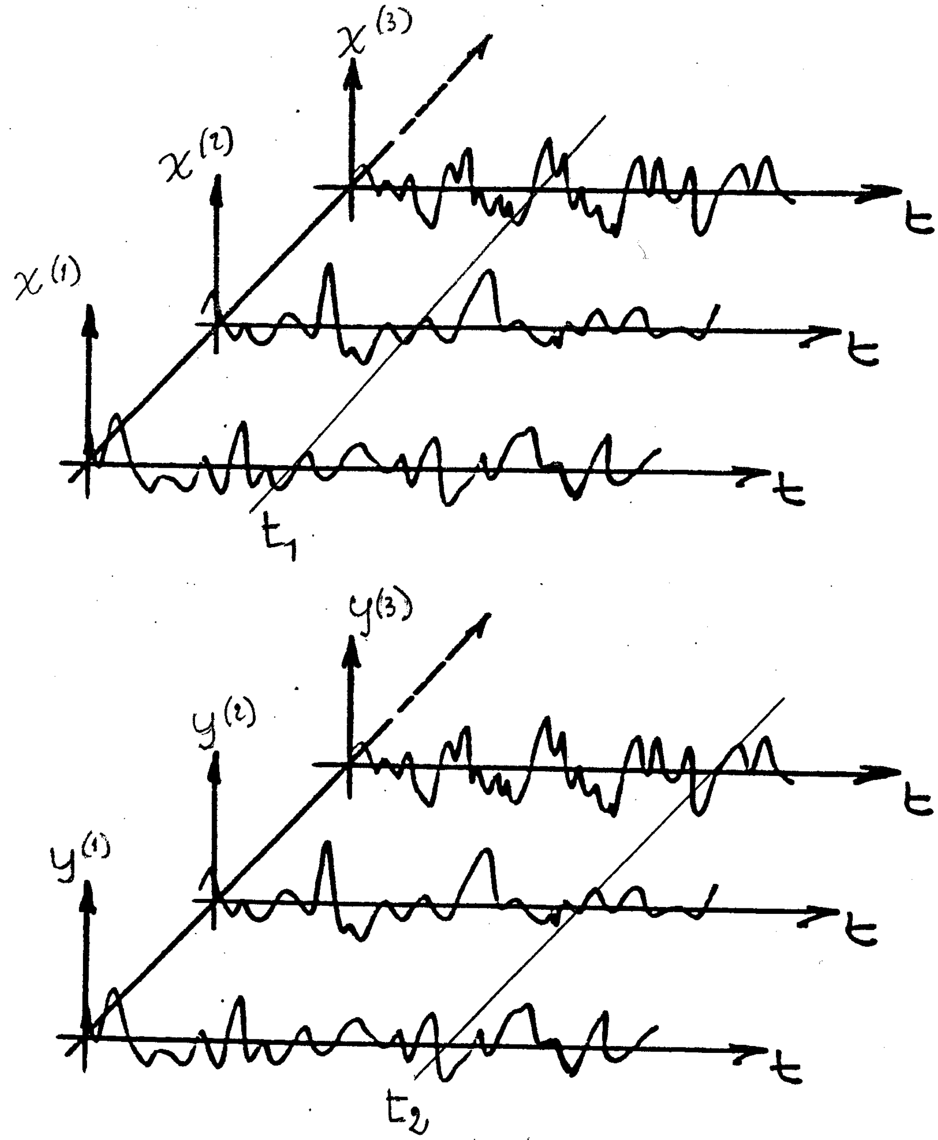

Fig. 1

$$p_X(x_1, t_1) = \int_{-\infty}^{\infty} p_{XY}(x_1, t_1; y_2, t_2)\, dy_2 \tag{1}$$

$$p_Y(y_2, t_2) = \int_{-\infty}^{\infty} p_{XY}(x_1, t_1; y_2, t_2)\, dx_1 \tag{2}$$

From the marginal density functions of the first order, it is immediate to derive the statistical averages of the first order already defined for each random process.

Statistical averages of the second order

The statistical averages of the second order are the joint indexes of $X_1 = X(t_1)$ e $Y_2 = Y(t_2)$; they can be derived from the joint density function of the second order $p_{XY}(x_1, t_1; y_2, t_2)$.

The cross–correlation function of $X(t)$ and $Y(t)$ is defined as:

$$\boxed{R_{XY}(t_1, t_2) = E\left[X(t_1)Y(t_2)\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 y_2 p_{XY}(x_1, t_1; y_2, t_2)\, dx_1 dy_2} \tag{3}$$

The cross–covariance function is defined as:

$$\boxed{\begin{aligned} C_{XY}(t_1, t_2) &= E\left[\left\{X(t_1) - \mu_X(t_1)\right\}\left\{Y(t_2) - \mu_Y(t_2)\right\}\right] = \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[x_1 - \mu_X(t_1)\right]\left[y_2 - \mu_Y(t_2)\right] p_{XY}(x_1, t_1; y_2, t_2)\, dx_1 dy_2 \end{aligned}} \tag{4}$$

It derives:

$$\boxed{C_{XY}(t_1, t_2) = R_{XY}(t_1, t_2) - \mu_X(t_1)\mu_Y(t_2)} \tag{5}$$

The normalised cross-covariance function is defined as:

$$\boxed{\rho_{XY}(t_1, t_2) = \frac{C_{XY}(t_1, t_2)}{\sigma_X(t_1)\sigma_Y(t_2)}} \tag{6}$$

The prefix “cross” indicates that $X(t_1)$ and $Y(t_2)$ are extracted from different processes (although associated with the same experiment). Eqs. (3), (4) and (6) involve the following properties:

$$\begin{aligned} R_{XY}(t_1, t_2) &= R_{YX}(t_2, t_1) \\ C_{XY}(t_1, t_2) &= C_{YX}(t_2, t_1) \\ \rho_{XY}(t_1, t_2) &= \rho_{YX}(t_2, t_1) \end{aligned} \tag{7}$$

Two random processes $X(t)$ and $Y(t)$ are defined as not correlated if:

$$\boxed{R_{XY}(t_1, t_2) = \mu_X(t_1)\mu_Y(t_2) \quad \forall t_1, t_2 \in \mathbf{R}} \tag{8}$$

$$\boxed{C_{XY}(t_1,t_2)=0 \quad \forall t_1,t_2 \in \mathbf{R}} \tag{9}$$

Two random processes $X(t)$ and $Y(t)$ are defined as <u>orthogonal</u> if:

$$\boxed{R_{XY}(t_1,t_2)=0 \quad \forall t_1,t_2 \in \mathbf{R}} \tag{10}$$

Stationary processes

A couple of random processes is defined as weakly stationary if the density functions of the first order and the joint density functions of the second order are independent of any translation τ of the origin of the axis of time:

$$p_X(x_1,t_1)=p_X(x_1,t_1+\tau) \tag{11a}$$
$$p_Y(y_2,t_2)=p_Y(y_2,t_2+\tau) \tag{11a}$$
$$p_{XY}(x_1,t_1;y_2,t_2)=p_{XY}(x_1,t_1+\tau;y_2,t_2+\tau) \tag{11b}$$

Assigning $\tau=-t_1$, it is immediate to show that Eq. (11) involves the following properties:

(a) the density function of the first order is independent of t_1;

(b) the joint density function of the second order depends on only the time interval (t_2-t_1).

Thus, the statistical averages of the first order are independent of time. The statistical averages of the second order depend on only the time lag $\tau=t_2-t_1$:

$$\boxed{R_{XY}(t_1,t_2)=R_{XY}(\tau)}$$
$$\boxed{C_{XY}(t_1,t_2)=C_{XY}(\tau)}$$
$$\boxed{\rho_{XY}(t_1,t_2)=\rho_{XY}(\tau)}$$

The <u>cross-correlation function</u> of two (weakly) stationary processes:

$$R_{XY}(\tau)=E\left[X(t)Y(t+\tau)\right] \tag{12}$$

has several noteworthy properties (Fig. 2):

1) Setting τ = 0 in Eq. (12):

$$R_{XY}(0)=E\left[X(t)Y(t)\right] \tag{13}$$

2) $R_{XY}(\tau)=C_{XY}(\tau)+\mu_X\mu_Y=\rho_{XY}(\tau)\sigma_X\sigma_Y+\mu_X\mu_Y$. Thus, since $|\rho_{XY}|\le 1$, it follows that:

$$-\sigma_X\sigma_Y+\mu_X\mu_Y \le R_{XY}(\tau) \le \sigma_X\sigma_Y+\mu_X\mu_Y \quad \forall \tau \in \mathbf{R} \tag{14}$$

3) For $|\tau|$ tending to infinite the couple of random variables $X(t)$, $X(t+\tau)$ tends to become not correlated $(\rho_{XY}=0)$. Thus:

$$\lim_{|\tau|\to\infty} R_{XY}(\tau) = \mu_X \mu_Y \tag{15}$$

4) Setting $\overline{t} = t + \tau$, Eq. (11) becomes $R_{XY}(\tau) = E\left[X(\overline{t}-\tau)Y(\overline{t})\right]$. Thus it results:

$$\boxed{R_{XY}(\tau) = R_{YX}(-\tau)} \tag{16}$$

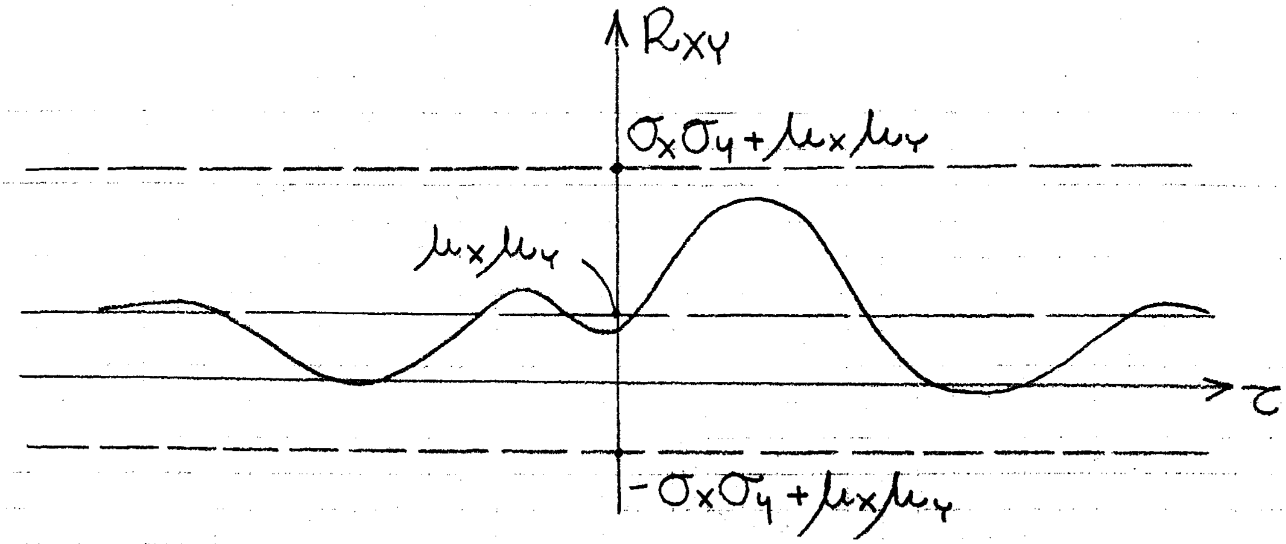

Fig. 2

The cross-covariance function of two (weakly) stationary processes:

$$C_{XY}(\tau) = E\left[\{X(t) - \mu_X\}\{Y(t+\tau) - \mu_Y\}\right] \tag{17}$$

has properties analogous to the cross-correlation function (Fig. 3):

$$C_{XY}(0) = E\left[\{X(t) - \mu_X\}\{Y(t) - \mu_Y\}\right] \tag{18}$$

$$-\sigma_X\sigma_Y \le C_{XY}(\tau) \le \sigma_X\sigma_Y \quad \forall \tau \in \mathbf{R} \tag{19}$$

$$\lim_{|\tau|\to\infty} C_{XY}(\tau) = 0 \tag{20}$$

$$\boxed{C_{XY}(\tau) = C_{YX}(-\tau)} \tag{21}$$

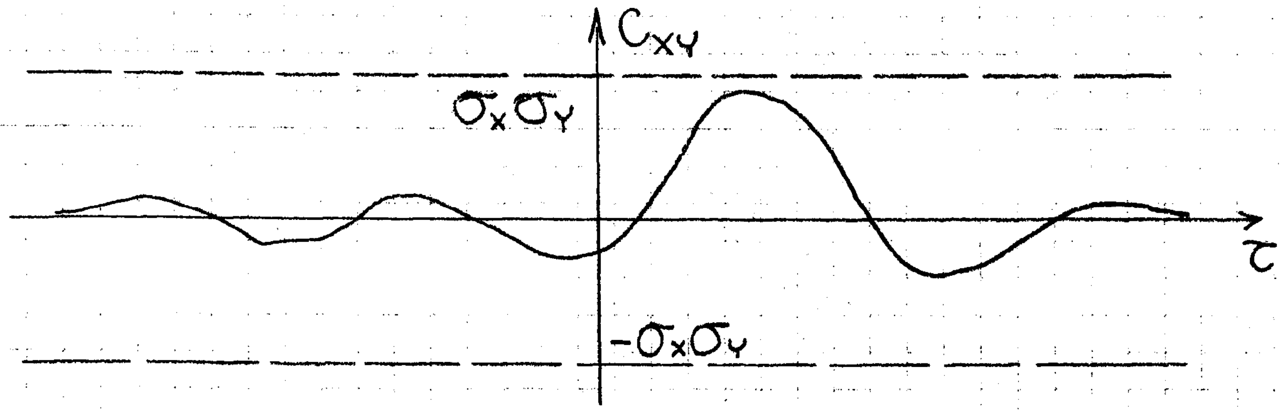

Fig. 3

If $X(t)$ and $Y(t)$ are <u>not correlated</u> then, due to Eq. (9):

$$\boxed{C_{XY}(\tau)=0 \quad \forall\tau\in\mathbf{R}} \tag{22}$$

If $X(t)$ and $Y(t)$ are <u>orthogonal</u> then, due to Eq. (10):

$$\boxed{R_{XY}(\tau)=0 \quad \forall\tau\in\mathbf{R}} \tag{23}$$

Bi-variate normal process

Two random stationary processes X(t) and Y(t) have a bi-variate normal distribution if their joint density function of the second order is given by the relationship:

$$\boxed{\begin{aligned} p_{XY}(x,y;\tau) &= \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho_{XY}^2(\tau)}} \cdot \\ &\cdot\exp\left\{-\frac{\sigma_Y^2(x-\mu_X)^2-2\sigma_X\sigma_Y\rho_{XY}(\tau)(x-\mu_X)(y-\mu_y)+\sigma_X^2(y-\mu_y)^2}{2\sigma_X^2\sigma_Y^2\left[1-\rho_{XY}^2(\tau)\right]}\right\}\end{aligned}} \tag{24}$$

where μ_X and μ_Y are the means of X(t) and Y(t), σ_X^2 e σ_Y^2 are the variances, $\rho_{XY}(\tau)$ is the normalised cross-covariance function.

Cross-power spectral density

Let us consider a couple of stationary random processes. The cross-power spectral density, or more simply the cross-power spectrum, $S_{XY}(\omega)$ of X(t) andY(t), is defined as:

$$\boxed{S_{XY}(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}C_{XY}(\tau)e^{-i\omega\tau}d\tau} \tag{25}$$

Unless the factor $1/2\pi$, it coincides with the Fourier transform of the cross-covariance function $C_{XY}(\tau)$. Thus, the cross-covariance function is the inverse Fourier transform (unless the factor 2π) of the cross-power spectral density $S_{XY}(\omega)$:

$$\boxed{C_{XY}(\tau)=\int_{-\infty}^{\infty}S_{XY}(\omega)e^{i\omega\tau}d\omega} \tag{26}$$

Also the Eqs. (25) and (26) are known as the Wiener-Khintchine equations.
$S_{XY}(\omega)$ exists if $C_{XY}(\tau)$ is absolutely integrable:

$$\int_{-\infty}^{\infty}\left|C_{XY}(\tau)\right|d\tau<\infty$$

If X(t) and Y(t) are zero mean, the cross-covariance function $C_{XY}(\tau)$ coincides with the cross-correlation function $R_{XY}(\tau)$. So, the above equations hold also replacing $C_{XY}(\tau)$ by $R_{XY}(\tau)$.
Since $C_{XY}(\tau)$ is in general a non symmetric function, $S_{XY}(\omega)$ is in general a complex function. As such it can be expressed as:

$$\boxed{S_{XY}(\omega)=S_{XY}^{C}(\omega)+iS_{XY}^{Q}(\omega)} \tag{27}$$

where $S_{XY}^{C}(\omega)=\mathrm{Re}\left[S_{XY}(\omega)\right]$ is referred to as the co-spectrum and $S_{XY}^{Q}(\omega)=\mathrm{Im}\left[S_{XY}(\omega)\right]$ is referred to as the quad-spectrum. Let us rewrite $C_{XY}(\tau)$ as:

$$C_{XY}(\tau)=\frac{1}{2}\left[C_{XY}(\tau)+C_{XY}(-\tau)\right]+\frac{1}{2}\left[C_{XY}(\tau)-C_{XY}(-\tau)\right] \tag{28}$$

where the two terms in the brackets are, respectively, symmetric and anti-symmetric functions of τ.
Let us execute the Fourier transform of both terms. It follows:

$$S_{XY}^{C}(\omega)=\frac{1}{4\pi}\int_{-\infty}^{\infty}\left[C_{XY}(\tau)+C_{XY}(-\tau)\right]e^{-i\omega\tau}d\tau \tag{29}$$

$$S_{XY}^{Q}(\omega)=\frac{1}{4\pi}\int_{-\infty}^{\infty}\left[C_{XY}(\tau)-C_{XY}(-\tau)\right]e^{-i\omega\tau}d\tau \tag{30}$$

$$S_{XY}^{C}(\omega)=\frac{1}{4\pi}\int_{-\infty}^{\infty}\left[C_{XY}(\tau)+C_{XY}(-\tau)\right]\cos\omega\tau\,d\tau \tag{31}$$

$$S_{XY}^{Q}(\omega)=\frac{1}{4\pi}\int_{-\infty}^{\infty}\left[C_{XY}(\tau)-C_{XY}(-\tau)\right]\sin\omega\tau\,d\tau \tag{32}$$

Eqs. (29) and (30) demonstrate that $S_{XY}^{C}(\omega)$ is the Fourier transform (unless the factor 4π) of the symmetric part of $C_{XY}(\tau)$; $S_{XY}^{Q}(\omega)$ is the Fourier transform (unless the factor 4π) of the anti-symmetric part of $C_{XY}(\tau)$. Thus, $S_{XY}(\omega)$ is real if $C_{XY}(\tau)$ is symmetric.
Eqs. (31) and (32) lead to the relationships:

$$S_{XY}^{C}(\omega) = S_{XY}^{C}(-\omega) \tag{33}$$

$$S_{XY}^{Q}(\omega) = -S_{XY}^{Q}(-\omega) \tag{34}$$

So, $S_{XY}^{C}(\omega)$ and $S_{XY}^{Q}(\omega)$ are, respectively, symmetric and anti-symmetric functions of ω.
Dealing with $S_{YX}(\omega)$ in an analogous way it results:

$$S_{XY}^{C}(\omega) = S_{YX}^{C}(\omega) \tag{35}$$

$$S_{XY}^{Q}(\omega) = -S_{YX}^{Q}(\omega) \tag{36}$$

from which it derives:

$$S_{XY}(\omega) = S_{YX}^{*}(\omega) \tag{37}$$

where S* is the complex conjugate of S.

Coherence, correlation and dependence

The coherence function of two stationary processes X(t) and Y(t) is defined as:

$$\boxed{\gamma_{XY}(\omega) = \frac{S_{XY}(\omega)}{\sqrt{S_{XX}(\omega) S_{YY}(\omega)}}} \tag{38}$$

The real and the imaginary part of the coherence function are referred to as, respectively, the co-coherence and the quad-coherence:

$$\gamma_{XY}^{C}(\omega) = \mathrm{Re}\left[\gamma_{XY}(\omega)\right] = \frac{S_{XY}^{C}(\omega)}{\sqrt{S_{XX}(\omega) S_{YY}(\omega)}} \tag{39}$$

$$\gamma_{XY}^{Q}(\omega) = \mathrm{Im}\left[\gamma_{XY}(\omega)\right] = \frac{S_{XY}^{Q}(\omega)}{\sqrt{S_{XX}(\omega) S_{YY}(\omega)}} \tag{40}$$

Thus:

$$\boxed{\gamma_{XY}(\omega) = \gamma_{XY}^{C}(\omega) + i\gamma_{XY}^{Q}(\omega)} \tag{41}$$

Once introduced the coherence function, the cross-power spectral density assumes the form:

$$\boxed{S_{XY}(\omega) = \sqrt{S_{XX}(\omega) S_{YY}(\omega)}\, \gamma_{XY}(\omega)} \tag{42}$$

which shows that the cross-power spectral density is known through the knowledge of the power spectral densities of each process and through the coherence function.

The coherence function may be interpred as the frequency domain counter-part of the normalised cross-correlation function of the processes X(t) and Y(t). It can be proved that:

$$\boxed{0 \le \left|\gamma_{XY}(\omega)\right| \le 1} \tag{43}$$

In particular, when:

$$\left|\gamma_{XY}(\omega)\right| = 1$$

the processes X(t) and Y(t) are referred to as perfectly not correlated at the circular frequency ω. If such a condition is satisfied for any ω, then the two processes are referred to as totally or identically coherent and $S_{XY}(\omega) = \sqrt{S_{XX}(\omega) S_{YY}(\omega)}$

Instead, when:

$$\left|\gamma_{XY}(\omega)\right| = 0$$

the two processes X(t) and Y(t) are referred to as not correlated at the circular frequency ω. If such a condition is satisfied for any ω, then the two processes are referred to as totally or identically incoherent and $S_{XY}(\omega) = 0$; consequently, $C_{XY}(\tau) = 0$.

Two stationary processes X(t) and Y(t) are defined as statistically independent if:

$$p_{XY}(x_1, t_1; y_2, t_2) = p_X(x_1, t_1)\, p_Y(y_2, t_2)$$

for any t_1 and t_2. It is easy to show that two independent random processes are also not correlated. The opposite is generally not true; however, if X(t) and Y(t) are normal processes, then the not correlation implies the independence.