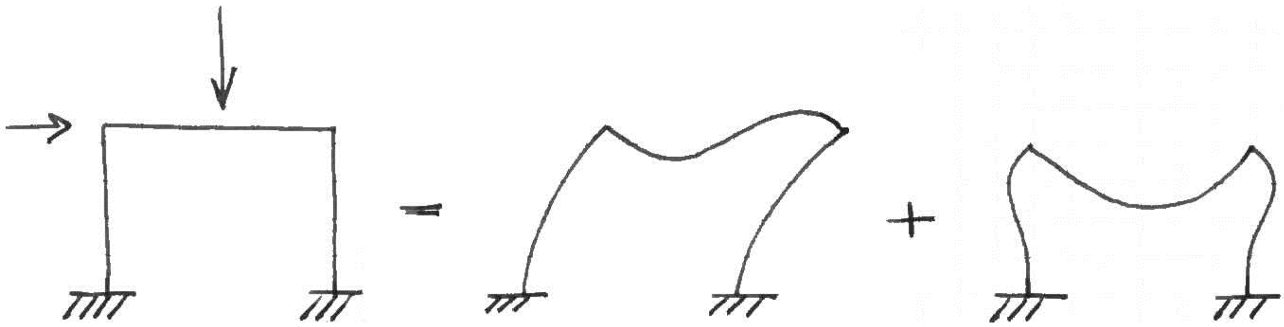
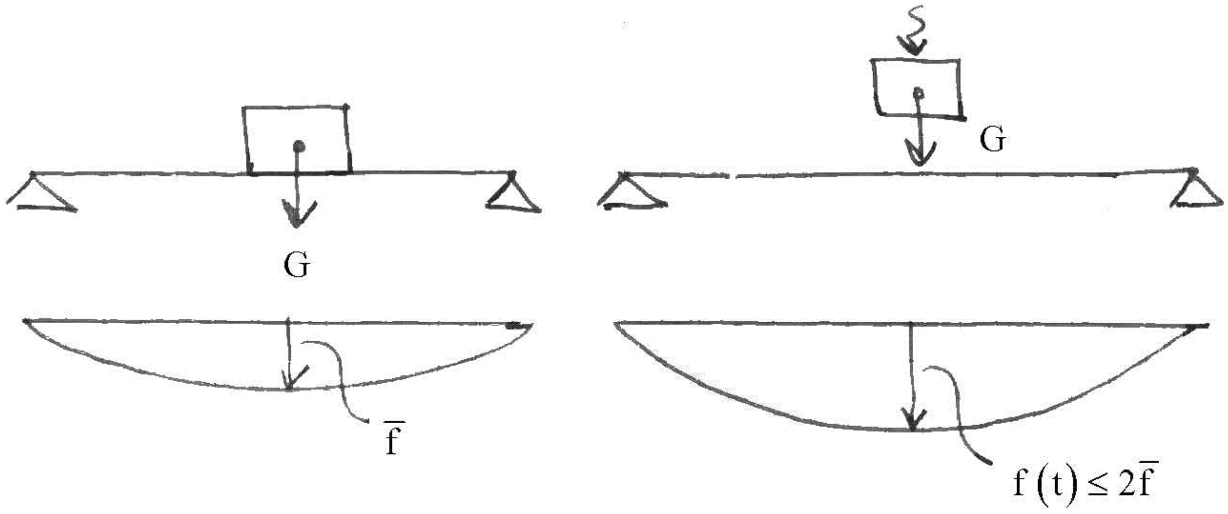
INTRODUCTION TO STRUCTURAL DYNAMICS

Structural mechanics frequently deals with static problems, i.e. problems which are independent of time. In these cases we are usual to consider an equilibrium configuration reached by increasing slowly forces and displacements. In other words, we consider that the evolution from the unloaded and undeformed configuration to the loaded and deformed configuration occurs through a series of equilibrium configurations.



In reality, everey physical phenomenon concerning structural mechanics depends on time.

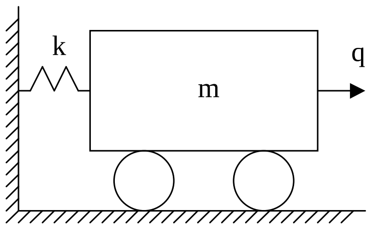


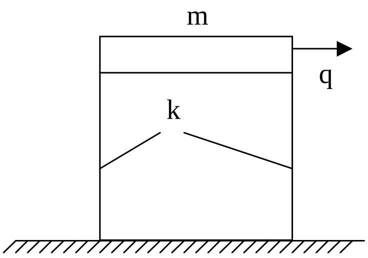
We can deal with a problem as static when the time dependence is slow, as in the case of the snow accumulation, or when it is limited to a small time interval (the application of permanent loads).

We shall deal with a problem as dynamic when rapid time variations occur, as in the case of wind and seismic actions.

SINGLE-DEGREE-OF-FREEDOM-SYSTEMS

Undamped free vibrations





2nd Newton law F = ma

F = - kq restoring elastic force; q = q(t)

a = $\ddot{q}$ absolute acceleration

$$\boxed{m\ddot{q}(t) + kq(t) = 0} \Rightarrow \text{(dividing both members by m)}$$

$$\ddot{q}(t) + \frac{k}{m}q(t) = 0 \Rightarrow \quad \text{Defining } \omega_0^2 = \frac{k}{m}$$

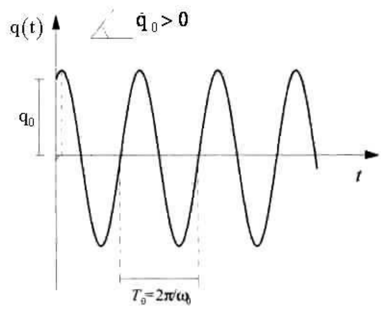
$$\boxed{\ddot{q}(t) + \omega_0^2 q(t) = 0} \Rightarrow$$ 2nd order, homogeneous, linear differential equation with constant coefficients

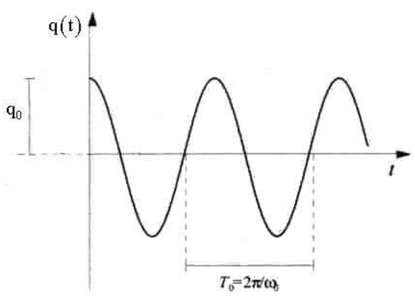
$q(0) = q_0$; $\dot{q}(0) = \dot{q}_0$ initial conditions

$$q(t) = A\cos\omega_0 t + B\sin\omega_0 t \Rightarrow q(0) = A = q_0$$

$$\dot{q}(t) = -\omega_0 A\sin\omega_0 t + \omega_0 B\cos\omega_0 t \Rightarrow \dot{q}(0) = \omega_0 B = \dot{q}_0 \Rightarrow B = \frac{\dot{q}_0}{\omega_0}$$

$$\boxed{q(t) = q_0\cos\omega_0 t + \frac{\dot{q}_0}{\omega_0}\sin\omega_0 t} \qquad \dot{q}_0 = 0 \Rightarrow q(t) = q_0\cos\omega_0 t$$





$\omega_0 =$ fundamental circular frequency

$n_0 = \omega_0 / 2\pi =$ fundamental frequency

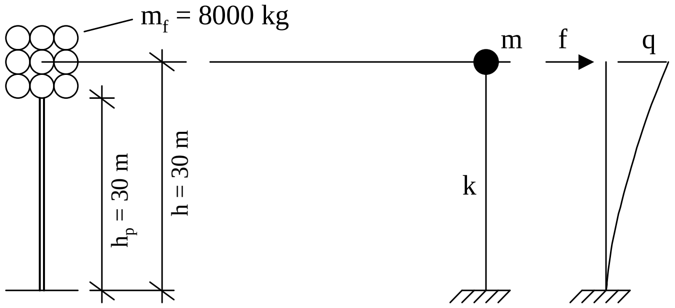
$T_0 = 1/n_0 = 2\pi/\omega_0 =$ fundamental period

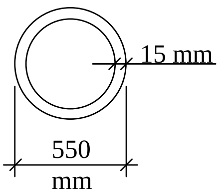
Indicatively:

$n_0 < 1\,\text{Hz}\,(T_0 > 1\text{s})$ - Dynamically flexible structure

$n_0 > 1\,\text{Hz}\,(T_0 < 1\text{s})$ - Dynamically rigid structure

Example: Arc lamp





$$E = 2.100.000 \;\; kgf/cm^2 = 0.21\times10^{12} \; N/m^2$$

$$J = 9.027\times10^{-4} \; m^4$$

$$A = 0.0252 \; m^2$$

$$k = \frac{f}{q} = \frac{f}{\frac{f\,h^3}{3EJ}} = \frac{3EJ}{h^3} = 21063 \; N/m$$

Mass of the pole $m_p = 0.0252\times27\times7850 = 5343 \;\; kg$

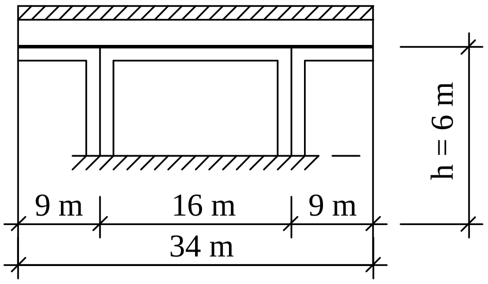
It is assumed $m = m_f + m_p/2 = 10671 \; kg$

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{21063}{10671}} = 1.405 \; rad/s$$

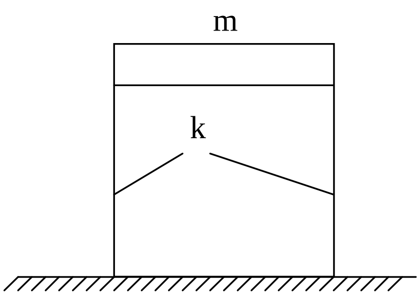
$$n_0 = \omega_0/2\pi = 0.223 \; Hz$$

$$T_0 = 1/n_0 = 4.47 \; s$$

Example: Single-storey reinforced concrete building



Frame step i = 3 m
Slab depth = 20 cm
Beam 30 ×150 cm
Columns 30 ×100 cm
Shear-type model



$$
\begin{aligned}
m = & \; 2500\times0.2\times34\times2.7 && \text{(slab)}\\
& + 2500\times34\times0.3\times1.5 && \text{(beam)}\\
& + 2\times2500\times5.25\times0.3\times1/2 && \text{(columns/2)}\\
& = 88087.5 \text{ kg}
\end{aligned}
$$

$$J_p = \frac{1}{12} b_p h_p^3 = \frac{1}{12}\times0.3\times(1)^3 = 0.025\,\mathrm{m}^4$$

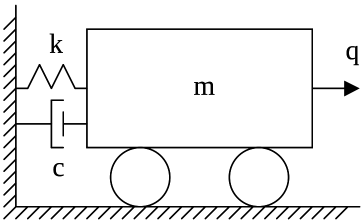
$$k = 2\times\frac{12\,E\,J_p}{h^3} = 2\times\frac{12\times0.3\times10^{11}\times0.025}{(6)^3} = 0.8333\times10^8 \quad \mathrm{N/m}$$

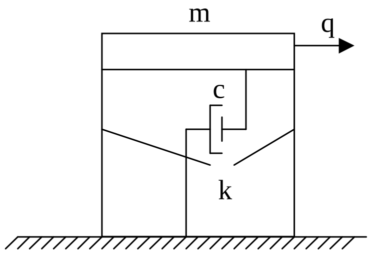
$$\omega_0 = \sqrt{\frac{k}{m}} = 30.76\ \mathrm{rad/s};\ n_0 = 4.9\,\mathrm{H_z};\ T_0 = 0.204\,\mathrm{s}$$

Considering a distributed vertical load with mass $1000\ \ \mathrm{kg/m^2} \Rightarrow$
$m = 88087.5 + 1000\times34\times3 = 190087.5$ kg

$$\omega_0 = \sqrt{\frac{k}{m}} = 20.94\ \mathrm{rad/s};\ n_0 = 3.33\,\mathrm{H_z};\ T_0 = 0.30\,\mathrm{s}$$

Damped free vibrations





2nd Newton law $F = ma$; $a = \ddot{q}$

$F = -kq - c\dot{q}$; $-c\dot{q} =$ damping viscous force

$\boxed{m\ddot{q}(t) + c\dot{q}(t) + kq(t) = 0}$ $\Rightarrow$ (dividing both members by m)

$$\ddot{q}(t) + \frac{c}{m}\dot{q}(t) + \frac{k}{m}q(t) = 0 \qquad \frac{k}{m} = \omega_0^2$$

$$\frac{c}{m} = \frac{c \cdot 2\sqrt{k}}{\sqrt{m}\sqrt{m} \cdot 2\sqrt{k}} = 2\underbrace{\frac{c}{2\sqrt{km}}}_{\xi}\underbrace{\sqrt{\frac{k}{m}}}_{\omega_0}$$

$\xi =$ damping ratio or damping coefficient

$$\boxed{\begin{aligned}&\ddot{q}(t) + 2\xi\omega_0\dot{q}(t) + \omega_0^2 q(t) = 0\\ &q(0) = q_0 \ ; \ \dot{q}(0) = \dot{q}_0\end{aligned}} \tag{1}$$

Eq. (1) admits three distinct solutions depending on whether $\xi < 1, \xi > 1, \xi = 1$. In structural engineering not only $\xi < 1$ but, even more $\xi \ll 1$. A structure with $\xi < 1$ is said “underdamped”. In this case the solution of Eq. (1) is given by:

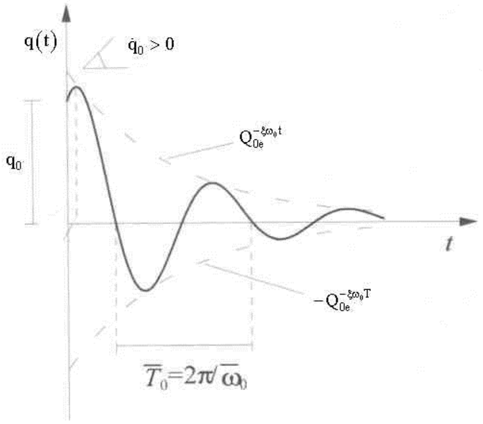
$$\boxed{q(t) = e^{-\xi\omega_0 t}\left(a_1 \cos\omega_0\sqrt{1-\xi^2}\ t + a_2 \sin\omega_0\sqrt{1-\xi^2}\ t\right)} \tag{2}$$

where a_1 and a_2 are constants depending on the initial conditions:

$$\boxed{a_1 = q_0 \ ; \quad a_2 = \frac{\dot{q}_0 + \xi\omega_0 q_0}{\omega_0\sqrt{1-\xi^2}}}$$

Eq. (2) may be rewritten as:

$$\begin{aligned}&q(t) = Q_0\, e^{-\xi\omega_0 t}\cos\left(\omega_0\sqrt{1-\xi^2}\ t + \varphi\right)\\ &Q_0 = \sqrt{q_0^2 + \left(\frac{\dot{q}_0 + \xi\omega_0 q_0}{\omega_0\sqrt{1-\xi^2}}\right)^2} \ ; \quad \varphi = -\text{arctg}\left(\frac{\dot{q}_0 + \xi\omega_0 q_0}{\omega_0 q_0\sqrt{1-\xi^2}}\right)\end{aligned} \tag{3}$$



$$\overline{\omega}_0 = \omega_0\sqrt{1-\xi^2}$$

<u>Remarks</u>

1. $q(t)$ defines a damped vibratory motion for which the relative maximum and minimum values occur every $T = 2\pi/\omega_0\sqrt{1-\xi^2}$; they lie on the symmetric curves $\pm Q_0 e^{-\xi\omega_0 t}$.
2. The absolute values of the relative maxima and minima correspond to a series with rate $e^{-\xi\omega_0 t}$; the logarithmic decrement is defined as:

$$\delta = \ell n\left[\frac{Q_0 e^{-\xi\omega_0 t}}{Q_0 e^{-\xi\omega_0 (t+T)}}\right] \Rightarrow \delta = \xi\omega_0 T = \frac{2\pi\xi}{\sqrt{1-\xi^2}}$$

3. The vibratory motion tends to vanish on increasing the time:

$$\ell\mathrm{im}_{t\to\infty}\, q(t) = 0$$

This tendence becomes faster on increasing the damping ratio ξ.

4. In the typical case $\xi << 1$, Eq. (2) becomes:

$$q(t) \cong e^{-\xi\omega_0 t}\left(a_1 \cos\omega_0 t + a_2 \sin\omega_0 t\right)$$
$$a_1 = q_0 \ ; \ a_2 \cong \frac{\dot{q}_0}{\omega_0} + \xi q_0$$

Furthermore: $T \cong 2\pi/\omega_0 = T_0$; $\delta \cong 2\pi\xi$

5. In the limit case ξ = 0, Eq. (2) becomes:

$$q(t) = a_1 \cos\omega_0 t + a_2 \sin\omega_0 t$$
$$a_1 = q_0 \ ; \quad a_2 = \dot{q}_0/\omega_0$$

Table F.2 —Approximate values of logarithmic decrement of structural damping in the fundamental mode, δ_s

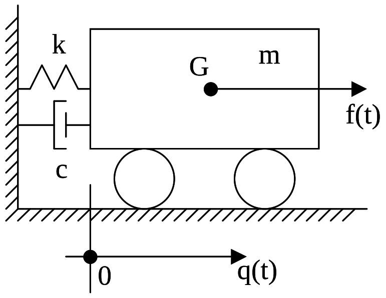
Structural type		structural damping, δ_s
reinforced concrete buildings		0,10
steel buildings		0,05
mixed structures concrete + steel		0,08
reinforced concrete towers and chimneys		0,03
unlined welded steel stacks without external thermal insulation		0,012
unlined welded steel stack with external thermal insulation		0,020
steel stack with one liner with external thermal insulation[a]	$h/b < 18$	0,020
	$20 \leq h/b < 24$	0,040
	$h/b \geq 26$	0,014
steel stack with two or more liners with external thermal insulation [a]	$h/b < 18$	0,020
	$20 \leq h/b < 24$	0,040
	$h/b \geq 26$	0,025
steel stack with internal brick liner		0,070
steel stack with internal gunite		0,030
coupled stacks without liner		0,015
guyed steel stack without liner		0,04
steel bridges + lattice steel towers	welded	0,02
	high resistance bolts	0,03
	ordinary bolts	0,05
composite bridges		0,04
concrete bridges	prestressed without cracks	0,04
	with cracks	0,10
Timber bridges		0,06 - 0,12
Bridges, aluminium alloys		0,02
Bridges, glass or fibre reinforced plastic		0,04 - 0,08
cables	parallel cables	0,006
	spiral cables	0,020

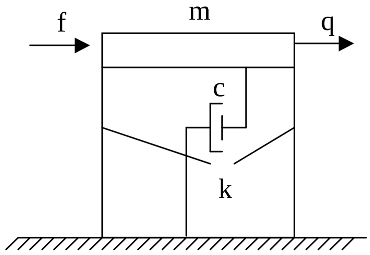
NOTE 1 The values for timber and plastic composites are indicative only. In cases where aerodynamic effects are found to be significant in the design, more refinded figures are needed through specialist advice (agreed if appropriate with the competent Authority.

NOTE 2 For cable supported bridges the values given in Table F.2 need to be factored by 0,75

[a] For intermediate values of h/b, linear interpolation may be used

Forced damped vibrations





2nd Newton law $F = ma$; $a = \ddot{q}$

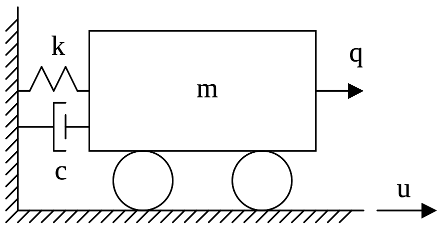
$F = -kq - c\dot{q} + f$; $f = f(t)$ = external force; $q = q(t)$.

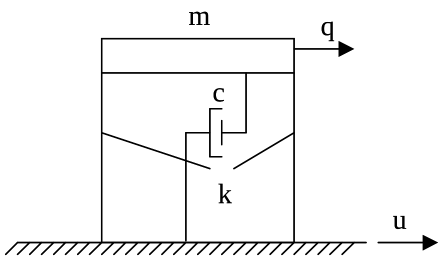
$$\boxed{m\ddot{q}(t) + c\dot{q}(t) + kq(t) = f(t)} \Rightarrow \text{(dividing both members by m)}$$

$$\ddot{q}(t) + \frac{c}{m}\dot{q}(t) + \frac{k}{m}q(t) = \frac{f}{m}(t) \Rightarrow$$

$$\boxed{\begin{array}{l} \ddot{q}(t) + 2\xi\omega_0\dot{q}(t) + \omega_0^2 q(t) = \dfrac{1}{m}f(t) \\ q(0) = q_0 \;\; ; \;\; \dot{q}(0) = \dot{q}_0 \end{array}} \qquad (4)$$

Motion induced by a rigid-base translation - Seismic motion

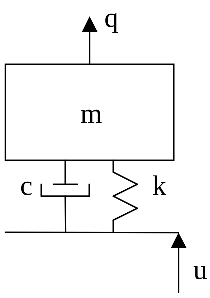




2nd Newton law $F = ma$

$F = -kq - c\dot{q}$

$a = \ddot{q} + \ddot{u}$ (absolute acceleration)



$$m\left[\ddot{q}(t) + \ddot{u}(t)\right] + c\dot{q}(t) + kq(t) = 0 \Rightarrow$$

$$\boxed{m\ddot{q}(t) + c\dot{q}(t) + kq(t) = -m\ddot{u}(t)} \Rightarrow$$

$$\boxed{\begin{array}{l} \ddot{q}(t) + 2\xi\omega_0\dot{q}(t) + \omega_0^2 q(t) = -\ddot{u}(t) \\ q(0) = q_0 \;\; ; \;\; \dot{q}(0) = \dot{q}_0 \end{array}}$$

$f = -m\ddot{u} =$ equivalent apparent force

$$\boxed{\begin{aligned}&\ddot{q}(t)+2\xi\omega_0\dot{q}(t)+\omega_0^2 q(t)=\frac{1}{m}f(t)\\&q(0)=q_0\ ;\ \dot{q}(0)=\dot{q}_0\end{aligned}} \tag{5}$$

Eq. (5) defines the general problem of the damped forced vibrations of a S.D.O.F. system. Due to linearity, the solution may be expressed as:

$$\boxed{q(t)=q'(t)+q''(t)} \tag{6}$$

Thus:

$$\begin{aligned}&\ddot{q}'(t)+\ddot{q}''(t)+2\xi\omega_0\dot{q}'(t)+2\xi\omega_0\dot{q}''(t)+\omega_0^2 q'(t)+\omega_0^2 q''(t)=\frac{1}{m}f(t)\\&q'(0)+q''(0)=q_0\ ;\ \dot{q}'(0)+\dot{q}''(0)=\dot{q}_0\end{aligned} \tag{7}$$

Let us assume that $q'(t)$ is the solution of the problem:

$$\boxed{\begin{aligned}&\ddot{q}'(t)+2\xi\omega_0\dot{q}'(t)+\omega_0^2 q'(t)=0\\&q'(0)=q_0\ ;\ \dot{q}'(0)=\dot{q}_0\end{aligned}} \tag{8}$$

Replacing Eq. (5) into Eq. (4):

$$\boxed{\begin{aligned}&\ddot{q}''(t)+2\xi\omega_0\dot{q}''(t)+\omega_0^2 q''(t)=\frac{1}{m}f(t)\\&q''(0)=0\ ;\ \dot{q}''(0)=0\end{aligned}} \tag{9}$$

Eq. (8) edefines the problem of the free vibrations with an initial perturbation. Eq. (9) defines the problem of the forced vibrations without initial perturbation.

It was demonstrated that, for $\forall\xi, \lim_{t\to\infty} q'(t)=0 \Rightarrow$

Thus, for $\forall\xi$ $t=\bar{t}$ exists such as, for $t>\bar{t}$ $|q'(t)|<\varepsilon$, with ε arbitrarily small. Thus, for $t>\bar{t}$ $q(t)=\ddot{q}(t)$.

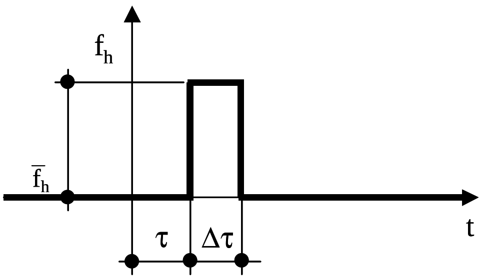
The following sections assume $\xi<1$.

SINGLE-DEGREE-OF-FREEDOM-SYSTEMS

Time-domain analyisis

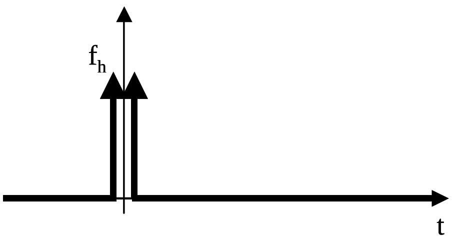
Impulsive force

An impulsive force $f_h(t)$ is a force with a constant value $\bar{f}_h$ over a short interval $\Delta\tau$, being nul in the remaining time.



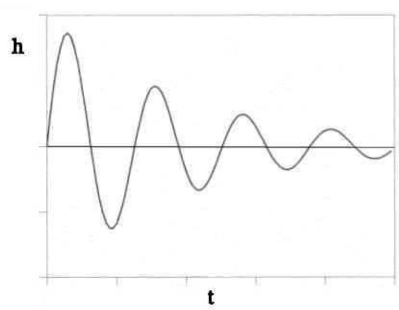
$$f_h(t) = \begin{cases} \bar{f}_h & \text{per} \quad \tau < t < \tau + \Delta\tau \\ 0 & \text{elsewhere} \end{cases}$$

Such a force has an impulse $I = \bar{f}_h \, \Delta\tau$.
An elementary impulsive force is a impulsive force characterised by the parameters $\tau = 0$, $\Delta\tau \to 0$, $I = 1 \left(\bar{f}_h \to \infty\right)$.



$f_h(t) = \delta(t) =$ Dirac' function

$$\begin{cases} \delta(t-a) = 0 \quad \text{per } t \neq a \\ \int_{-\infty}^{\infty} \varepsilon(t)\, \delta(t-a) dt = \varepsilon(a) \end{cases}$$



The equation of motion q(t) of a S.D.O.F. system subjected to an elementary impulsive force is denoted by the symbol h(t).

The action of an elementary impulsive force is equivalent to the effect of an initial velocity $\dot{q}_0$. The value of $\dot{q}_0$ is obtained by equating the impulse I with the momentum $Q = m\dot{q}_0$. It follows that $\dot{q}_0 = 1/m$.

Thus, the equation of motion h(t):

$$\boxed{\begin{array}{l} \ddot{h}(t) + 2\xi\omega_0\dot{h}(t) + \omega_0^2 h(t) = \frac{1}{m}\delta(t) \\ h(0) = \dot{h}(0) = 0 \end{array}}$$

may be solved by studyng the problem:

$$\begin{array}{l} \ddot{h}(t) + 2\,\xi\,\omega_0\dot{h}(t) + \omega_0^2 h(t) = 0 \\ h(0) = 0\,;\, \dot{h}(0) = 1/m \end{array}$$

It results:

$$\boxed{h(t) = e^{-\xi\omega_0 t}\,\frac{1}{m\,\omega_0\sqrt{1-\xi^2}}\,\sin\omega_0\sqrt{1-\xi^2}\;t} \qquad (1)$$

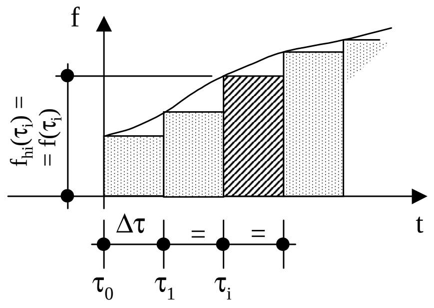
h(t) is defined as impulse response function.

Thanks to linearity, the response q(t) to a force with an impulse I applied at time τ results:

$$\boxed{q(t) = I\,h(t-\tau)} \qquad (2)$$

Impulsive force method

A generic force f(t) may be approximated by a superposition of a series of suitable impulsive forces $f_{hi}(t)$ (i = 1, 2, …): $f(t) \cong \sum_{0}^{i} f_{hi}(t)$.



Therefore, the response q(t) can be expressed as the superposition of the responses $q_i(t)$ to each impulsive component force $f_{hi}(t)$:

$$q(t) \cong \sum_{0}^{i} q_i(t), \text{ where } q_i(t) = I_1\, h(t-\tau_i) =$$
$$= \bar{f}_{hi}(\tau_i)\Delta\tau\, h(t-\tau_i) = f(\tau_i)\Delta\tau\, h(t-\tau_i) \Rightarrow$$
$$q(t) \cong \sum_{0}^{i} f(\tau_i)\, h(t-\tau_i)\,\Delta\tau$$

This expression is rigorous at the limit for $\Delta t \to 0$:

$$\boxed{q(t) = \int_0^t f(\tau)\, h(t-\tau)\, d\tau} \qquad (3)$$

Due to Eq. (3) q(t) is the convolution integral of f(t) and h(t):

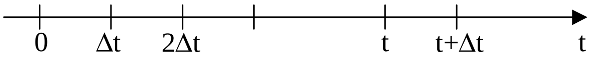
$$\boxed{q(t) = f(t) * h(t)} \qquad (4)$$

where * denotes the convolution product. Eq. (3) is also called Duhamel’s integral.

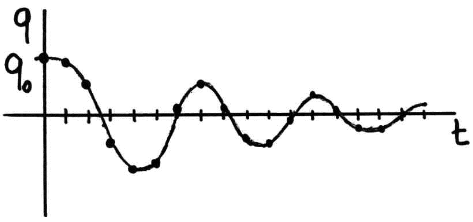
Numerical integration

$$\ddot{q}(t) + 2\xi\omega_0\dot{q}(t) + \omega_0^2 q(t) = \frac{1}{m} f(t)$$
$$q(0) = q_0 \ ; \ \dot{q}(0) = \dot{q}_0$$

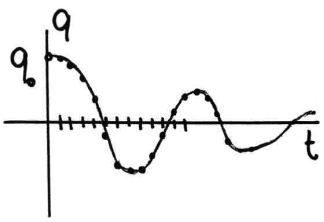
- Instead of solving the equation of motion at any time t, it is satisfied at discrete time intervals Δt. So, the dynamic balance is imposed in a finite number of points along the time axis.
- The solution is searched by recursive algorithms. Knowing the solution at times 0, Δt, 2Δt, … t, the algorithm provides the solution at time + Δt.

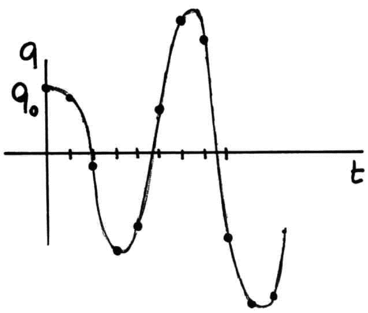


- An explicit integration method is a method formulated by imposing the dynamic balance at time t. An implicit integration method is a method formulated by imposing the dynamic balance at time t + Δt.
- The accuracy, the stability and the burdensome of the algorithm depend on the choice of the time interval Δt and by the way in which $q, \dot{q}, \ddot{q}$ are assumed to vary within Δt.
- A numerical integration method is defined as unconditionally stable if the solution to any initial condition does not increse without limits on increasing t, for any choice of Δt.



- A numerical integration method is said conditionally stable if the above condition holds for Δt < $\Delta t_{critico}$, where $\Delta t_{critico}$ is a stability limit.





SINGLE-DEGREE-OF-FREEDOM-SYSTEMS

Frequency domain analysis

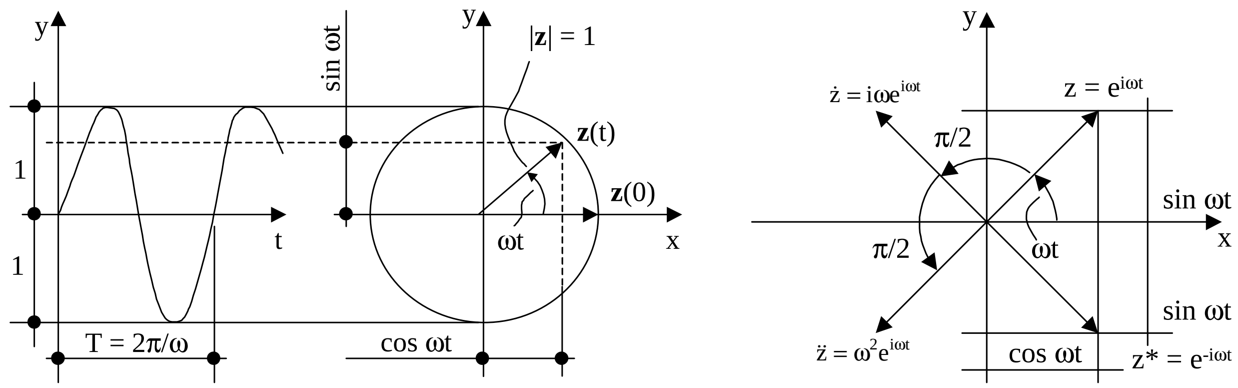
Elementary harmonic force

A harmonic force f(t) is defined as elementary when it has a unit amplitude. This condition is satisfied by the real expression $f(t) = \sin \omega t$ and by the complex expression $f(t) = e^{i\omega t}$:

$$f(t) = e^{i\omega t} = \cos \omega t + i \sin \omega t \Rightarrow$$
$$\left|e^{i\omega t}\right| = \sqrt{(\sin \omega t + i \cos \omega t)\ (\sin \omega t - i \cos \omega t)} = \sqrt{\sin^2 \omega t + \cos^2 \omega t} = 1$$

Observation: the elementary harmonic function $\sin \omega t$ may be regarded as the projection on the axis y of the ordinates of a vector **Z** with unit modulus, rotating around the origin of a Cartesian reference system (x, y) with uniform angular velocity ω and nil initial phase.

Interpreting (x, y) as an Argand-Gauss plane, the vector **Z** is associated with a complex number z whose real and imaginary parts are respectively the projections on x and y of **Z** : z = x + iy, x = Re(z) = cos ωt, y = Im(z) = sin ωt; thus, z = cos ωt + isinωt. Using Euler's formula $z = e^{i\omega t}$.



Let us consider the equation of motion:

$$\boxed{\begin{aligned} &\ddot{q}(t) + 2\xi\omega_0\dot{q}(t) + \omega_0^2 q(t) = \frac{1}{m} f(t) = \frac{1}{m} e^{i\omega t} = \frac{1}{m}(\cos \omega t + i \sin \omega t) \\ &q(0) = q_0\ ;\ \dot{q}(0) = \dot{q}_0 \end{aligned}} \qquad (1)$$

Since f(t) is a complex quantity, thus also q(t) is complex and may be written as:

q(t) = x(t) + i y(t) (2)

where the real functions x(t) = Re[q(t)] and y(t) = Im[q(t)] are, respectively, the solutions of the two problems:

$$\boxed{\begin{aligned}&\ddot{x}(t)+2\xi\omega_0\dot{x}(t)+\omega_0^2x(t)=\frac{1}{m}\cos\omega t\\&x(0)=x_0=\mathrm{Re}(q_0)\ ;\ \dot{x}(0)=\dot{x}_0=\mathrm{Re}(\dot{q}_0)\end{aligned}} \tag{3}$$

$$\boxed{\begin{aligned}&\ddot{y}(t)+2\xi\omega_0\dot{y}(t)+\omega_0^2y(t)=\frac{1}{m}\sin\omega t\\&y(0)=y_0=\mathrm{Im}(q_0);\ \dot{y}(0)=\dot{y}_0=\mathrm{Im}(\dot{q}_0)\end{aligned}} \tag{4}$$

The response y(t) to the real elementary harmonic force f(t) = sin ωt is the imaginary part of the response q(t) to the complex elementary harmonic force $f(t)=e^{i\omega t}$:

$$\begin{aligned}&\ddot{q}(t)+2\xi\omega_0\dot{q}(t)+\omega_0^2q(t)=\frac{1}{m}f(t)=\frac{1}{m}e^{i\omega t}=\frac{1}{m}(\cos\omega t+i\sin\omega t)\\&q(0)=q_0\ ;\ \dot{q}(0)=\dot{q}_0\end{aligned} \tag{5}$$

The solution of Eq. (5) is the sum of the integral $q'(t)$ of the homogeneous associated equation and of any particular integral $q''(t)$ of the complete equation:

$$q(t)=q'(t)+q''(t) \tag{6}$$

From Eq. (2) it results:

$$\begin{aligned}&q'(t)=x'(t)+iy'(t) \qquad && x(t)=x'(t)+x''(t)\\&q''(t)=x''(t)+iy''(t) && y(t)=y'(t)+y''(t)\end{aligned}$$

where $x'(t)$ and $y'(t)$ are the integrals of the homogeneous equations associated with Eqs. (3) and (4), respectively:

$$\begin{aligned}&x'(t)=e^{-\xi\omega_0 t}\left(a_{x1}\cos\omega_0\sqrt{1-\xi^2}\,t+a_{x2}\sin\omega_0\sqrt{1-\xi^2}\,t\right)\\&y'(t)=e^{-\xi\omega_0 t}\left(a_{y1}\cos\omega_0\sqrt{1-\xi^2}\,t+a_{y2}\sin\omega_0\sqrt{1-\xi^2}\,t\right)\end{aligned}$$

The integration constants depend on the initial conditions.

It is easy to demonstrate that an expression of $q''(t)$ is given by:

$$q''(t)=H(\omega)e^{i\omega t} \tag{7}$$

Substituting Eq. (7) into Eq. (5):

$$-\omega^2H(\omega)\cancel{e^{i\omega t}}+2\xi\omega_0\cdot i\omega H(\omega)\cancel{e^{i\omega t}}+\omega_0^2H(\omega)\cancel{e^{i\omega t}}=\frac{1}{m}\cancel{e^{i\omega t}}\Rightarrow$$

$$\boxed{H(\omega) = \frac{1}{m\,\omega_0^2}\,\frac{1}{1 - \dfrac{\omega^2}{\omega_0^2} + 2i\xi\,\dfrac{\omega}{\omega_0}}} \tag{8}$$

$H(\omega)$ is the complex frequency response function and may be rewritten as:

$$H(\omega) = R(\omega) + i\,I(\omega) \tag{9}$$

$$H(\omega) = |H(\omega)|e^{i\psi(\omega)} \tag{10}$$

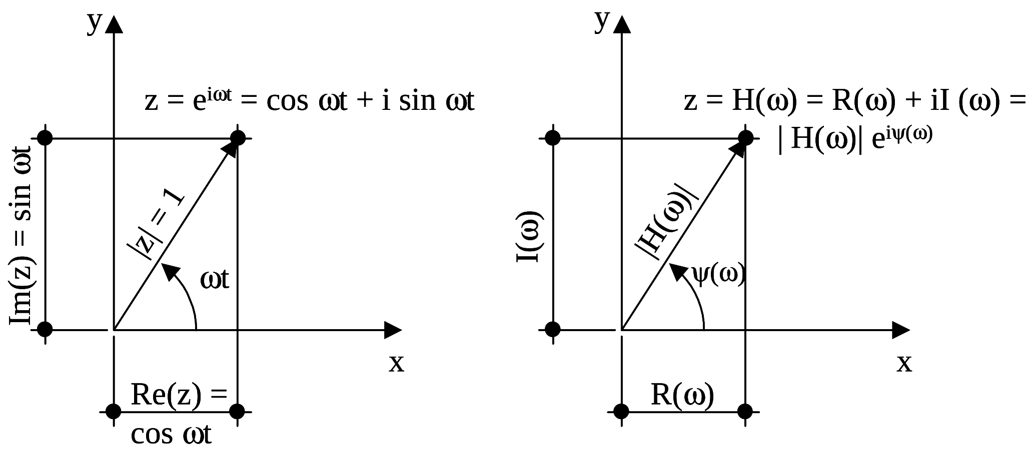
where:

$$R(\omega) = \mathrm{Re}[H(\omega)] = \frac{1}{m\,\omega_0^2}\,\frac{1 - \omega^2/\omega_0^2}{\left(1 - \omega^2/\omega_0^2\right) + 4\xi\,\omega^2/\omega_0^2} \tag{11}$$

$$I(\omega) = \mathrm{Im}[H(\omega)] = \frac{1}{m\,\omega_0^2}\,\frac{-2\zeta\,\omega/\omega_0}{\left(1 - \omega^2/\omega_0^2\right)^2 + 4\xi^2\,\omega^2/\omega_0^2} \tag{12}$$

$$|H(\omega)| = \sqrt{R^2(\omega) + I^2(\omega)} = \frac{1}{m\omega_0^2}\;\frac{1}{\sqrt{\left(1 - \omega^2/\omega_0^2\right)^2 + 4\xi^2\omega^2/\omega_0^2}} \tag{13}$$

$$\psi(\omega) = \mathrm{arctg}\;\frac{I(\omega)}{R(\omega)} = \mathrm{arctg}\left(-\frac{2\zeta\omega/\omega_0}{1 - \omega^2/\omega_0^2}\right) \tag{14}$$



Neglecting the initial transient stage of the motion, i.e. $q'(t) = 0 \Rightarrow q(t) = q''(t) \Rightarrow$

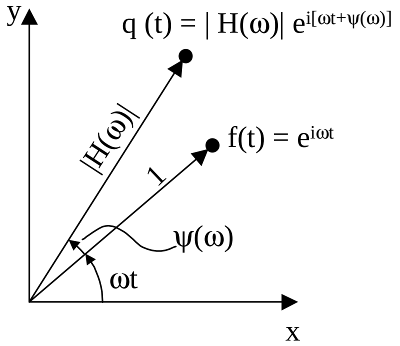
$$\boxed{q(t) = H(\omega)e^{i\omega t}} \tag{15}$$

Eq. (15) provides the steady-state response of a S.D.O.F. subjected to the complex elementary harmonic force $f(t)=e^{i\omega t}$; $H(\omega)$ is the ratio between the time-dependent response and force. Thus it has the meaning of the inverse of a dynamic stiffness.

Substituting Eq. (10) into Eq. (15):

$$\boxed{q(t)=|H(\omega)|e^{i[\omega t+\psi(\omega)]}} \qquad (16)$$

Thus, $|H(\omega)|$ is the amplitude of the dynamic response, $\psi(\omega)$ is the phase delay of the response q(t) with respect to the force $f(t)=e^{i\omega t}$.



$f(t)=e^{i\omega t} \Rightarrow$

$$q(t)=|H(\omega)|e^{i[\omega t+\psi(\omega)]}$$

$$|H(\omega)|=\frac{1}{m\,\omega_0^2}\frac{1}{\sqrt{\left(1-\frac{\omega^2}{\omega_0^2}\right)^2+4\xi^2\frac{\omega^2}{\omega_0^2}}}$$

$$\psi(\omega)=\text{arctg}\left\{-\frac{2\xi\frac{\omega}{\omega_0}}{1-\frac{\omega^2}{\omega_0^2}}\right\}$$

for $\omega=0 \Rightarrow$ $f(t)=1=$ static force with unit amplitude

$$\psi(0)=0\,; \quad |H(0)|=H(0)=\frac{1}{m\omega_0^2}=\frac{1}{k}\Rightarrow$$

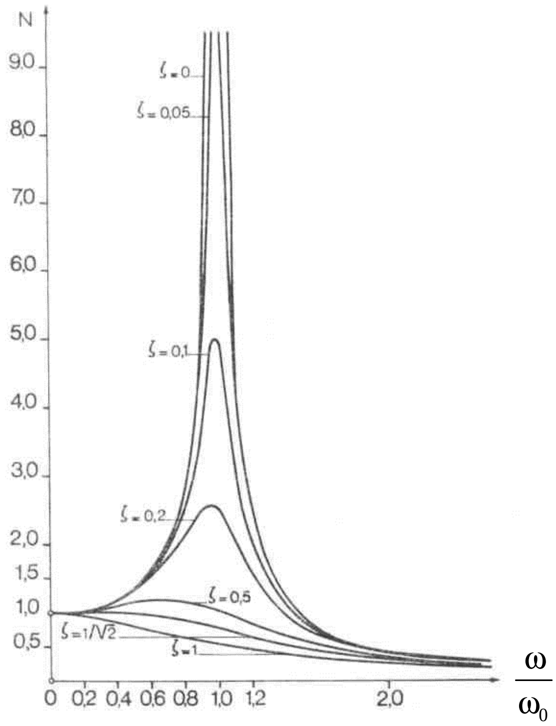
$$q(t)=\frac{1}{m\omega_0^2}= \text{ static response to a unit static force}$$

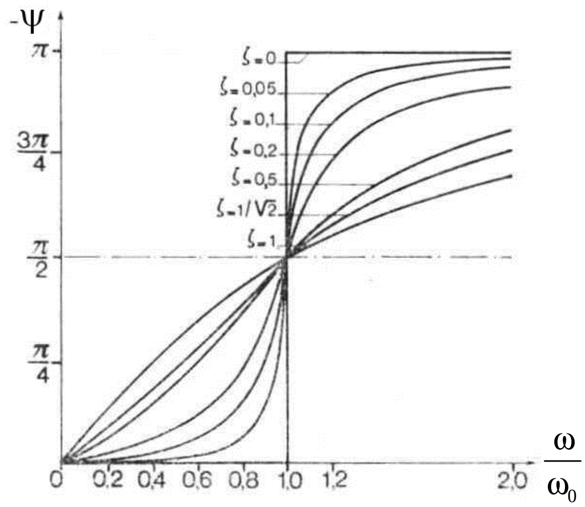
The magnification factor $N(\omega)$ is defined as the ratio between the amplitude $|H(\omega)|$ of the dynamic response and the amplitude $H(0)$ of the static response:

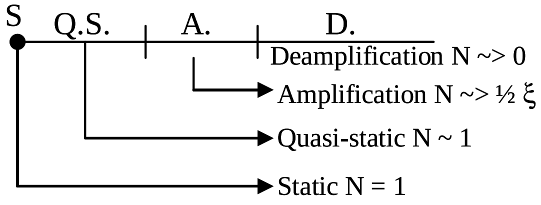
$$\boxed{N(\omega)=\frac{|H(\omega)|}{H(0)}\frac{1}{\sqrt{\left(1-\frac{\omega^2}{\omega_0^2}\right)^2+4\xi^2\frac{\omega^2}{\omega_0^2}}}} \qquad (17)$$

In conclusion: $q(t)=N(\omega)\frac{1}{m\,\omega_0^2}e^{i\omega t}e^{i\psi(\omega)}$

- $N(\omega)$: dynamic magnification of response amplitude
- $\frac{1}{m\,\omega_0^2}e^{i\omega t}$: static response
- $e^{i\omega t}$: force
- $e^{i\psi(\omega)}$: dynamic phase delay







Real elementary harmonic force f(t) = sinωt

$$\ddot{q}(t)+2\xi\omega_0\dot{q}(t)+\omega_0^2 q(t)=\frac{1}{m}f(t)\;;\; q(0)=q_0\;,\;\dot{q}(0)=\dot{q}_0$$

$$\boxed{f(t)=e^{i\omega t}} = \cos\omega t + i\sin\omega t$$

$$q(t)=q'(t)+q''(t)$$

$$q(t)=x'(t)+iy'(t)$$

$$x'(t)=e^{-\xi\omega_0 t}\left(a_{x1}\cos\omega_0\sqrt{1-\xi^2}\,t+a_{x2}\sin\omega_0\sqrt{1-\xi^2}\,t\right)$$

$$y'(t)=e^{-\xi\omega_0 t}\left(a_{y1}\cos\omega_0\sqrt{1-\xi^2}\,t+a_{y2}\sin\omega_0\sqrt{1-\xi^2}\,t\right)$$

$$q''(t)=H(\omega)e^{i\omega t}$$

$$\boxed{f(t)=\sin\omega t}$$

$$q(t)=q'(t)+q''(t)$$

$$q'(t)=y'(t)$$

$$q''(t)=\operatorname{Im}\left[H(\omega)e^{i\omega t}\right]=$$

$$=\operatorname{Im}\left[\left|H(\omega)\right|e^{i\{\omega t+\psi(\omega)\}}\right]=$$

$$=\left|H(\omega)\right|\sin\left[\omega t+\psi(\omega)\right]$$

In steady-state conditions: $q'(t)=0 \Rightarrow q(t)=q''(t) \Rightarrow$

$$\boxed{q(t)=H(\omega)e^{i\omega t}}$$

$$\boxed{q(t)=\left|H(\omega)\right|\sin\left[\omega t+\psi(\omega)\right]}$$

$$\boxed{f(t)=F\sin\omega t} \Rightarrow$$

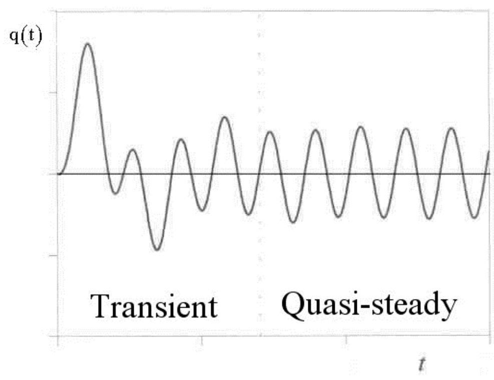
$$q(t)=F\left|H(\omega)\right|\sin\left[\omega t+\psi(\omega)\right]$$

$$N(\omega)=\frac{\left|H(\omega)\right|}{H(0)}=m\omega_0^2\left|H(\omega)\right|=k\left|H(\omega)\right| \Rightarrow$$

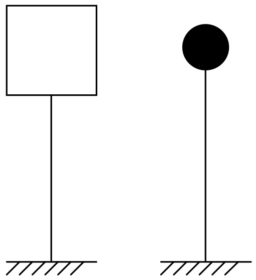
$$q(t)=\frac{F}{k}N(\omega)\sin\left[\omega t+\psi(\omega)\right] \Rightarrow$$

$$\boxed{q(t)=q_{st}\,N(\omega)\sin\left[\omega t+\psi(\omega)\right]}$$

The following figure shows how the steady-state regime is approached after a transient vibration.



Example: Arc lamp

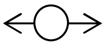


k = 21063 N/m
m = 10671 kg

$\omega_0 = 1.405$ rad/s

$n_0 = 0.223$ Hz

Vibrodyne



$f(t) = F\sin\omega t$ F = 100 N

$$q(t) = \underbrace{q_{st}N(\omega)}_{Q = \frac{F}{k}N(\omega)} \sin\left[\omega t + \psi(\omega)\right]$$

$$\omega \approx 0 \Rightarrow N(\omega) = 1 \Rightarrow Q = 0.00475\,\text{m} = 4.75\,\text{mm}$$

$$\omega = \omega_0 \Rightarrow N(\omega) = 1/2\xi \Rightarrow \quad \xi = 0.01 \Rightarrow N = 50 \Rightarrow Q = 0.237\,\text{m}$$

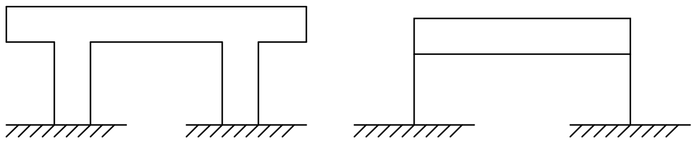
$$\xi = .005 \Rightarrow N = 100 \Rightarrow Q = 0.475\,\text{m}$$

$$\xi = .002 \Rightarrow N = 250 \Rightarrow Q = 1.187\,\text{m}$$

$$\xi = .001 \Rightarrow N = 500 \Rightarrow Q = 2.37\,\text{m}$$

$$\omega = 10\,\text{rad/s} \Rightarrow N(\omega) \approx 0.142 \Rightarrow Q = 6.74 \times 10^{-4}\,\text{m} = 0.674\,\text{mm}$$

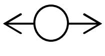
Example: Single-storey R.C. building



$k = 0.8333 \times 10^8$ N/m
m = 88087.5 kg

$\omega_0 = 30.76$ rad/s

$n_0 = 4.9$ Hz



$f(t) = F\sin\omega t$ F = 1000 N

$$\omega \approx 0 \Rightarrow N(\omega) = 1 \Rightarrow Q = 1.2 \times 10^{-5}\,\text{m}$$

$$\omega = \omega_0 \Rightarrow \quad \xi = 0.05 \Rightarrow N = 10 \Rightarrow Q = 1.2 \times 10^{-4}\,\text{m}$$

$$\xi = 0.02 \Rightarrow N = 25 \Rightarrow Q = 3.0 \times 10^{-4}\,\text{m}$$

Periodic force

A function f(t) is defined as periodic with period T when f(t) = f(t + T) for $\forall t \in \mathbf{R}$, with T > 0. The minimum period, or simply the period, is the minimum value of T for which above condition is satisfied.

Under very general conditions, a periodic function f(t) can be expanded according to the following Fourier series:

$$f(t) = \frac{a_0}{2} + \sum_{1}^{\infty}{}_k \left(a_k \cos \omega_k t + b_k \sin \omega_k t \right) \tag{18}$$

where:

$$\begin{aligned} a_k &= \frac{2}{T}\int_{-T/2}^{T/2} f(t) \cos \omega_k t \, dt \qquad k = 0, 1, 2, ... \\ b_k &= \frac{2}{T}\int_{-T/2}^{T/2} f(t) \sin \omega_k t \, dt \qquad k = 1, 2, ... \\ \omega_k &= k\frac{2\pi}{T} \qquad k = 0,1,2, ... \infty \end{aligned} \tag{19}$$

The mean value of f(t) is $a_0 / 2$:

The Fourier series:

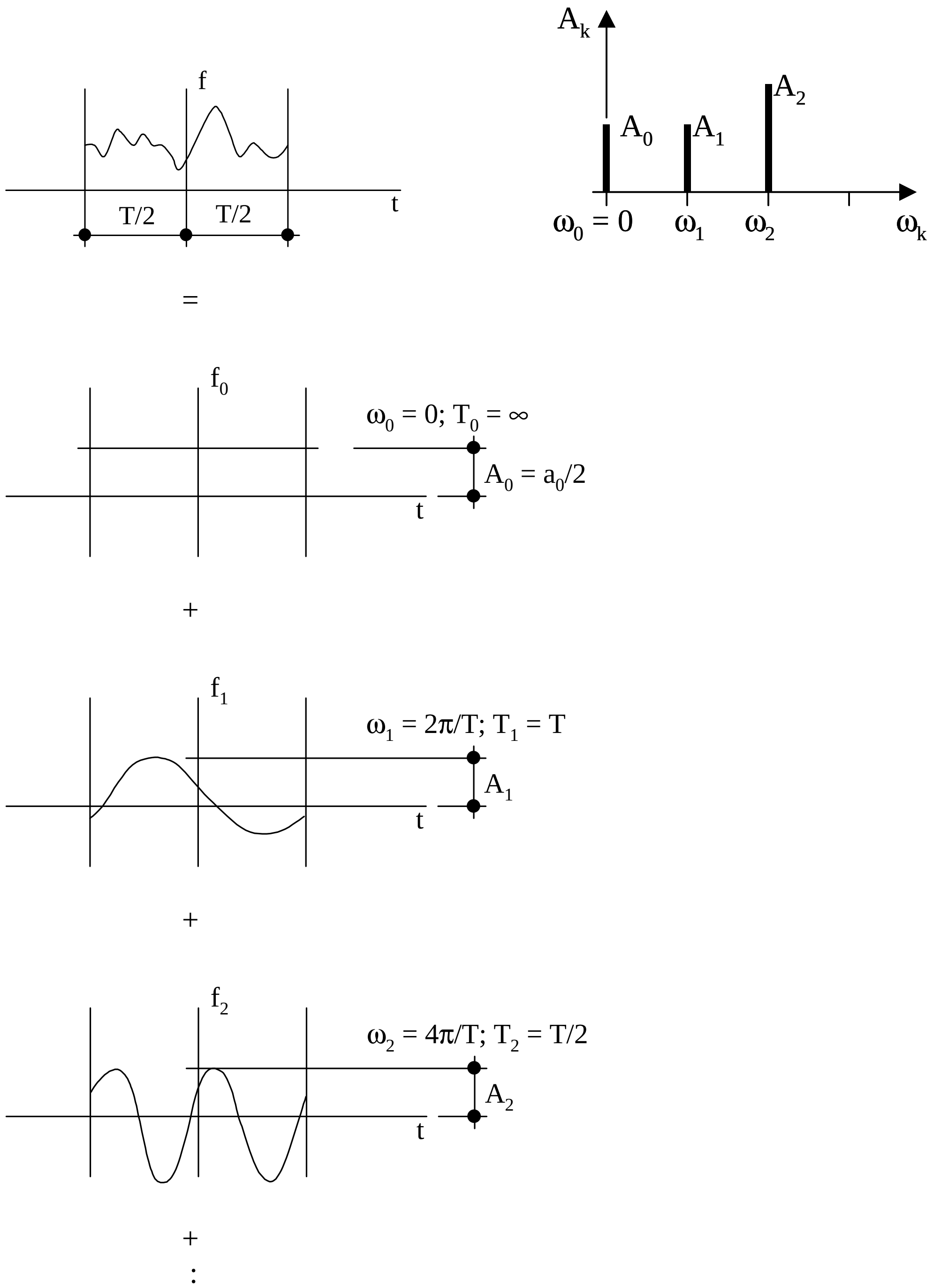
$$f(t) = \frac{a_0}{2} + \sum_{1}^{\infty}{}_k \left(a_k \cos \omega_k t + b_k \sin \omega_k t \right)$$

may be rewritten as:

$$f(t) = \frac{a_0}{2} + \sum_{1}^{\infty}{}_k A_k \sin\left(\omega_k t + \varphi_k \right) \tag{20}$$

where:

$$\begin{aligned} A_k &= \sqrt{a_k^2 + b_k^2} \\ \varphi_k &= \operatorname{arctg}\left(\frac{a_k}{b_k} \right) \end{aligned} \tag{21}$$



Moreover, the Fourier series may be rewritten using the following exponential complex notation:

$$f(t) = \sum_{1}^{\infty} {}_k\, c_k e^{i\omega_K t} \quad \omega_k = k\frac{2\pi}{T} \tag{22}$$

$$c_k = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\, e^{-i\omega_K t} dt \tag{23}$$

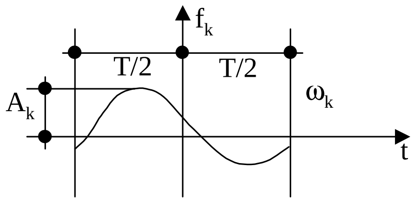
Rewriting Eq. (22) as:

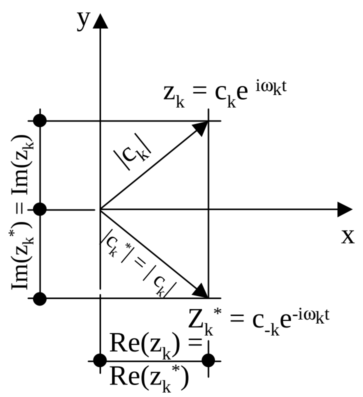
$$f\left(t\right)=c_0+\sum_{1}^{\infty}{}_k\left(c_k\,e^{i\omega_k t}+c_{-k}\,e^{-i\omega_k t}\right) \qquad (24)$$

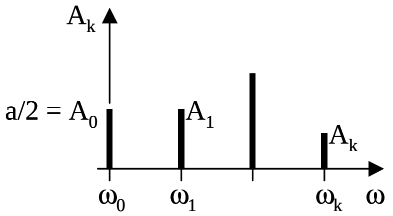
the correspondence with Eq. (20) is apparent. Each real harmonic term in Eq. (20) corresponds to a couple of complex harmonic terms in Eq. (24). In particular:

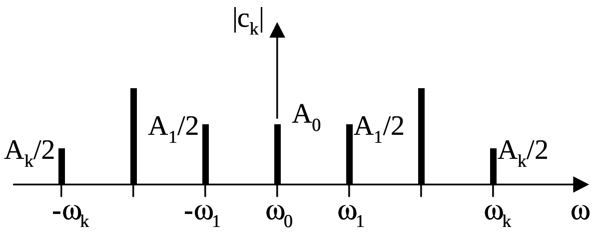
$c_0=a_0/2$; $\left|c_k\right|=\left|c_{-k}\right|=A_k/2=\sqrt{a_k^2+b_k^2}/2$;
$c_k\,e^{i\omega_k t}$ is the complex conjugate of $c_{-k}e^{-i\omega_k t}$.

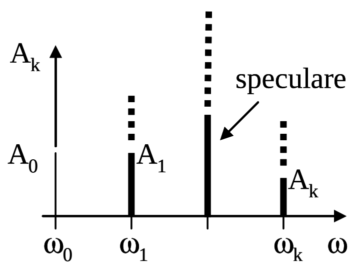
$$f\left(t\right)=\underbrace{\frac{a_o}{2}}_{f_o=\bar{f}}+\sum_{1}^{\infty}{}_k\underbrace{A_k\sin\left(\omega_k t+\varphi_k\right)}_{f_k(t)}=\underbrace{c_o}_{f_o=\bar{f}}+\sum_{1}^{\infty}{}_k\underbrace{\left(c_k\,e^{i\omega_k t}+c_{-k}\,e^{-i\omega_k t}\right)}_{f_k(t)}$$











Dynamic response to a periodic force

$$\ddot{q}(t) + 2\xi\omega_0\dot{q}(t) + \omega_0^2 q(t) = \frac{1}{m} f(t)$$

The steady-state response q(t) to a periodic force $f(t) = \sum_k f_k(t)$ may be expressed as the superposition of the responses $q_k(t)$ to the component elementary harmonic responses $f_k(t)$, i.e. $q(t) = \sum_k q_k(t)$:

$$f_k(t) = e^{i\omega_k t} \Rightarrow \qquad q_k(t) = H(\omega_k)\ e^{i\omega_k t}$$
$$f_k(t) = c_k e^{i\omega_k t} \Rightarrow \qquad q_k(t) = c_k H(\omega_k)\ e^{i\omega_k t}$$

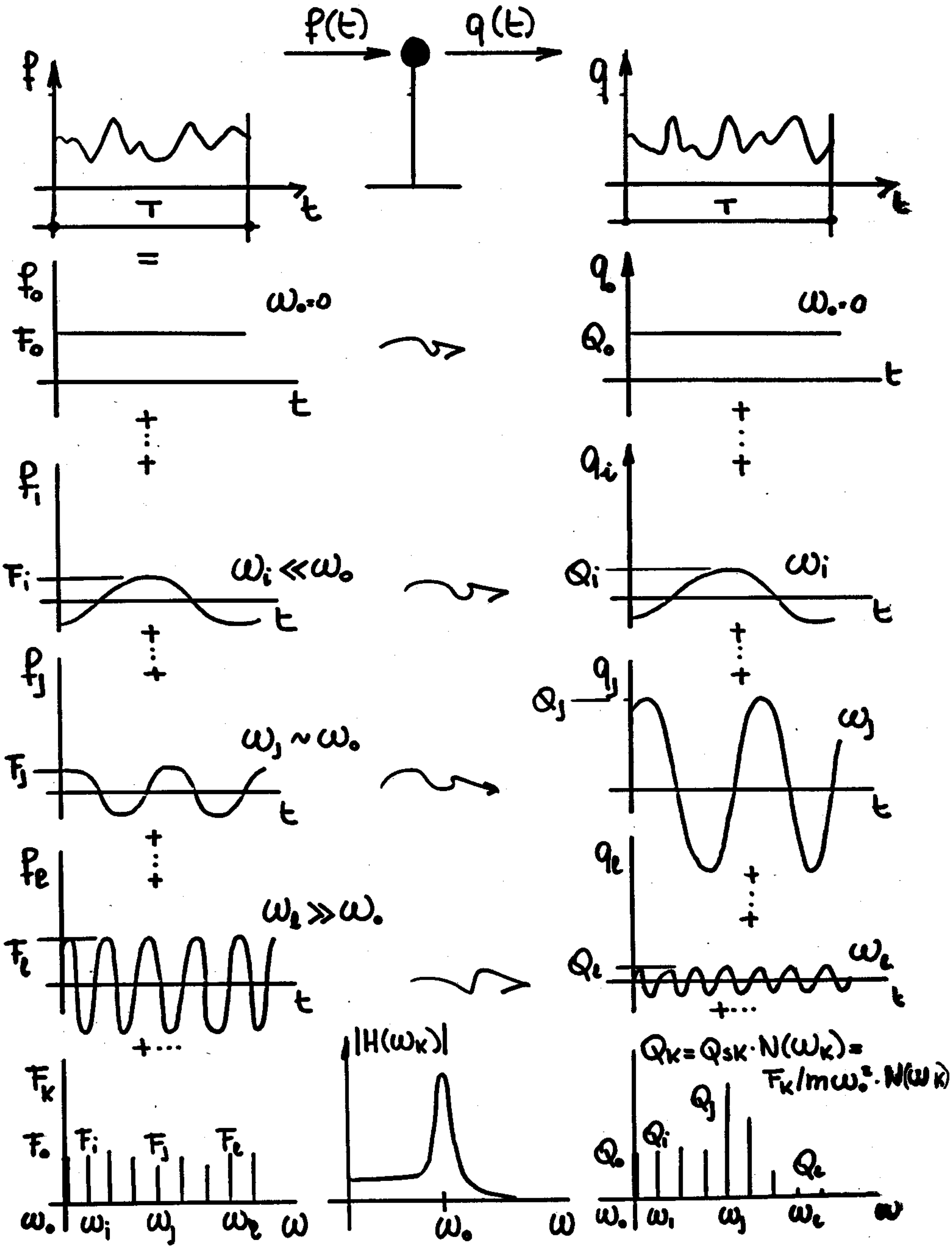
$$f_k(t) = \sum_{-\infty}^{\infty}{}_k f_k(t) = \sum_{-\infty}^{\infty}{}_k c_k e^{i\omega_k t} \Rightarrow \tag{25}$$

$$q(t) = \sum_{-\infty}^{\infty}{}_k q_k(t) = \sum_{-\infty}^{\infty}{}_k c_k H(\omega_k) e^{i\omega_k t} \tag{26}$$

The structural system operates a filtering effect related to its complex frequency response function. $F_k = 2|c_k|$ is the amplitude of the k-th component harmonic force $f_k(t)$. The amplitude of the k-th component harmonic response $q_k(t)$ to $f_k(t)$ is given by:

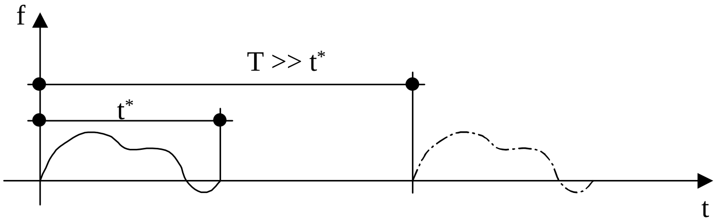
$$Q_k = 2|c_k|\,|H(\omega_k)| = F_k|H(\omega_k)| = F_k H(0)\,N(\omega_k) =$$
$$F_k / m\omega_o^2 \cdot N(\omega_k) \Rightarrow Q_k = Q_{sk}\,N(\omega_k)$$

where $Q_{sk} = F_k / m\omega_o^2$ is the amplitude of the static response to a static force F_k.


f(t)
q(t)
f
q
T
t
=
f₀
F₀
ω₀=0
q₀
Q₀
fᵢ
Fᵢ
ωᵢ≪ω₀
qᵢ
Qᵢ
ωᵢ
fⱼ
Fⱼ
ωⱼ∼ω₀
qⱼ
Qⱼ
ωⱼ
fₗ
Fₗ
ωₗ≫ω₀
qₗ
Qₗ
ωₗ
+...
Fₖ
|H(ωₖ)|
Qₖ=Qsₖ·N(ωₖ)=
Fₖ/mω₀²·N(ωₖ)
ω

Generic force

A generic force $f(t)$ may be dealt with as a periodic force with period $T \to \infty$.



Let us consider the complex exponential Fourier series:

$$f(t) = \sum_{-\infty}^{\infty}{}_k c_k e^{i\omega_k t}$$
$$c_k = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\, e^{i\omega_k t} dt$$
$$\omega_k = k \cdot 2\pi / T$$

Assuming $\Delta\omega = \omega_{k+1} - \omega_k = 2\pi / T \Rightarrow 1/T = \Delta\omega / 2\pi \Rightarrow$

$$f(t) = \sum_{-\infty}^{\infty}{}_k e^{i\omega_k t} \frac{1}{T}\int_{-T/2}^{T/2} f(\eta)\, e^{-i\omega_k \eta} d\eta =$$
$$= \frac{1}{2\pi}\sum_{-\infty}^{\infty}{}_k e^{i\omega_k t} \Delta\omega \int_{-T/2}^{T/2} f(\eta)\, e^{-i\omega_k \eta} d\eta$$

For $T \to \infty$, $\Delta\omega \to 0$, ω tends to become a continuos variable $\Rightarrow$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t} \left[\int_{-\infty}^{\infty} f(\eta)\, e^{i\omega\eta} d\eta\right] d\omega$$

and the exponential Fourier series tends to become the Fourier integral:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{i\omega t}\, d\omega \qquad (27)$$

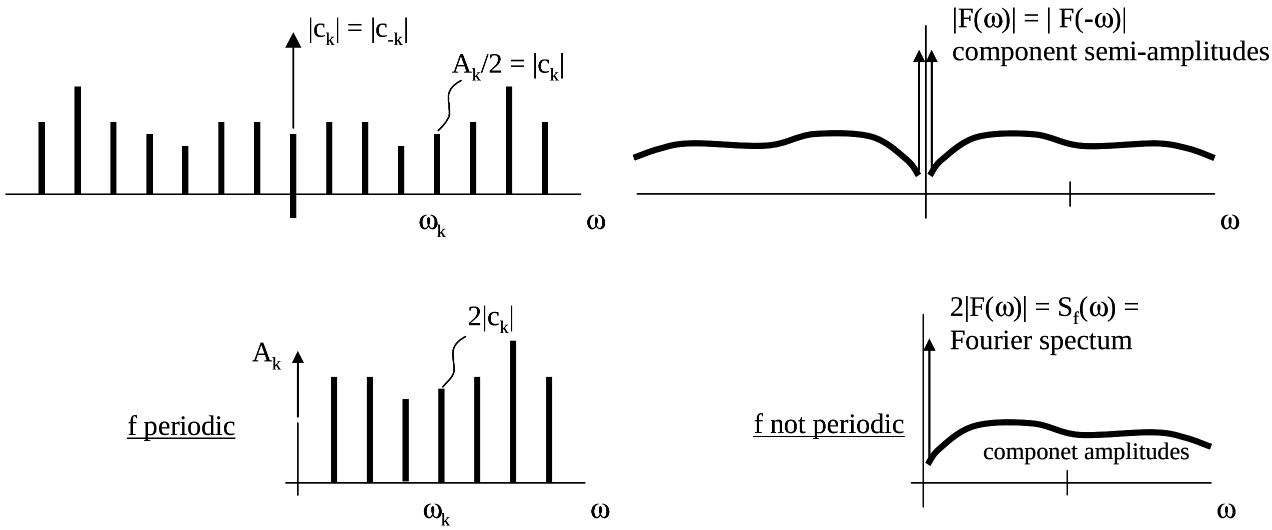
$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt \qquad (28)$$

$F(\omega)$ is a complex function called Fourier transform; $f(t)$ is consequently called inverse Fourier transform. The uniqueness of a Fourier couple, $f(t)$ and $F(\omega)$, is demonstrated under wide conditions. $F(\omega)$ exists provided that:

$\int_{-\infty}^{\infty} |f(t)|\, dt$ is finite.

It can be shown that:

$$F(\omega_k) = \lim_{T\to\infty} c_k;\ f(t) = f_0 \Rightarrow F(\omega) = f_0\delta(\omega)$$



$$\ddot{q}(t) + 2\xi\omega_0\dot{q}(t) + \omega_0^2 q(t) = \frac{1}{m} f(t)$$

The steady-state response q(t) to a generic force $f(t)$ can be expressed as the integral of the elementary component responses to the elementary component harmonic forces:

$$f(t) = e^{i\omega t} \Rightarrow \qquad q(t) = H(\omega)e^{i\omega t}$$
$$f(t) = F(\omega)e^{i\omega t} \Rightarrow \qquad q(t) = F(\omega)\,H(\omega)e^{i\omega t}$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}d\omega \Rightarrow \tag{29}$$

$$q(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\,H(\omega)e^{i\omega t}d\omega \tag{30}$$

Moreover, using the definition of Fourier transform and inverse Fourier response of the response:

$$q(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Q(\omega)e^{i\omega t}d\omega \tag{31}$$

$$Q(\omega) = \int_{-\infty}^{\infty} q(t)e^{-i\omega t}dt \tag{32}$$

Comparing Eqs. (30) and (32):

$$\boxed{Q(\omega) = H(\omega)\,F(\omega)} \qquad (33)$$

Eq. (33) is the basic relationship between $f(t)$ and $q(t)$ in the frequency domain.

Summarising, the frequency domain analysis consists of 4 steps:

(1) Starting from $f(t)$ its Fourier transform is calculated $F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt$;

(2) The structural system is characterised by its complex frequency response function:
$H(\omega) = \frac{1}{m\,\omega_o^2}\,\frac{1}{\left(1-\omega^2/\omega_o^2\right) + 2i\xi\omega/\omega_o}$;

(3) The Fouriem of $q(t)$ is determined: $Q(\omega) = H(\omega)\,F(\omega)$;

(4) The inverse Fourier transform of $Q(\omega)$ is calculated: $q(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Q(\omega)e^{i\omega t}d\omega$.

It is easy to demonstrate that:

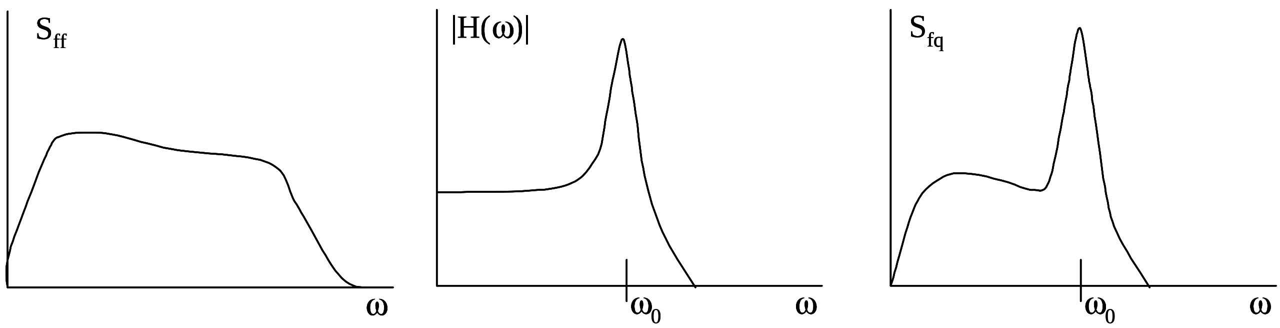
$$\boxed{\left|Q(\omega)\right| = \left|H(\omega)\right|\left|F(\omega)\right|} \qquad (34)$$

$S_{ff}(\omega) = 2\left|F(\omega)\right|$ = Fourier spectrum of the force =
= amplitude of the harmonic components of $f(t)$

$S_{fq}(\omega) = 2\left|Q(\omega)\right|$ = Fourier spectrum of the response =
= amplitude of the harmonic components of $q(t)$

$\left|H(\omega)\right| = N(\omega)/m\omega_o^2$ = Ratio between the amplitudes of the harmonic components of the response and of the force

Thus: $S_{fq}(\omega) = \left|H(\omega)\right| \cdot S_{ff}(\omega)$



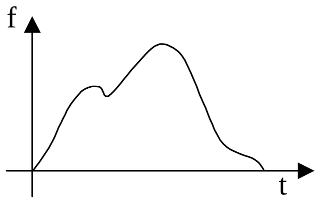
SINGLE-DEGREE-OF-FREEDOM-SYSTEMS

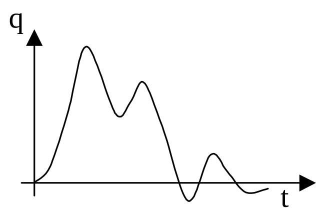
Response spectrum analysis

$$\ddot{q}(t) + 2\xi\omega_0\dot{q}(t) + \omega_0^2 q(t) = \frac{1}{m} f(t)$$
$$q(0) = \dot{q}(0) = 0$$

$f(t) =$ short duration action

$$q(t) = Q\left\{t; \omega_0, \xi, \frac{f(t)}{m}\right\}$$



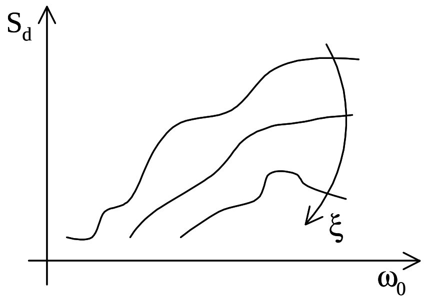


The response spectrum S is a diagram that furnishes a maximum value of the dynamic response (the displacement, the velocity, the acceleration, …) as a function of the structural parameters, for a given loading history.

Response spectrum of the displacement:

$$S_d = \left|q(t)\right|_{max} = S\left\{\omega_0, \xi; \frac{f(t)}{m}\right\}$$

fixed $f(t)/m \Rightarrow$

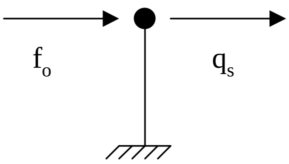


Let us define:

$$f_o = |f(t)|_{max}$$

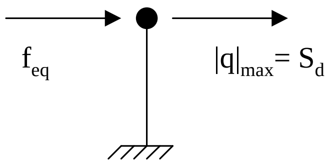
$q_s = f_o / k =$ displacement due to f_o

$$\boxed{f_o = kq_s}$$



The equivalent static force f_{eq} is a force that statically applied on the structure causes the maximum displacement due to the effective dynamic action.

$$\boxed{f_{eq} = k|q|_{max} = kS_d}$$



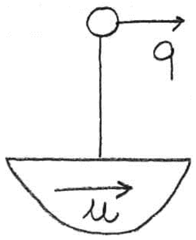
The dynamic magnification factor β, or more simply the dynamic factor, is the non-dimensional ratio:

$$\boxed{\beta = \frac{f_{eq}}{f_o} = \frac{S_d}{q_s}} \qquad \Rightarrow \qquad \boxed{f_{eq} = \beta f_o} \qquad \boxed{|q|_{max} = \beta q_s}$$

$\beta < 1 \Rightarrow$ dynamics reduce the response
$\beta = 1 \Rightarrow$ quasi-static behaviour
$\beta > 1 \Rightarrow$ dynamics amplify the response

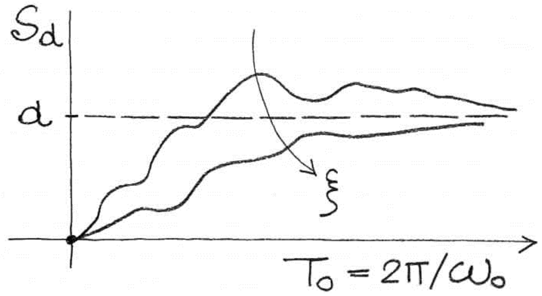
Seismic response spectrum

$$\ddot{q}(t) + 2\xi\omega_0\dot{q}(t) + \omega_0^2 q(t) = -\ddot{u}(t)$$
$$q(0) = \dot{q}(0) = 0$$



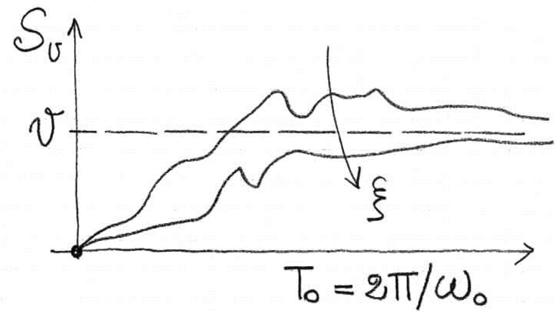
Response spectrum of the relative displacement

$$S_d = |q(t)|_{max}$$
$$d = |u(t)|_{max}$$



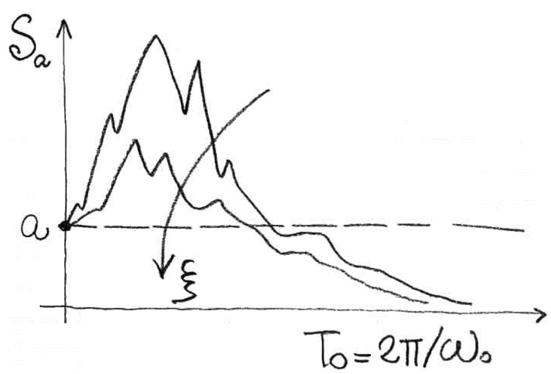
Response spectrum of the relative velocity

$$S_v = |\dot{q}(t)|_{max}$$
$$v = |\dot{u}(t)|_{max}$$

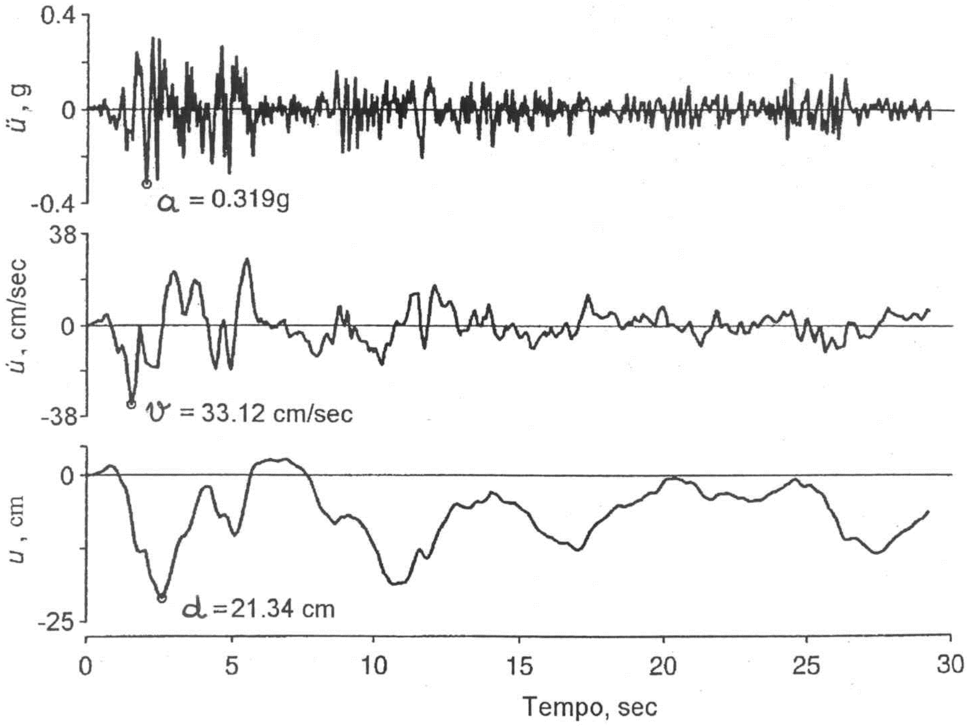


Response spectrum of the absolute acceleration

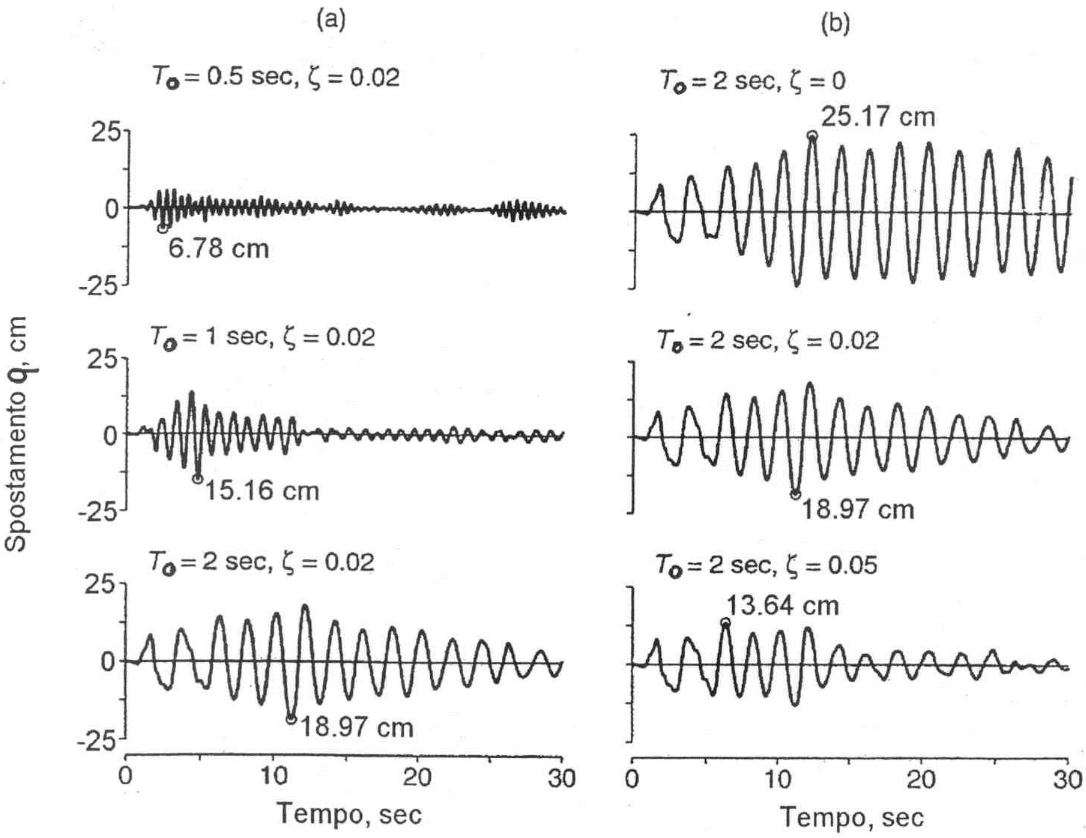
$$S_a = |\ddot{q}(t) + \ddot{u}(t)|_{max}$$
$$a = |\ddot{u}(t)|_{max}$$



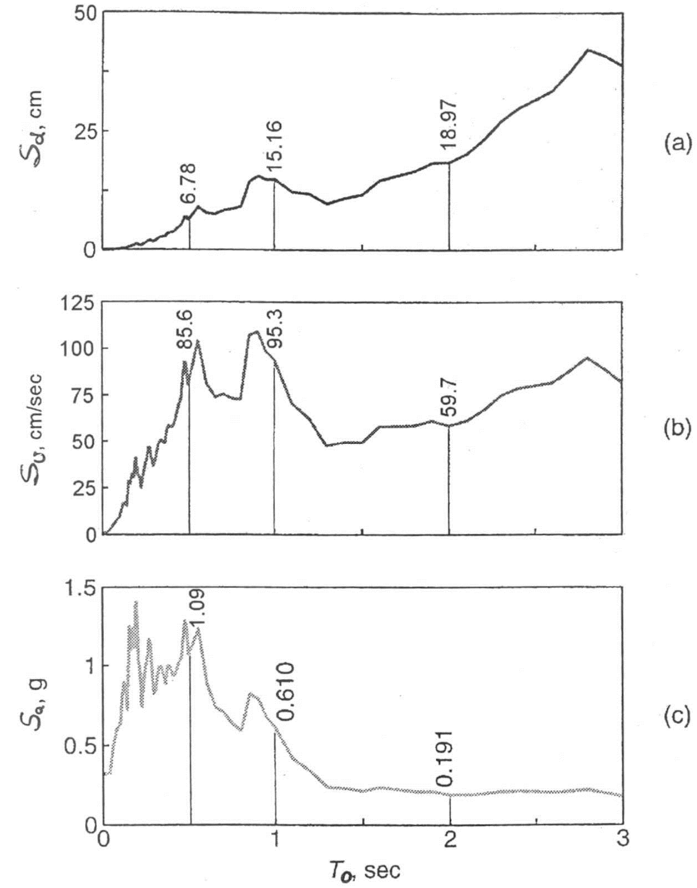
El Centro earthquake (component N-S) 18 May 1940



Dynamic response of S.D.O.F. systems to the El Centroearthquake



Response spectra (ζ = 2%)



<u>Response spectrum of the relative pseudo-velocity</u>

$S_{pv} = \omega_0 S_d$

<u>Response spectrum of the absolute pseudo-acceleration</u>

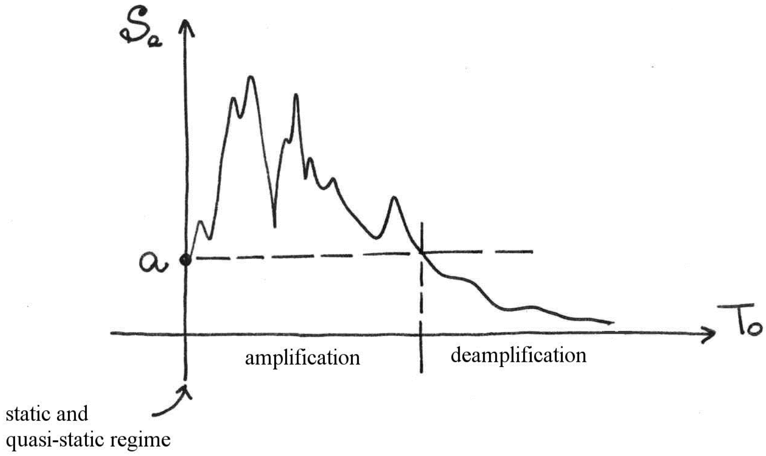
$S_{pa} = \omega_0^2 S_d$

It is possible to show that, for $\xi << 1$:

$S_{pv} \simeq S_v$ (a)

$S_{pa} \simeq S_a$ (b)

Thus, the knowledge of any spectrum among S_d, S_v and S_a allows to derive the other two.



a) For $\xi << 1$ any response cycle, in particular that corresponding to reaching $|q|_{max}$, can be represented by a harmonic with circular frequency ω_0

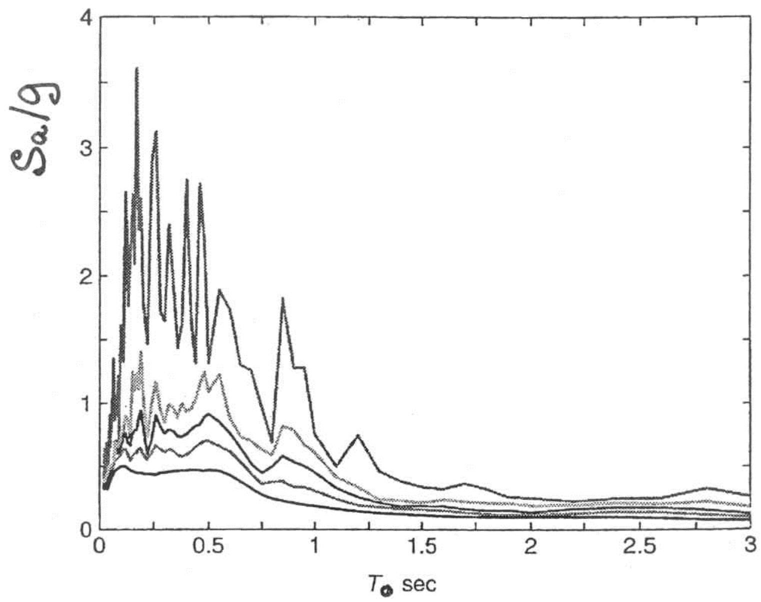
$$q(t) \cong |q|_{max} \sin\omega_0 t \Rightarrow$$
$$\dot{q}(t) \cong |q|_{max}\, \omega_0 \sin\omega_0 t \simeq |\dot{q}|_{max} \cos\omega_0 t \Rightarrow$$
$$S_d\omega_0 \simeq S_v$$

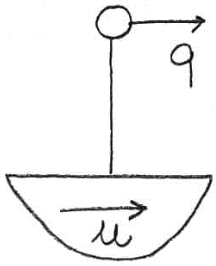
b) For $\xi = 0$ the equation of motion results:

$$\ddot{q}(t) + \omega_0^2 q(t) = -\ddot{u}(t) \Rightarrow$$
$$\left|\ddot{q}(t) + \ddot{u}(t)\right| = \omega_0^2 \left|q(t)\right| \Rightarrow$$
$$\left|\ddot{q}(t) + \ddot{u}(t)\right|_{max} = \omega_0^2 \left|q(t)\right|_{max} \Rightarrow$$
$$S_a = \omega_0^2 S_d$$

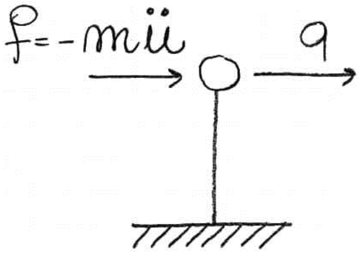
Therefore, for $\xi << 1$, $S_a \simeq \omega_0^2 S_d$

Response spectrum of the absolute pseudo-acceleration
for the El Centro earthquake- ζ = 0, 2, 5, 10, 20 %





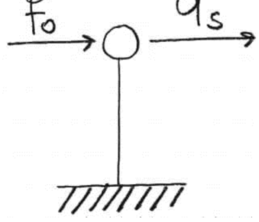
$\Rightarrow$



$$f_o = \left|f(t)\right|_{max} = \left|-m\ddot{u}(t)\right|_{max} \Rightarrow$$

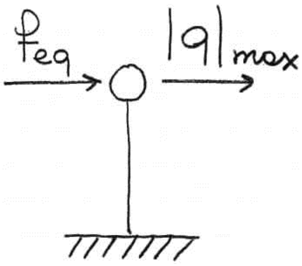
$$f_o = ma$$

$$q_s = \frac{f_o}{k} = \frac{ma}{k} \Rightarrow q_S = \frac{a}{\omega_o^2}$$



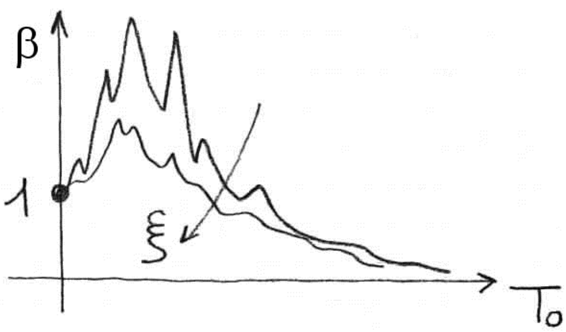
$$f_{eq} = k\left|q(t)\right|_{max} = kS_d = k\frac{S_a}{\omega_0^2} \Rightarrow$$

$$f_{eq} = mS_a$$



$$\beta = \frac{f_{eq}}{f_o} = \frac{\cancel{m}\ S_a}{\cancel{m}\ a} \Rightarrow$$

$$f_{eq} = \beta f_o$$



Eurocode 8 – Seismic actions

$$F_e = mS_e(T, \xi)$$

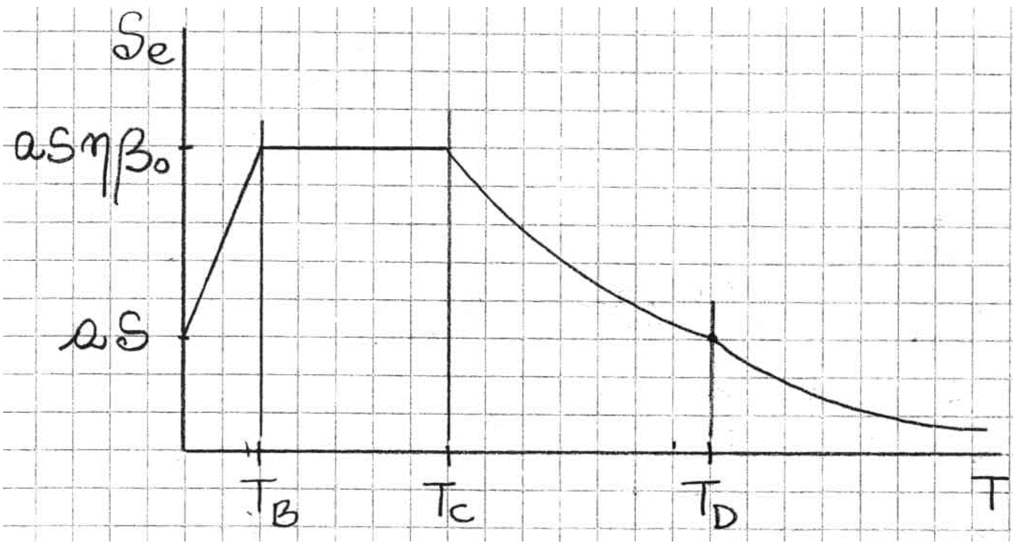
$$S_e(T, \xi) = aS\left[1 + \frac{T}{T_B}(\eta\beta_o - 1)\right] \qquad \text{for } 0 < T < T_B$$

$$S_e(T, \xi) = aS\eta\beta_o \qquad \text{for } T_B < T < T_C$$

$$S_e(T, \xi) = aS\eta\beta_o\left[\frac{T_C}{T}\right]^{k_1} \qquad \text{for } T_C < T < T_D$$

$$S_e(T, \xi) = aS\eta\beta_o\left[\frac{T_C}{T_D}\right]^{k_1}\left[\frac{T_D}{T}\right]^{k_2} \qquad \text{for } T_D < T$$

$$\eta = \sqrt{0.07/(0.02 + \xi)} \quad > 0.7 \qquad (\xi = 0.05 \Rightarrow \eta = 1)$$



Soil	S	β_o	K_1	K_2	$K_B(s)$	$K_C(s)$	$K_D(s)$
A	1.0	2.5	1.0	2.0	0.10	0.40	3.0
B	1.0	2.5	1.0	2.0	0.15	0.60	3.0
C	0.9	2.5	1.0	2.0	0.20	0.80	3.0

Soil A: rocky or very compact
Soil B: mean consistency
Soil C: soft

Italian Code:	zone 1:	a = 0.35g
	zone 2:	a = 0.25g
	zone 3:	a = 0.15g
	zone 4:	a = 0.05g

Example: Arc lamp

Zone 2 $\Rightarrow$ a = 0.25 g = 2.45 m/s^2
Soil B $\Rightarrow S = 1, \beta_o = 2.5, k_1 = 1, k_2 = 2, T_B = 0.15s, T_C = 0.6s, T_D = 3s$

$\omega_0 = 1.405 rad/s$, $n_0 = 0.223 Hz$, $T_0 = 4.47$ s, $m = 10671 kg$

$\xi = 0.02 \Rightarrow \eta = 1.32$

$T_0 > T_D \Rightarrow S_e = 0.297a = 0.728 \quad m/s^2$

$F_e = m S_e = 10671 \times 0.728 = 7772 N$

$|q|_{max} = F_e / k = 7772 / 21063 = 0.37 m$

Base bending moment $M = F_e h = 7772 \times 30 = 233160 Nm$

Maximum stress $\sigma = M\phi / 2 / J = 233160 \times 0.275 / 9.027 \times 10^{-4} =$

$= 71.030.243 N/m^2 = 724 kgf/cm^2$

Example: Single-storey r.c. building

Zone 2 $\Rightarrow$ a = 0.25 g = 2.45 m/s^2
Soil B $\Rightarrow S = 1, \beta_o = 2.5, k_1 = 1, k_2 = 2, T_B = 0.15s, T_C = 0.6s, T_D = 3s$

$\omega_0 = 20.94 rad/s$, $n_0 = 3.33 Hz$, $T_0 = 0.30s$, $m = 10671 kg$

$\xi = 0.05 \Rightarrow \eta = 1$

$T_B < T_0 < T_C \Rightarrow S_e = 2.5a = 6.125 m/s^2$

$F_e = m S_e = 190.087.5 \times 6.125 = 1.164.285 N$

$|q|_{max} = F_e / k = 1.164.285 / 0.833 \times 10^8 = 0.014 m$

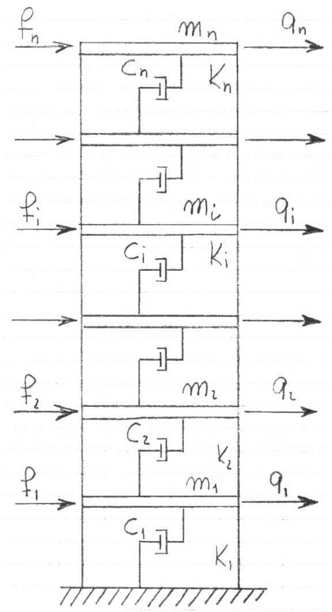
Base bending moment $M = 6EJ_p |q|_{max} / h^2 =$

$= 6 \times 0.3 \times 10^{11} \times 0.025 \times 0.014 / \overline{6}^2 = 1.750.000 Nm$

N-DEGREES-OF-FREEDOM-SYSTEMS

Equations of motion

Shear-type system – damped forced vibrations



Equation of motion of the i-th mass; 2nd Newton law:

$$F_i = m_i a_i$$

$$F_i = -k_i(q_i - q_{i-1}) - k_{i+1}(q_i - q_{i+1}) -$$ restoring elastic forces

$$-c_i(\dot{q}_i - \dot{q}_{i-1}) - c_{i+1}(\dot{q}_i + \dot{q}_{i+1}) +$$ viscous damping forces

$$+f_i$$ external force

$a_i = \ddot{q}_i$ absolute acceleration

1) $$m_1\ddot{q}_1 + (c_1 + c_2)\dot{q}_1 - c_2\dot{q}_2 - + (k_1 + k_2)q_1 - k_2 q_2 = f_1$$

i) $$m_i\ddot{q}_i - c_i\dot{q}_{i-1} + (c_i + c_{i+1})\dot{q}_i - c_{i+1}\dot{q}_{i+1} - - k_i q_{i-1} + (k_i + k_{i+1})q_i - k_{i+1}q_{i+1} = f_i$$

n) $$m_n\ddot{q}_n - c_n\dot{q}_{n-1} + c_n\dot{q}_n - k_n q_{n-1} + k_n q_n = f_n$$

In matrix form:

$$\mathbf{M}\ddot{\mathbf{q}}(t)+\mathbf{C}\dot{\mathbf{q}}(t)+\mathbf{K}\mathbf{q}(t)=\mathbf{f}(t)$$
$$\mathbf{q}(0)=\mathbf{q}_o \; ; \; \dot{\mathbf{q}}(0)=\dot{\mathbf{q}}_o$$

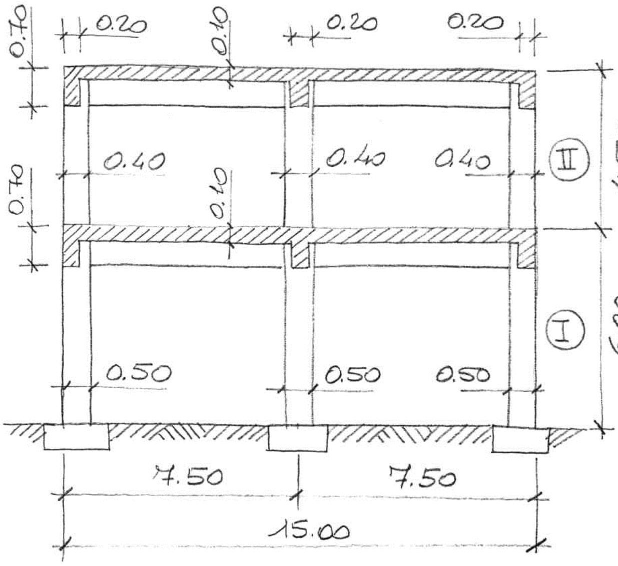
$$\mathbf{q}=\begin{Bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{Bmatrix} \quad ; \quad \mathbf{f}=\begin{Bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{Bmatrix} \quad ; \quad \mathbf{M}=\begin{bmatrix} m_1 & 0 & \cdots & 0 \\ 0 & m_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & m_n \end{bmatrix}$$

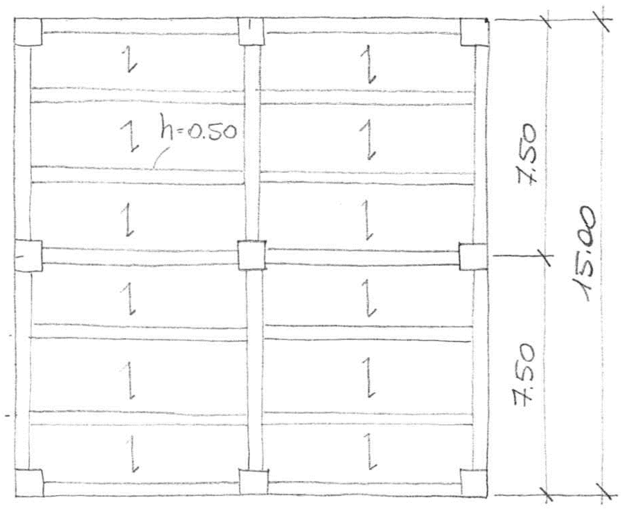
$$\mathbf{K}=\begin{bmatrix} k_1+k_2 & -k_2 & & & & \\ -k_2 & k_2+k_3 & -k_3 & & 0 & \\ & & & -k_{n-1} & k_{n-1}+k_n & -k_n \\ & 0 & & & -k_n & k_n \end{bmatrix}$$

$$\mathbf{C}=\begin{bmatrix} c_1+c_2 & -c_2 & & & & \\ -c_2 & c_2+c_3 & -c_3 & & 0 & \\ & & & -c_{n-1} & c_{n-1}+c_n & -c_n \\ & 0 & & & -c_n & c_n \end{bmatrix}$$

In this case **M, K**, **C** are real, symmetric, positive definite matrices; **M** is also diagonal.

<u>Example: Shear-Type building with 2 D.O.F.</u>





Mass of the column per unit length

1st order: $m_I = 0.50 \times 0.50 \times 2500 = 625 \text{ kg/m}$

2nd order: $m_{II} = 0.40 \times 0.40 \times 2500 = 400 \text{ kg/m}$

Mass of the beams (outside the slab) per each floor

Main beams: $m_{tp} = 0.20 \times 0.60 \times 7.50 \times 2500 = 2250 \text{ kg}$

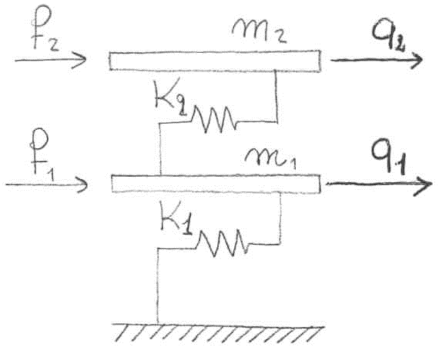
Secundary beams: $m_{tps} = 0.20 \times 0.40 \times 7.50 \times 2500 = 1500 \text{ kg}$

Total mass: $m_t = 2500 \times 12 + 1500 \times 8 = 39000 \text{ kg}$

Mass of the slab per each floor

$m_s = 0.10 \times 15 \times 15 \times 2500 = 56250 \text{ kg}$

Structural scheme



$$\mathbf{q} = \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} \quad ; \quad \mathbf{f} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix}$$

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \quad ; \quad \mathbf{K} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}$$

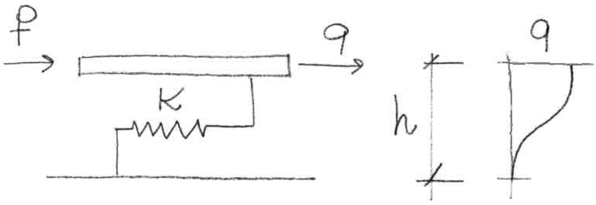
Mass at the 1st level

–	columns:	$(625 \times 3 + 400 \times 2.25) \times 9$	= 24975 kg
–	beams:		= 39000 “
–	slab:		= 56250 “
–	sottofondo:	$0.03 \times 15 \times 15 \times 1700$	= 11475 “
–	pavement :	$15 \times 15 \times 40$	= 9000 “
–	walls :	$80 \times 15 \times 15$	= 18000 “
–	accidental load:	$500 \times 15 \times 15$	= 112500 “
			m_1 = 271200 kg

Mass at the 2nd level

–	columns:	$400 \times 2.25 \times 9$	= 8100 kg
–	beams:		= 39000 “
–	slab:		= 56250 “
–	sottofondo:		= 11475 “
–	pavement:		= 9000 “
–	accidental load:	$100 \times 15 \times 15$	= 22500 “
			m_2 = 146325 kg

Inter-storey stiffness



$$f = \sum_{1}^{N} {}_k f_k = \sum_{1}^{N} {}_k \left(\frac{12EJ}{h^3} \right) q = N \frac{12EJ}{h^3} q \Rightarrow$$

$$k = \frac{f}{q} = N \frac{12EJ}{h^3}$$

N = number of columns = 9

E = $.3 \times 10^{11}$ N/m^2

$$J_I = .5 \times .\overline{5}^3 / 12 = 5.2083 \times 10^{-3} \text{ m}^4 \quad ; \; J_{II} = .4 \times .\overline{4}^3 / 12 = 2.1333 \times 10^{-3} \text{ m}^4$$

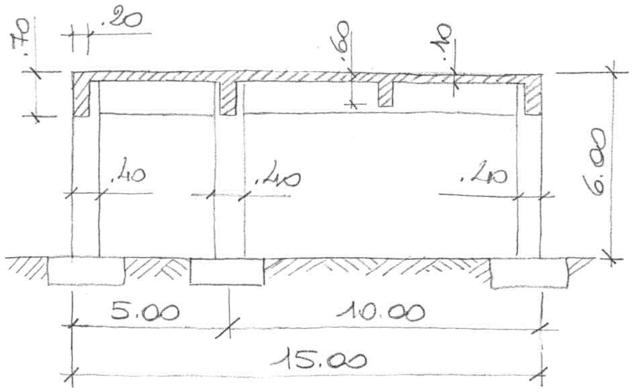
$$h_I = 5.65 \text{ m} \quad ; \; h_{II} = 4.50 \text{ m}$$

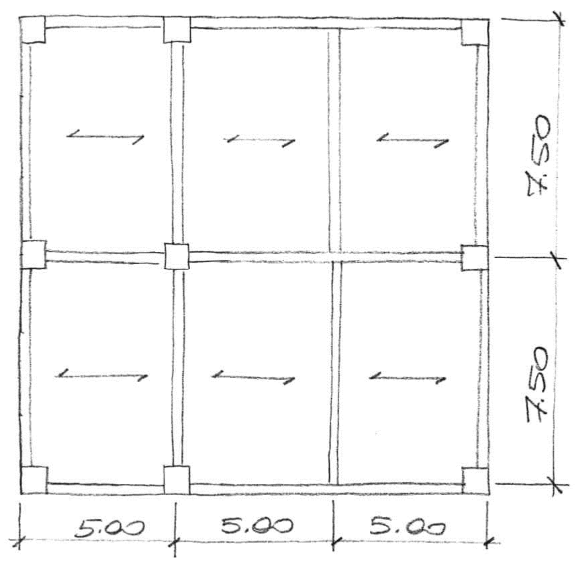
$$k_1 = \frac{9 \times 12 \times 3 \times 10^{11} \times 5.2083 \times 10^{-3}}{\overline{5.65}^3} = 0.9356 \times 10^8 \text{ N/m}$$

$$k_2 = \frac{9 \times 12 \times 3 \times 10^{11} \times 2.1333 \times 10^{-3}}{\overline{4.50}^3} = 0.7585 \times 10^8 \text{ N/m}$$

$$\mathbf{M} = \begin{bmatrix} 271200 & 0 \\ 0 & 146325 \end{bmatrix} \text{kg} \quad ; \quad \mathbf{K} = \begin{bmatrix} 16941 \times 10^8 & -0.7585 \times 10^8 \\ -0.7585 \times 10^8 & 0.7585 \times 10^8 \end{bmatrix} \text{N/m}$$

Example: Shear-Type single-storey building with 3 D.O.F.





Mass of the column per unit length:

$m_p = 0.40 \times 0.40 \times 2500 = 400 \text{ kg/m}$

Mass of the beams (outside the slab):

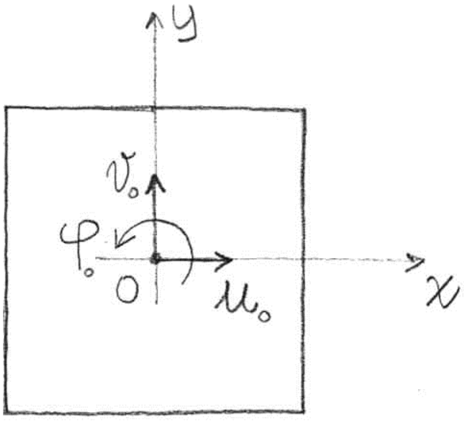
Main beams (per unit length): $m_{tp} = 0.20 \times 0.60 \times 2500 = 300 \text{ kg/m}$

Secundary beams (per unit length): $m_{ts} = 0.20 \times 0.50 \times 2500 = 250 \text{ kg/m}$

Total mass: $m_t = 300 \times 15 \times 6 + 250 \times 15 = 30750 \text{ kg}$

Mass of the slab of each floor: $m_s = 0.10 \times 15 \times 15 \times 2500 = 56250 \text{ kg}$

Structural scheme



O = center of mass

$$\mathbf{q}_o = \begin{Bmatrix} u_o \\ v_o \\ \vartheta_o \end{Bmatrix} \quad ; \quad \mathbf{f}_o = \begin{Bmatrix} f_{xo} \\ f_{yo} \\ m_{\vartheta o} \end{Bmatrix}$$

$$\mathbf{M} = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & I \end{bmatrix}$$

Mass m

– columns:	$400 \times 3 \times 9$	=	10800 kg
– beams:		=	30750 "
– slab:		=	56250 "
– sottofondo:		=	11475 "
– pavement:		=	9000 "
– accidental load:		=	22500 "
		m =	140775 kg

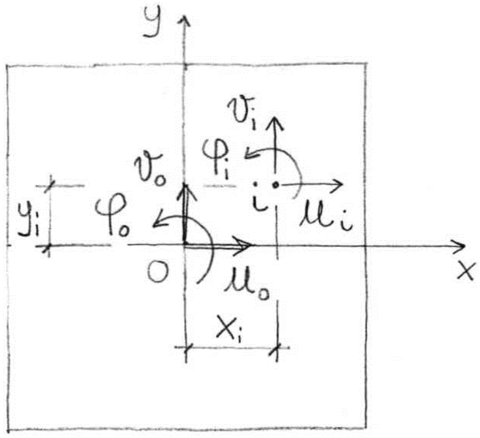
Rotatory mass moment of inertia I

$$I = \frac{m}{A} J_p = \frac{140775}{225} \times 8437.5 = 5279062 \text{ kg m}^2$$

A = surface area = 15 × 15 = 225 m^2

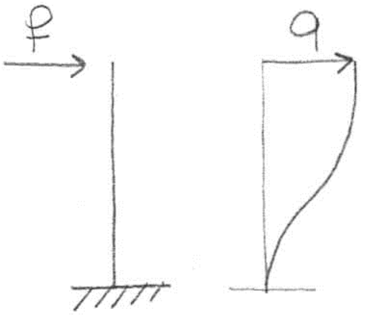
J_p = polar moment of inertia = $J_x + J_y = 2 \times 15 \times \overline{15}^3 / 12 = 8437.5 \text{ m}^4$

Kinematic of the rigid body



$$\begin{cases} u_i = u_o - \varphi_o y_i \\ v_i = v_o + \varphi_o x_i \\ \vartheta_i = \vartheta_o \end{cases}$$

Translational stiffness of columns



$$f = \frac{12EJ}{h^3} q \Rightarrow k = \frac{f}{q} = \frac{12EJ}{h^3} \Rightarrow$$

$$k = k_x = k_y = \frac{12 \times 3 \times 10^{11} \times 2.1333 \times 10^{-3}}{\overline{5.65}^3} = 0.4258 \times 10^7 \text{ N/m}$$

It is assumed that the torsional stiffness of columns is negligible.

Equilibrium equations with respect to 0

$$f_{xo} = \sum_{1}^{N}{}_i f_{xi} = \sum_{1}^{N}{}_i k_{xi}\, u_i = \sum_{1}^{N}{}_i k_{xi}\left(u_o - \varphi_o y_i\right) =$$

$$= u_o \sum_{1}^{N}{}_i k_{xi} - \varphi_o \sum_{1}^{N}{}_i k_{xi} y_i$$

$$f_{yo} = \sum_{1}^{N}{}_i f_{yi} = \sum_{1}^{N}{}_i k_{yi} v_i = \sum_{1}^{N}{}_i k_{yi}\left(v_o + \varphi_o x_i\right) =$$

$$= v_o \sum_{1}^{N}{}_i k_{yi} + \varphi_o \sum_{1}^{N}{}_i k_{yi} x_i$$

$$m_{\vartheta o} = \sum_{1}^{N}{}_i m_{\vartheta i} + \sum_{1}^{N}{}_i \left(-f_{xi} y_i + f_{yi} x_i\right) = \sum_{1}^{N}{}_i \left(-k_{xi} u_i y_i + k_{yi} v_i x_i\right) =$$

$$= -\sum_{1}^{N}{}_i k_{xi}\left(u_o - \varphi_o y_i\right) y_i + \sum_{1}^{N}{}_i k_{yi}\left(v_o + \varphi_o x_i\right) x_i =$$

$$= -u_o \sum_{1}^{N}{}_i k_{xi} y_i + \varphi_o \sum_{1}^{N}{}_i k_{xi} y_i^2 + v_o \sum_{1}^{N}{}_i k_{yi} + \varphi_o \sum_{1}^{N}{}_i k_{yi} x_i^2$$

In matrix form:

$$\underbrace{\begin{Bmatrix} f_{xo} \\ f_{yo} \\ m_{\vartheta o} \end{Bmatrix}}_{\mathbf{f}_o} = \underbrace{\begin{bmatrix} \sum_{1}^{N}{}_i \; k_{yi} x_i & 0 & -\sum_{1}^{N}{}_i \; k_{xi} y_i \\ 0 & \sum_{1}^{N}{}_i \; k_{yi} & \sum_{1}^{N}{}_i \; k_{yi} x_i \\ -\sum_{1}^{N}{}_i \; k_{xi} y_i & \sum_{1}^{N}{}_i \; k_{yi} x_i & \sum_{1}^{N}{}_i \; \left(k_{xi} y_i^2 + k_{yi} x_i^2\right) \end{bmatrix}}_{\mathbf{K}_o} \underbrace{\begin{Bmatrix} u_o \\ v_o \\ \varphi_o \end{Bmatrix}}_{\mathbf{q}_o}$$

In this case:

$$\underbrace{\begin{Bmatrix} f_{xo} \\ f_{yo} \\ m_{\vartheta o} \end{Bmatrix}}_{\mathbf{f}_o} = \underbrace{\begin{bmatrix} Nk & 0 & \cancel{-k\sum_{1}^{N}{}_i y_i} \\ 0 & Nk & k\sum_{1}^{N}{}_i x_i \\ \cancel{-k\sum_{1}^{N}{}_i y_i} & k\sum_{1}^{N}{}_i x_i & k\sum_{1}^{N}{}_i \left(x_i^2 + y_i^2\right) \end{bmatrix}}_{\mathbf{K}_o} \underbrace{\begin{Bmatrix} u_o \\ v_o \\ \varphi_o \end{Bmatrix}}_{\mathbf{q}_o}$$

$$\mathbf{K}_o = 0.4258 \times 10^7 \times \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & -7.5 \\ 0 & -7.5 & 658.23 \end{bmatrix} \left(N,\ m,\ rad\right)$$

N-DEGREES-OF-FREEDOM-SYSTEMS

Damped forced vibrations

$$\mathbf{M}\ddot{\mathbf{q}}(t)+\mathbf{C}\dot{\mathbf{q}}(t)+\mathbf{K}\mathbf{q}(t)=\mathbf{f}(t)$$

Undamped free vibrations

$$\boxed{\begin{array}{l}\mathbf{M}\,\ddot{\mathbf{q}}(t)+\mathbf{K}\,\mathbf{q}(t)=0 \\ \mathbf{q}(0)=\mathbf{q}_o\ ;\ \dot{\mathbf{q}}(0)=\dot{\mathbf{q}}_o\end{array}} \qquad \begin{array}{r}(1a)\\(1b)\end{array}$$

Let us search the solution of Eq. (1) in the class of the functions:

$$\mathbf{q}(t)=\boldsymbol{\psi}f(t) \qquad (2)$$

where $\boldsymbol{\psi}$ is a vector of n constant components and f is a function of time which is common, unless a factor, to all the degrees of freedom. Substituting Eq. (2) into Eq. (1a) it results:

$$\mathbf{M}\boldsymbol{\psi}\ddot{f}(t)+\mathbf{K}\boldsymbol{\psi}f(t)=\mathbf{0} \qquad (3)$$

This expression can be rewritten as:

$$\sum_1^n{}_j m_{ij}\,\psi_j\,\ddot{f}(t)+\sum_1^n{}_j k_{ij}\,\psi_j\,f(t)=0 \qquad (i = 1, 2, \dots n) \qquad (4)$$

Thus:

$$-\frac{\ddot{f}(t)}{f(t)}=\frac{\sum_1^n{}_j\,k_{ij}\psi_j}{\sum_1^n{}_j\,m_{ij}\psi_j} \qquad (i = 1, 2, \dots n) \qquad (5)$$

Since the first member of Eq. (5) is a scalar quantity independent of index i, and the second member is independent of time t, then both members are eqaul to a constant λ. Therefore, Eq. (5) reduces to a couple of equations:

$$\boxed{\ddot{f}(t)+\lambda f(t)=0} \qquad (6)$$

$$\sum_1^n{}_j\,(k_{ij}-\lambda m_{ij})\,\psi_j=0 \qquad (i = 1, 2, \dots n) \qquad (7)$$

From Eq. (7) it results:

$$\boxed{(\mathbf{K}-\lambda\mathbf{M})\boldsymbol{\psi}=\mathbf{0}} \qquad (8)$$

which represents a system of n linear homogeneous equations in the n unknowns ψ_i (i = 1, 2, … n). Obviously, it involves the trivial solution $\boldsymbol{\psi} = \mathbf{0}$.
In order to obtain non-trivial solutions, $\boldsymbol{\psi} \neq \mathbf{0}$, it is necessary that the determinant of the matrix of the coefficients is null:

$$D = \det\left(\mathbf{K} - \lambda\mathbf{M}\right) = 0 \tag{9}$$

This leads to an algebraic equation of order n in λ, called characteristic equation, from which n roots may be obtained, called characteristic values or eigenvalues $\left(\lambda = \lambda_1, \lambda_2, \ldots \lambda_n\right)$.

Since **K** and **M** are real and symmetric matrices, then the eigenvalues $\lambda_1, \lambda_2, \ldots \lambda_n$ are real. Let us assume $\lambda_1 < \lambda_2 < \ldots < \lambda_n$.
M is also positive definite. If also **K** is positive definite, then the eigenvalues are not only real but also positive: $0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n$.
If **K** is semi-positive definite, then the eigenvalues are not negative (thus null eigenvalues may exist).

For each eigenvalue, the system (8) involves a non-trivial solution called characteristic vector or eigenvector $\left(\boldsymbol{\psi} = \boldsymbol{\psi}_1, \boldsymbol{\psi}_2, \ldots \boldsymbol{\psi}_n\right)$. Each eigenvector is defined unless an arbitrary factor.
The eigenvectors are linearly independent (for any **K**, **M** matrices if the eigenvalues are distinct; can be transformed as such when eigenvalues are multiple and **K**, **M** are real, symmetric matrices). Thus, they constitute a basis in the space of the Lagrangian coordinates. The eigenvectors of real and symmetric matrices are real.

Let us consider the k-th eigenvalue λ_k and the corresponding eigenvector ψ_k. From Eq. (8):

$$\mathbf{K}\boldsymbol{\psi}_k = \lambda_k \mathbf{M}\boldsymbol{\psi}_k \Rightarrow \boldsymbol{\psi}_k^T \mathbf{K}\boldsymbol{\psi}_k = \lambda_k \boldsymbol{\psi}_k^T \mathbf{M}\boldsymbol{\psi}_k \tag{10}$$

Let us define as k-th modal mass and k-th modal stiffness the quantities:

$$\boxed{m_k = \boldsymbol{\psi}_k^T \mathbf{M}\boldsymbol{\psi}_k \;;\; k_k = \boldsymbol{\psi}_k^T \mathbf{K}\boldsymbol{\psi}_k} \tag{11}$$

From Eq. (10) it results:

$$\boxed{\lambda_k = \frac{k_k}{m_k}} \tag{12}$$

Since **M** is positive definite, then $m_k > 0$. If **K** is positive definite, then $k_k > 0$.

Distinct eigenvalues
In structural engineering, usually, all eigenvalues are distinct, i.e. $\lambda_i \neq \lambda_j$ for $\forall i \neq j$. From Eq. (8):

$$\mathbf{K}\psi_i = \lambda_i \mathbf{M}\psi_i \ ; \ \mathbf{K}\psi_j = \lambda_j \mathbf{M}\psi_j \tag{13}$$

Pre-multiplying Eq. (13a) by ψ_j^T and Eq. (13b) by ψ_i^T:

$$\psi_j^T\mathbf{K}\psi_i = \lambda_i \psi_j^T \mathbf{M}\psi_i \ ; \ \psi_j^T\mathbf{K}\psi_j = \lambda_j \psi_j^T \mathbf{M}\psi_j \Rightarrow$$

$$\left(\psi_j^T\mathbf{K}\psi_j\right)^T = \lambda_j \left(\psi_j^T\mathbf{M}\psi_j\right)^T \Rightarrow$$

$$\psi_j^T\mathbf{K}\psi_i = \lambda_j \psi_j^T \mathbf{M}\psi_i$$

Subtracting member to member:

$$\left(\lambda_i - \lambda_j\right)\psi_j^T \mathbf{M}\psi_i = 0$$

from which, since $\lambda_i \neq \lambda_j$:

$$\boxed{\psi_j^T\mathbf{M}\psi_i = 0} \ ; \ \boxed{\psi_j^T\mathbf{K}\psi_i = 0} \quad (i \neq j) \tag{14}$$

Thus, the eigenvectors related to distinct eigenvalues are orthogonal with respect to matrices **M** and **K**. Furthermore, using Eq. (11):

$$\boxed{\psi_j^T\mathbf{M}\psi_i = m_i\,\delta_{ij}} \ ; \ \boxed{\psi_j^T\mathbf{K}\psi_i = k_i\,\delta_{ij}} \tag{15}$$

Let us define as modal matrix:

$$\boldsymbol{\Psi} = \left[\psi_1\ \psi_2 \dots \psi_n\right] \tag{16}$$

From Eq. (15) it results:

$$\mathbf{L} = \boldsymbol{\Psi}^T\mathbf{M}\boldsymbol{\Psi} = \mathrm{diag}\left[m_k\right] = \begin{bmatrix} m_1 & 0 & \cdots & 0 \\ 0 & m_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & m_n \end{bmatrix} \tag{17}$$

$$\mathbf{N} = \boldsymbol{\Psi}^T\mathbf{K}\,\boldsymbol{\Psi} = \mathrm{diag}\ \left[k_k\right] = \begin{bmatrix} k_1 & 0 & \cdots & 0 \\ 0 & k_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & k_n \end{bmatrix} \tag{18}$$

where **L** is the matrix of the modal masses and **N** is the matrix of the modal stiffnesses. Moreover, from Eq. (12):

$$\mathbf{\Lambda} = \mathbf{L}^{-1}\mathbf{N} = \mathrm{diag}\ \left[k_k / m_k \right] = \mathrm{diag}\ \left[\lambda_k \right] = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \tag{19}$$

is the matrix of the eigenvalues. **L**, **N**, **Λ** are diagonal matrices.

Since the eigenvectors are defined unless an arbitrary factor, it is possible to assume $m_k = 1$; thus, due to Eq. (12), $k_k = \lambda_k$. In this case, Eq. (15) becomes:

$$\boldsymbol{\psi}_j^T \mathbf{M} \boldsymbol{\psi}_i = \delta_{ij} \ ; \ \boldsymbol{\psi}_j^T \mathbf{K} \boldsymbol{\psi}_i = \lambda_i \, \delta_{ij} \tag{20}$$

and the eigenvectors are said to be orthonormal with respect to **M** and **K**. Moreover, due to Eqs. (17), (18), (19):

$$\mathbf{L} = \mathbf{I} \quad ; \quad \mathbf{N} = \mathbf{\Lambda} \tag{21}$$

where **I** is the identity matrix.

K positive definite

In structural engineering, usually, all the eigenvalues are positive. Thus, let us assume $\lambda_k = \omega_k^2$ and let us examine Eq. (6). It becomes:

$$\ddot{f}(t) + \omega_k^2 f(t) = 0 \tag{22}$$

from which:

$$f(t) = f_k(t) = A_k \cos\omega_k t + B_k \sin\omega_k t \tag{23}$$

where A_k, B_k are real constants. Thus, the eigenvalue λ_k has a fundamental mechanical meaning: it is the square of the circular frequency ω_k.

Thus, Eq. (1) involves n linearly independent solutions analogous to Eq. (2):

$$\mathbf{q}(t) = \mathbf{q}_k(t) = \boldsymbol{\psi}_k f_k(t) = \boldsymbol{\psi}_k (A_k \cos\omega_k t + B_k \sin\omega_k t) \tag{24}$$

The general solution of Eq. (1) is then a linear combination of the n solutions provided by Eq. (24):

$$\boxed{\mathbf{q}(t) = \sum_{1}^{n}{}_k \boldsymbol{\psi}_k (A_k \cos\omega_k t + B_k \sin\omega_k t)} \tag{25}$$

where the 2n arbitrary constants A_k, B_k $(k = 1, 2, ..n)$ shall be set based on the initial conditions:

$$\dot{\mathbf{q}}(t) = \sum_{1}^{n}{}_k \boldsymbol{\psi}_k \omega_k (-A_k \sin\omega_k t + B_k \cos\omega_k t)$$

$$\mathbf{q}(0) = \sum_{1}^{n}{}_k \boldsymbol{\psi}_k A_k = \mathbf{q}_0 \quad ; \quad \dot{\mathbf{q}}(0) = \sum_{1}^{n}{}_k \boldsymbol{\psi}_k B_k = \dot{\mathbf{q}}_0 \Rightarrow$$

$$\sum_{1}^{n}{}_k A_k \boldsymbol{\psi}_k^T = \mathbf{q}_0^T \quad ; \quad \sum_{1}^{n}{}_k \omega_k B_k \boldsymbol{\psi}_k^T = \dot{\mathbf{q}}_0^T \Rightarrow$$

$$\sum_{1}^{n}{}_k A_k \boldsymbol{\psi}_k^T \mathbf{M} \boldsymbol{\psi}_i = \mathbf{q}_0^T \mathbf{M} \boldsymbol{\psi}_i \quad ; \quad \sum_{1}^{n}{}_k \omega_k B_k \boldsymbol{\psi}_k^T \mathbf{M} \boldsymbol{\psi}_i = \dot{\mathbf{q}}_0^T \mathbf{M} \boldsymbol{\psi}_i \Rightarrow$$

$$A_i \underbrace{\boldsymbol{\psi}_i^T \mathbf{M} \boldsymbol{\psi}_i}_{m_i} = \mathbf{q}_0^T \mathbf{M} \boldsymbol{\psi}_i \quad ; \quad \omega_k B_k \underbrace{\boldsymbol{\psi}_i^T \mathbf{M} \boldsymbol{\psi}_i}_{m_i} = \dot{\mathbf{q}}_0^T \mathbf{M} \boldsymbol{\psi}_i \Rightarrow$$

$$\boxed{A_i = \frac{1}{m_i} \mathbf{q}_0^T \mathbf{M} \boldsymbol{\psi}_i} \; ; \; \boxed{B_i = \frac{1}{m_i \omega_i} \dot{\mathbf{q}}_0^T \mathbf{M} \boldsymbol{\psi}_i} \quad (i = 1,2 .. n) \tag{26}$$

Finally, if the eigenvectors are orthonormal, i.e. $m_i = 1$, if:

$$\boxed{A_i = \mathbf{q}_0^T \mathbf{M} \boldsymbol{\psi}_i} \; ; \; \boxed{B_i = \frac{1}{\omega_i} \dot{\mathbf{q}}_0^T \mathbf{M} \boldsymbol{\psi}_i} \quad (i = 1,2 .. n) \tag{27}$$

The physical meaning of the eigenvector can be explained assuming:

$$\mathbf{q}_0 = \boldsymbol{\psi}_j \quad ; \quad \dot{\mathbf{q}}_0 = \mathbf{0} \tag{28}$$

i.e. deformating the structure in accordance with its j-th eigenvector and leaving its free to oscillate. Substituting Eq. (28) into Eq. (26) it results:

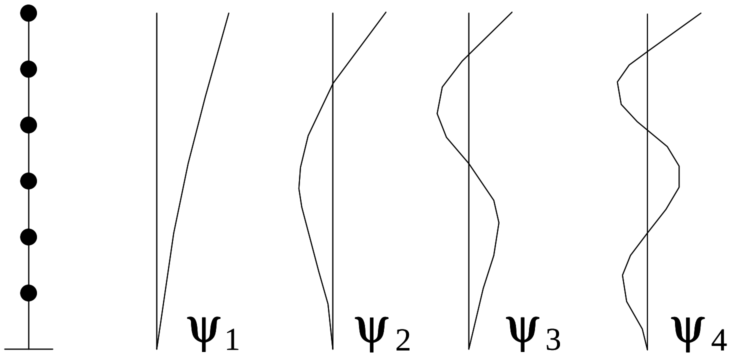
$$A_i = \boldsymbol{\psi}_J^T \mathbf{M} \boldsymbol{\psi}_i = \delta_{ij} \quad ; \quad B_i = 0 \tag{29}$$

Thus, substituting Eq. (29) into Eq. (25):

$$\mathbf{q}(t) = \sum_1^n {}_k \, \boldsymbol{\psi}_k \delta_{kj} \cos \omega_k t \Rightarrow$$

$$\mathbf{q}(t) = \boldsymbol{\psi}_j \cos \omega_j t \tag{30}$$

It follows that the k-th eigenvector is that special pattern of the initial displacement which causes the oscillation of all the DOFs of the structure with the same circular frequency ω_k. For this reason the eigenvectors represent proper/natural/elementary modes/shapes of vibration. Each eigenvalue is the square of a proper/natural/elementary circular frequency of vibration.



	$\boldsymbol{\psi}_1$	$\boldsymbol{\psi}_2$	$\boldsymbol{\psi}_3$	$\boldsymbol{\psi}_4$
eigenvalues	λ_1	λ_2	λ_3	λ_4
circular frequencies	ω_1	ω_2	ω_3	ω_4
frequencies	n_1	n_2	n_3	n_4
periods	T_1	T_2	T_3	T_4

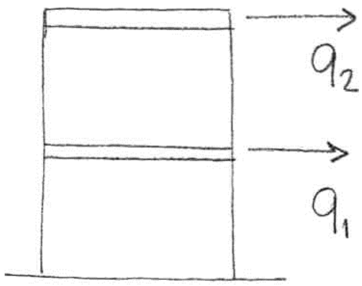
$$\omega_k^2 = \lambda_k \; ; \; n_k = \omega_k / 2\pi \; ; \; T_k = 1 / n_k \; (k = 1, 2, .. n)$$

Based upon Eq. (25) any free vibration may be regarded as a linear combination of proper/natural oscillations. More generally, since the set of the eigenvectors represents a basis in the space of the Lagrangian coordinates, **q**(t) may be expressed as a linear combinations of the modes $\boldsymbol{\psi}_k$:

$$\boxed{\mathbf{q}(t) = \sum_1^n {}_k \, \boldsymbol{\psi}_k \, p_k(t)} \tag{31}$$

This expression is called principal transformation rule.

Example: 2 D.O.F. shear-type building



$$\mathbf{M} = \begin{bmatrix} 271200 & 0 \\ 0 & 146325 \end{bmatrix} \quad (\mathrm{kg})$$

$$\mathbf{K} = \begin{bmatrix} 1.6941 & -0.7585 \\ -0.7585 & 0.7585 \end{bmatrix} \times 10^{8} \quad (\mathrm{N/m})$$

$$\mathrm{D} = \det\left(\mathbf{K} - \omega^2 \mathbf{M}\right) = 0 \Rightarrow$$

$$\mathrm{D} = \det\left(\begin{bmatrix} 1.6941 & -0.7585 \\ -0.7585 & 0.7585 \end{bmatrix} \times 10^{8} - \omega^2 \begin{bmatrix} 271200 & 0 \\ 0 & 146325 \end{bmatrix} \right) =$$

$$= \det \begin{bmatrix} 1.6941\times10^{8} - \omega^2 \cdot 271200 & -0.7585\times10^{8} \\ -0.7585\times10^{8} & 0.7585\times10^{8} - \omega^2 \cdot 271200 \end{bmatrix} =$$

$$= \left(1.6941\times10^{8} - 271200\,\omega^2\right)\left(0.7585\times10^{8} - 146325\,\omega^2\right) - \overline{0.7585\times10^{8}}^{\,2} =$$

$$= 3.968\times10^{10}\,\omega^4 - 4.5359\times10^{13}\,\omega^2 + 7.0965\times10^{15} = 0 \Rightarrow$$

$$\omega^2 = \frac{4.5359\times10^{13} \mp \sqrt{\left(4.5359\times10^{13}\right)^2 - 4\times3.968\cdot10^{10}\times7.0965\cdot10^{15}}}{2\times3.968\cdot10^{10}} \Rightarrow$$

$\omega_1^2 = 187.063\,;\ \omega_2^2 = 956.056 \Rightarrow$

$\omega_1 = 13.677\ \mathrm{rad/s}\,;\ \omega_2 = 30.920\ \mathrm{rad/s}$

$n_1 = 2.177\ \mathrm{Hz}\,;\ n_2 = 4.921\ \mathrm{Hz}$

$T_1 = 0.459\,\mathrm{s}\,;\ T_2 = 0.203\,\mathrm{s}$

$$\left(\mathbf{K} - \omega_k^2 \mathbf{M}\right)\psi_k = \mathbf{0}$$

k = 1

$$\begin{bmatrix} 1.6941\times10^{8} - \omega_1^2 \times 271200 & -0.7585\times10^{8} \\ -0.7585\times10^{8} & 0.7585\times10^{8} - \omega_1^2 \times 146325 \end{bmatrix} \begin{Bmatrix} \psi_{11} \\ \psi_{12} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

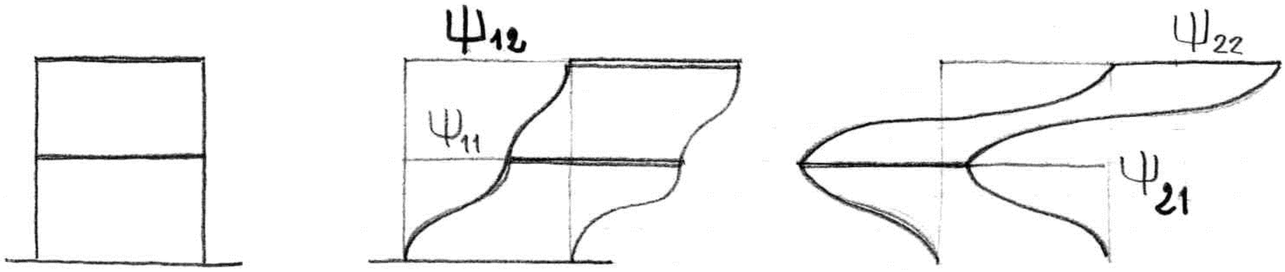
$\psi_{12} = 1 \Rightarrow -0.7585\times10^{8}\ \psi_{11} + 0.4848\times10^{8} = 0 \Rightarrow \psi_{11} = 0.639$

k = 2

$$\begin{bmatrix} 1.6941\times10^8-\omega_2^2\times271200 & -0.7585\times10^8 \\ -0.7585\times10^8 & 0.7585\times10^8-\omega_2^2\times146325 \end{bmatrix}\begin{Bmatrix}\psi_{21}\\ \psi_{22}\end{Bmatrix}=\begin{Bmatrix}0\\0\end{Bmatrix}$$

$$\psi_{22}=1\Rightarrow -0.7585\times10^8\ \psi_{21}+0.6404\times10^8=0\Rightarrow \psi_{21}=0.844$$

$$\mathbf{\Psi}=\left[\boldsymbol{\psi}_1\ \boldsymbol{\psi}_2\right]=\begin{bmatrix}\psi_{11} & \psi_{21}\\ \psi_{12} & \psi_{22}\end{bmatrix}=\begin{bmatrix}0.639 & -0.844\\ 1. & 1.\end{bmatrix}$$



$$\mathbf{\Psi}^T\mathbf{M}\mathbf{\Psi}=\begin{bmatrix}0.639 & 1.\\ -0.844 & 1.\end{bmatrix}\begin{bmatrix}271200 & 0\\ 0 & 146325\end{bmatrix}\begin{bmatrix}0.639 & -0.844\\ 1. & 1.\end{bmatrix}=$$

$$=\begin{bmatrix}257062 & 0\\ 0 & 339511\end{bmatrix}\Rightarrow$$

$$m_1=\mathbf{\Psi}_1^T\mathbf{M}\mathbf{\Psi}_1=257062$$

$$m_2=\mathbf{\Psi}_2^T\mathbf{M}\mathbf{\Psi}_2=339511$$

In order to make the eigenvectors orthonormal:

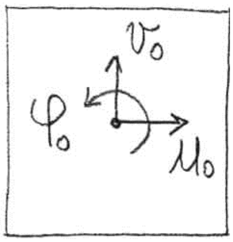
$$\mathbf{\Psi}_1=\frac{1}{\sqrt{m_1}}\begin{Bmatrix}0.639\\ 1.\end{Bmatrix}=\begin{Bmatrix}1.260\times10^{-3}\\ 1.972\times10^{-3}\end{Bmatrix}$$

$$\mathbf{\Psi}_2=\frac{1}{\sqrt{m_2}}\begin{Bmatrix}-0.844\\ 1.\end{Bmatrix}=\begin{Bmatrix}-1.448\times10^{-3}\\ 1.716\times10^{-3}\end{Bmatrix}$$

$$\mathbf{\Psi}=\left[\boldsymbol{\psi}_1\ \boldsymbol{\psi}_2\right]=\begin{bmatrix}1.260 & -1.448\\ 1.972 & 1.716\end{bmatrix}\times10^{-3}\Rightarrow \mathbf{\Psi}^T\mathbf{M}\mathbf{\Psi}=\mathbf{I}$$

$$\mathbf{\Psi}^T\mathbf{M}\mathbf{\Psi}=\mathbf{\Lambda}=\begin{bmatrix}187.063 & 0\\ 0 & 956.056\end{bmatrix}$$

Example: 3 D.O.F. shear type building



$$\mathbf{M} = \begin{bmatrix} 140775 & 0 & 0 \\ 0 & 140775 & 0 \\ 0 & 0 & 5279062 \end{bmatrix} \; ; \quad \mathbf{K} = \begin{bmatrix} 0.3832\times10^{8} & 0 & 0 \\ 0 & 0.3832\times10^{8} & -0.3193\times10^{8} \\ 0 & -0.3193\times10^{8} & 0.2802\times10^{10} \end{bmatrix}$$

$$D = \det\begin{bmatrix} 0.3832\times10^{8} - \omega^2 \cdot 140775 & 0 & 0 \\ 0 & 0.3832\times10^{8} - \omega^2 \cdot 140775 & -0.3193\times10^{8} \\ 0 & -0.3193\times10^{8} & 0.2802\times10^{10} - \omega^2 \cdot 5279062 \end{bmatrix}$$

$$= \left(0.3832\times10^{8} - 140775\,\omega^2\right)\left[\left(0.3832\times10^{8} - 140775\,\omega^2\right)\cdot\right.$$
$$\left.\left(0.2802\times10^{10} - 5279062\,\omega^2\right) - 0.3193\cdot10^{8}\times0.3193\cdot10^{8}\right] = 0 \Rightarrow$$

$\omega^2 = 272.207$

$7.4316\times10^{11}\,\omega^4 - 5.9675\times10^{14}\,\omega^2 + 1.0635\times10^{17} = 0 \Rightarrow$

$$\omega^2 = \frac{5.9675\times10^{14} \pm \sqrt{\left(5.9675\times10^{14}\right)^2 - 4\times7.4316\cdot10^{11}\times1.0635\cdot10^{17}}}{2\times7.4316\cdot10^{11}} \Rightarrow$$

$\omega^2 = \begin{cases} 266.984 \\ 536.005 \end{cases}$

$\omega_1^2 = 266.984$; $\omega_2^2 = 272.207$; $\omega_3^2 = 536.005$

$\omega_1 = 16.340$ rad / s ; $\omega_2 = 16.499$ rad / s ; $\omega_3 = 23.152$ rad / s

$n_1 = 2.601$ Hz ; $n_2 = 2.626$ Hz ; $n_3 = 3.685$ Hz

$T_1 = 0.385$ s ; $T_2 = 0.381$ s ; $T_3 = 0.271$ s

$$\left(\mathbf{K} - \omega_k^2\mathbf{M}\right)\psi_k = \mathbf{0}$$

k = 1

$$\begin{bmatrix} 0.3832\times10^{8} - \omega_1^2 \cdot 140775 & 0 & 0 \\ 0 & 0.3832\times10^{8} - \omega_1^2 \cdot 140775 & -0.3193\times10^{8} \\ 0 & -0.3193\times10^{8} & 0.2802\times10^{10} - \omega_1^2 \cdot 5279062 \end{bmatrix} \begin{Bmatrix} \psi_{11} \\ \psi_{12} \\ \psi_{13} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$\psi_{11} = 0$; $\psi_{12} = 1 \Rightarrow -0.3193\times10^{8}\,\psi_{11} + 1.3926\times10^{9}\,\psi_{13} = 0 \Rightarrow \psi_{13} = 0.0229$

k = 2

$$\begin{bmatrix} 0.3832\times10^{8}-\omega_2^2\cdot140775 & 0 & 0 \\ 0 & 0.3832\times10^{8}-\omega_1^2\cdot140775 & -0.3193\times10^{8} \\ 0 & -0.3193\times10^{8} & 0.2802\times10^{10}-\omega_2^2\cdot5279062 \end{bmatrix}\begin{Bmatrix}\psi_{21}\\ \psi_{22}\\ \psi_{23}\end{Bmatrix}=\begin{Bmatrix}0\\0\\0\end{Bmatrix}$$

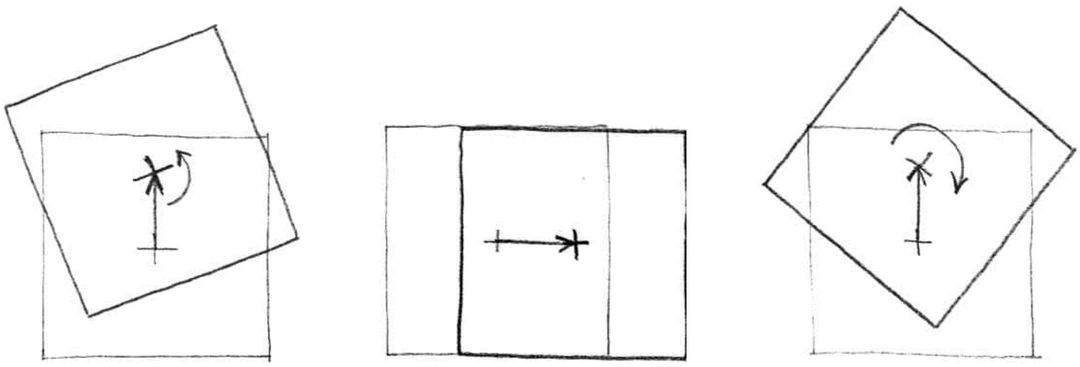
$\psi_{21}=1$; $\psi_{22}=0$; $\psi_{23}=0$

k = 3

$$\begin{bmatrix} 0.3832\times10^{8}-\omega_3^2\cdot140775 & 0 & 0 \\ 0 & 0.3832\times10^{8}-\omega_3^2\cdot140775 & -0.3193\times10^{8} \\ 0 & -0.3193\times10^{8} & 0.2802\times10^{10}-\omega_3^2\cdot5279062 \end{bmatrix}\begin{Bmatrix}\psi_{31}\\ \psi_{32}\\ \psi_{33}\end{Bmatrix}=\begin{Bmatrix}0\\0\\0\end{Bmatrix}$$

$\psi_{31}=0$; $\psi_{32}=1\Rightarrow -0.3193\times10^{8}-0.276\times10^{8}\,\psi_{33}=0\Rightarrow \psi_{33}=-1.1567$

$$\boldsymbol{\Psi}=\left[\boldsymbol{\psi}_1\ \boldsymbol{\psi}_2\ \boldsymbol{\psi}_3\right]=\begin{bmatrix}\psi_{11} & \psi_{12} & \psi_{13}\\ \psi_{21} & \psi_{22} & \psi_{23}\\ \psi_{31} & \psi_{32} & \psi_{33}\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0.0229 & 0 & -1.1567\end{bmatrix}$$

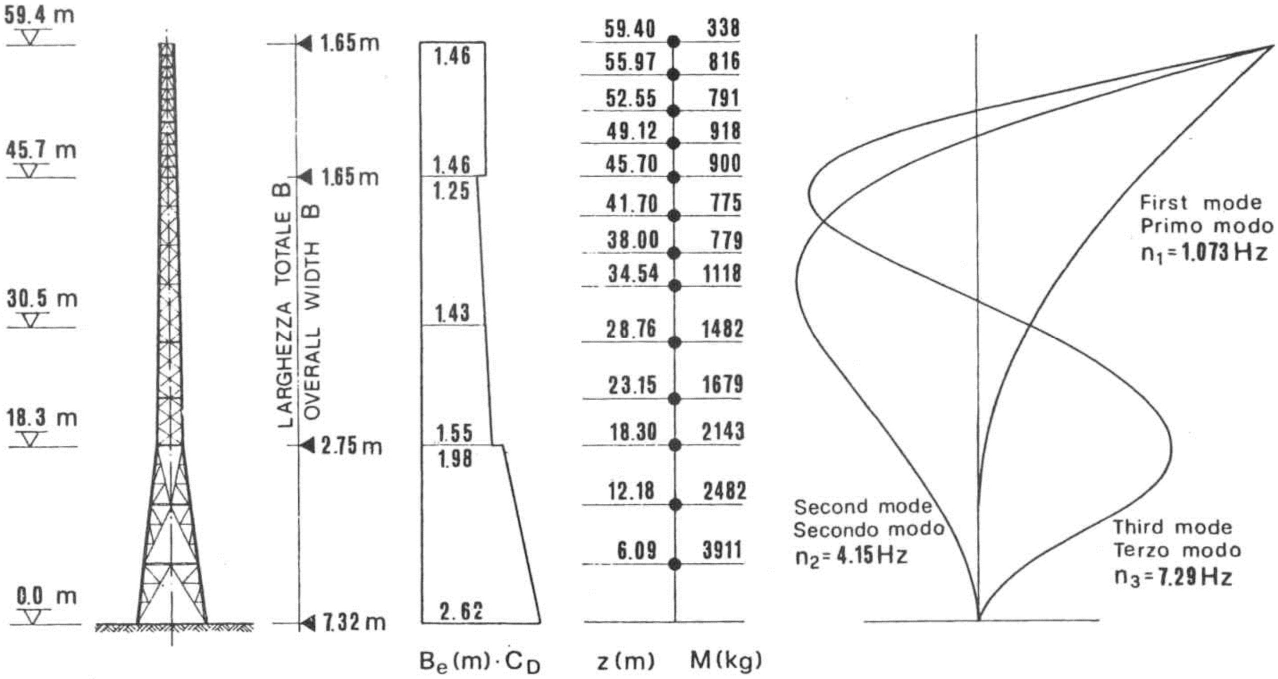


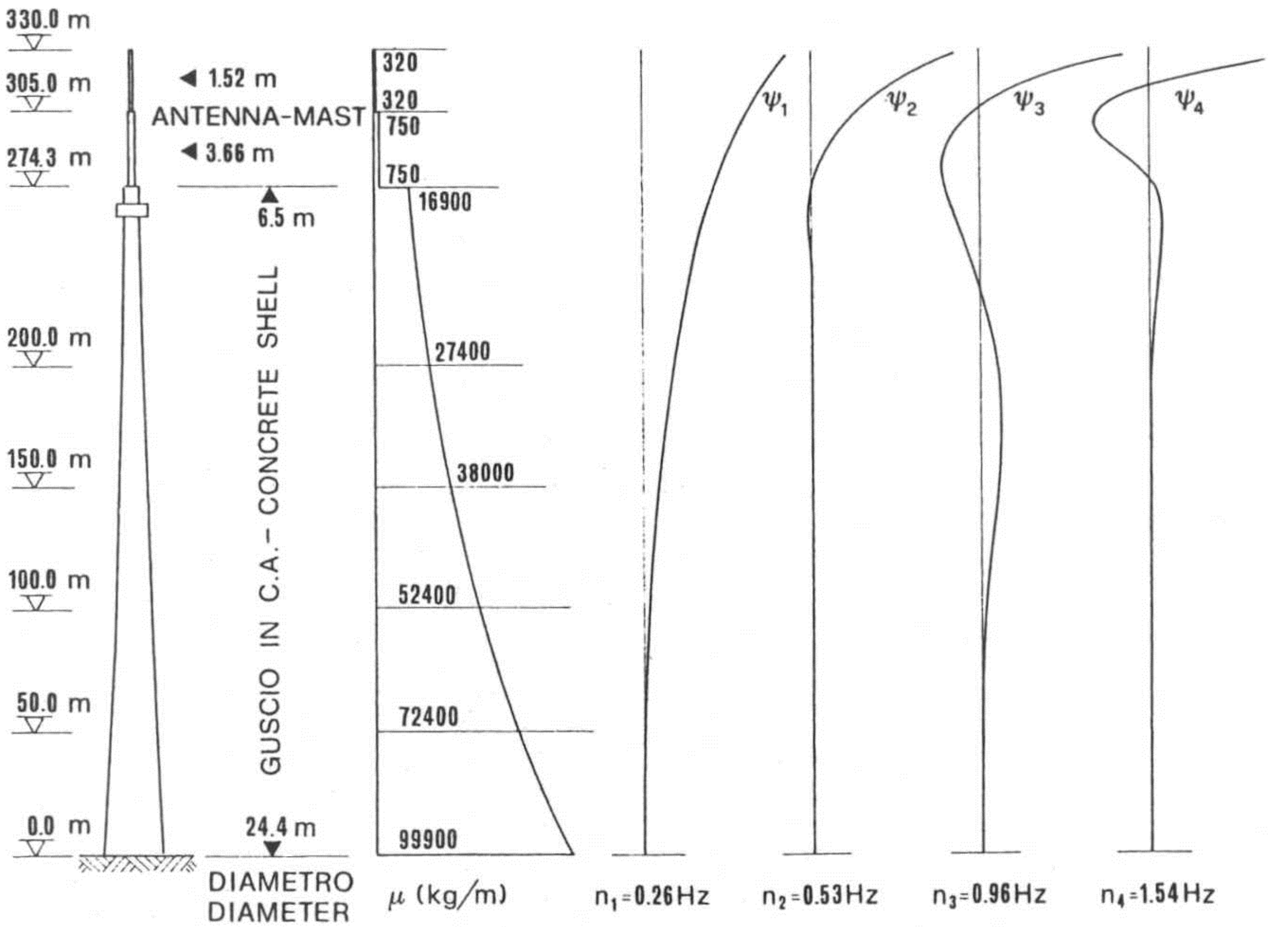
1st mode 2nd mode 3rd mode

$$\boldsymbol{\Psi}^{T}\mathbf{M}\boldsymbol{\Psi}=\begin{bmatrix}m_1 & 0 & 0\\ 0 & m_2 & 0\\ 0 & 0 & m_3\end{bmatrix}=\begin{bmatrix}143543 & 0 & 0\\ 0 & 140775 & 0\\ 0 & 0 & 7203922\end{bmatrix}$$

In order to make the eigenvectors orthonormal $\boldsymbol{\psi}_k \rightarrow \boldsymbol{\psi}_k/\sqrt{m_k}$:

$$\boldsymbol{\Psi}=\begin{bmatrix}0 & 2.665\times10^{-3} & 0\\ 2.639\times10^{-3} & 0 & 3.726\times10^{-4}\\ 6.044\times10^{-5} & 0 & -4.309\times10^{-4}\end{bmatrix}$$


59.4 m
45.7 m
30.5 m
18.3 m
0.0 m
1.65m
1.65m
2.75m
7.32m
LARGHEZZA TOTALE B
OVERALL WIDTH B
1.46
1.46
1.25
1.43
1.55
1.98
2.62
B_e(m)·C_D
59.40 338
55.97 816
52.55 791
49.12 918
45.70 900
41.70 775
38.00 779
34.54 1118
28.76 1482
23.15 1679
18.30 2143
12.18 2482
6.09 3911
z(m) M(kg)
First mode
Primo modo
n_1=1.073 Hz
Second mode
Secondo modo
n_2=4.15 Hz
Third mode
Terzo modo
n_3=7.29 Hz


330.0 m
305.0 m
274.3 m
200.0 m
150.0 m
100.0 m
50.0 m
0.0 m
1.52 m
ANTENNA-MAST
3.66 m
6.5 m
GUSCIO IN C.A. - CONCRETE SHELL
24.4 m
DIAMETRO
DIAMETER
320
320
750
750
16900
27400
38000
52400
72400
99900
μ (kg/m)
ψ_1
ψ_2
ψ_3
ψ_4
n_1=0.26Hz
n_2=0.53Hz
n_3=0.96Hz
n_4=1.54Hz

N-DEGREES-OF-FREEDOM-SYSTEMS

Undamped forced vibrations

Since the eigenvectors constitute a base in the space of Lagrangian coordinates, the displacement **q** at the time t can be expressed as a linear combination of the modes $\boldsymbol{\psi}_k$:

$$\boxed{\mathbf{q}(t) = \sum_{1}^{n} {}_k\boldsymbol{\psi}_k p_k(t)} \tag{1}$$

$$\boxed{\mathbf{q}(t) = \boldsymbol{\Psi}\mathbf{p}(t)} \tag{2}$$

where $\boldsymbol{\Psi} = \{\boldsymbol{\psi}_1\ \boldsymbol{\psi}_2\ \dots\ \boldsymbol{\psi}_n\}$ is the modal matrix and $\mathbf{p}(t) = \{p_1(t)\ p_2(t) .. p_n(t)\}^T$ is the vector of the principal coordinates. Eqs. (1) and (2) are referred to as the principal transformation law. Based on Eqs. (1) and (2) the principal coordinates have the property that, when the system oscillates on the i-th principal mode, only the principal coordinate p_i varies with time, being null all the others.
Due to the property that the eigenvectors are linearly independent, $\boldsymbol{\Psi}$ is non singular and the principal transformation may be inverted. In particular, pre-multiplying Eq. (2) by $\boldsymbol{\Psi}^T\mathbf{M}$ (and assuming $m_k = 1$):

$$\mathbf{p}(t) = \boldsymbol{\Psi}^T\mathbf{M}\,\mathbf{q}(t) \tag{3}$$

Let us consider now the equation of motion:

$$\boxed{\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{f}(t)} \tag{4}$$

and let us apply the principal transformation law:

$$\mathbf{M}\boldsymbol{\Psi}\ddot{\mathbf{p}}(t) + \mathbf{K}\boldsymbol{\Psi}\,\mathbf{p}(t) = \mathbf{f}(t) \tag{5}$$

Then, let us pre-multiply by $\boldsymbol{\Psi}^T$:

$$\boldsymbol{\Psi}^T\mathbf{M}\boldsymbol{\Psi}\,\ddot{\mathbf{p}}(t) + \boldsymbol{\Psi}\mathbf{K}\boldsymbol{\Psi}^T\mathbf{p}(t) = \boldsymbol{\Psi}^T\mathbf{f}(t) \Rightarrow$$
$$\mathbf{L}\,\ddot{\mathbf{p}}(t) + \mathbf{N}\,\mathbf{p}(t) = \boldsymbol{\Psi}^T\mathbf{f}(t)$$

Finally, let us pre-multiply by $\mathbf{L}^{-1}$:

$$\ddot{\mathbf{p}}(t) + \mathbf{L}^{-1}\mathbf{N}\mathbf{p}(t) = \mathbf{L}^{-1}\boldsymbol{\Psi}^T\mathbf{f}(t) \Rightarrow$$
$$\ddot{\mathbf{p}}(t) + \boldsymbol{\Lambda}\,\mathbf{p}(t) = \mathbf{L}^{-1}\boldsymbol{\Psi}^T\mathbf{f}(t) \tag{6}$$

Eq. (6) represents a set un uncoupled equations:

$$\boxed{\ddot{\mathbf{p}}_k(t) + \omega_k^2 p_k(t) = \frac{1}{m_k}\boldsymbol{\Psi}_k^T\mathbf{f}(t) = \frac{1}{m_k}\sum_{1}^{n} {}_i\psi_{ki} f_i(t) \quad (k = 1, 2, \dots n)} \tag{7}$$

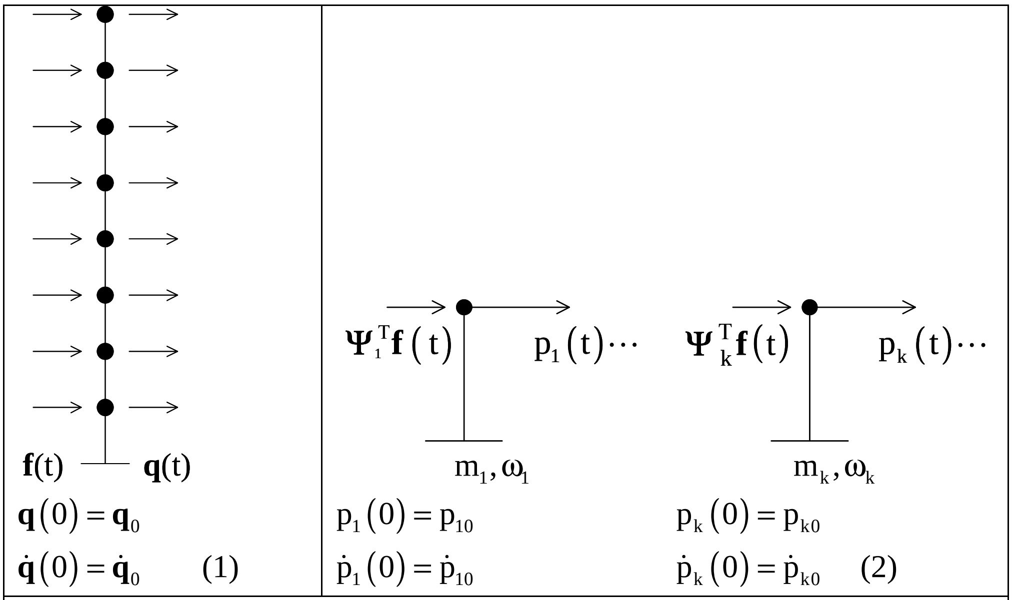
where $\boldsymbol{\Psi}_k^T \mathbf{f}(t)$ is the k-th modal force. It is the k-th component of the generalised forces in the principal system.
Thus, the undamped forced vibrations of a n-D.O.F. may be studied as the undamped forced vibrations of n S.D.O.F.s. The D.O.F. of the k-th oscillator is the k-th principal coordinate. The fundamental circular frequency of the k-th oscillator is the k-th principal circular frequency. The mass is the k-th modal mass. The external force is the k-th modal force.

As far as concern the initial conditions:

$$\mathbf{q}(0) = \mathbf{q}_o \; ; \; \dot{\mathbf{q}}(0) = \dot{\mathbf{q}}_o$$

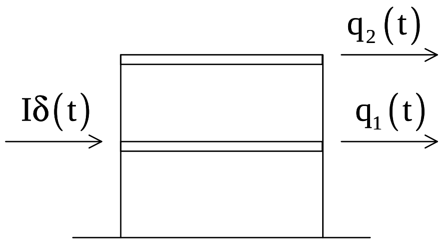
the initial conditions related to the principal coordinates can be found by applying the principal transformation law:

$$\boldsymbol{\Psi}\mathbf{p}(0) = \mathbf{q}_0 \; ; \qquad \boldsymbol{\Psi}\dot{\mathbf{p}}(0) = \dot{\mathbf{q}}_0 \Rightarrow$$
$$\boldsymbol{\Psi}^T\mathbf{M}\boldsymbol{\Psi}\,\dot{\mathbf{p}}(0) = \boldsymbol{\Psi}^T\mathbf{M}\mathbf{q}_0 \; ; \qquad \boldsymbol{\Psi}^T\mathbf{M}\boldsymbol{\Psi}\,\dot{\mathbf{p}}(0) = \boldsymbol{\Psi}^T\mathbf{M}\dot{\mathbf{q}}_0 \Rightarrow$$
$$\mathbf{p}(0) = \mathbf{L}^{-1}\boldsymbol{\Psi}^T\mathbf{M}\mathbf{q}_0 \; ; \quad \dot{\mathbf{p}}(0) = \mathbf{L}^{-1}\boldsymbol{\Psi}^T\mathbf{M}\dot{\mathbf{q}}_0 \Rightarrow$$
$$p_k(0) = p_{k0} \; ; \quad \dot{p}_k(0) = \dot{p}_{k0} \qquad (k = 1, \dots n)$$



$$q(t) = \sum_1^n {}_k\boldsymbol{\Psi}_k p_k(t) \qquad (3)$$

Example: 2-D.O.F. shear-type building



$$\mathbf{M}\,\ddot{\mathbf{q}}(t)+\mathbf{K}\,\mathbf{q}(t)=\mathbf{f}(t)$$

$$\mathbf{f}(t)=\begin{Bmatrix} f_1(t) \\ f_2(t) \end{Bmatrix}=\begin{Bmatrix} I\delta(t) \\ 0 \end{Bmatrix}$$

$$\mathbf{q}(t)=\mathbf{\Psi}\,\mathbf{p}(t)\Rightarrow$$

$$\begin{cases} \ddot{p}_1(t)+\omega_1^2\,p_1(t)=\dfrac{1}{m_1}\boldsymbol{\psi}_1^T\mathbf{f}(t)=\psi_{11}I\delta(t);\ p_{10}=\dot{p}_{10}=0 \\ p_2(t)+\omega_2^2\,p_2(t)=\dfrac{1}{m_2}\boldsymbol{\psi}_2^T\mathbf{f}(t)=\psi_{21}I\delta(t);\ p_{20}=\dot{p}_{20}=0 \end{cases}$$

$$\ddot{h}(t)+2\xi\omega_0\,\dot{h}(t)+\omega_0^2\,h(t)=\frac{1}{m}\delta(t);\ h(0)=\dot{h}(0)=0\Rightarrow$$

$$h(t)=e^{-\xi\omega_0 t}\,\frac{1}{m\,\omega_0\sqrt{1-\xi^2}}\sin\omega_0\sqrt{1-\xi^2}\ t$$

$$\begin{cases} p_1(t)=\dfrac{\psi_{11}I}{\omega_1}\sin\omega_1 t \\ p_2(t)=\dfrac{\psi_{21}I}{\omega_2}\sin\omega_2 t \end{cases}$$

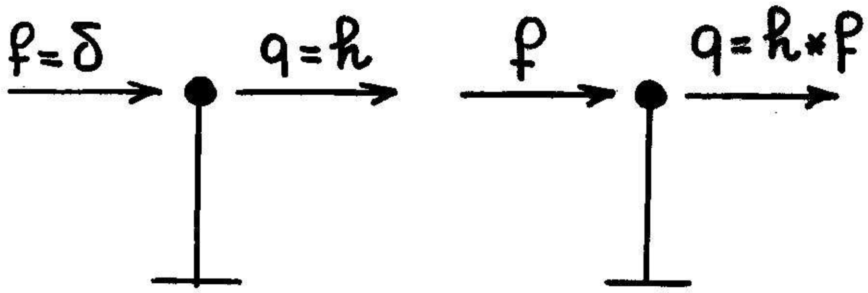
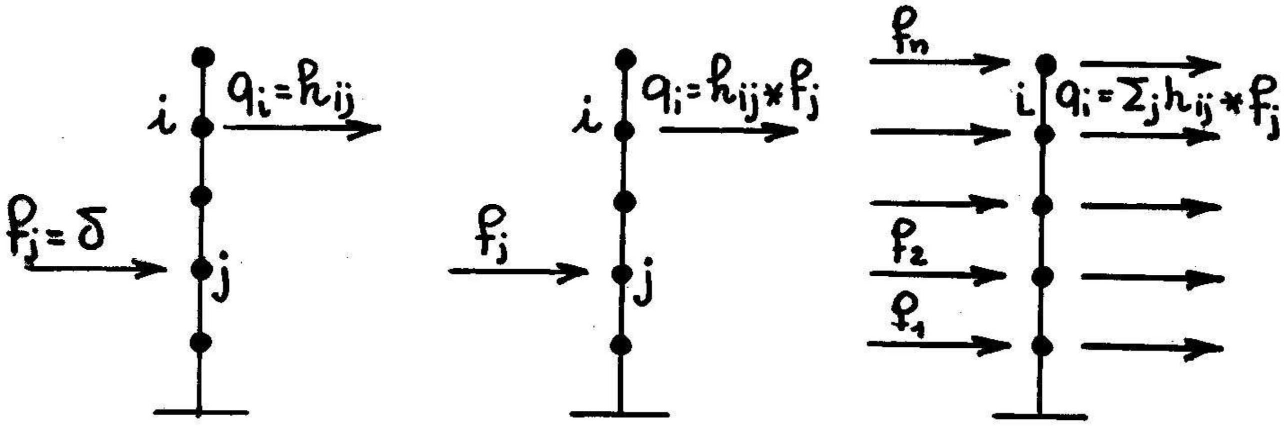
Using the principal transformation law:

$$\begin{Bmatrix} q_1(t) \\ q_2(t) \end{Bmatrix}=\begin{bmatrix} \psi_{11} & \psi_{21} \\ \psi_{12} & \psi_{22} \end{bmatrix}\begin{Bmatrix} p_1(t) \\ p_2(t) \end{Bmatrix}\Rightarrow$$

$$q_1(t)=I\left(\frac{\psi_{11}^2}{\omega_1}\sin\omega_1 t+\frac{\psi_{21}^2}{\omega_2}\sin\omega_2 t\right)$$

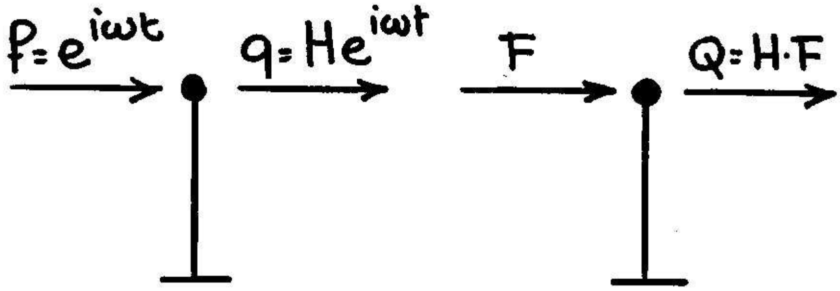
$$q_2(t)=I\left(\frac{\psi_{11}\,\psi_{12}}{\omega_1}\sin\omega_1 t+\frac{\psi_{21}\,\psi_{22}}{\omega_2}\sin\omega_2 t\right)$$

TIME-DOMAIN ANALYSIS

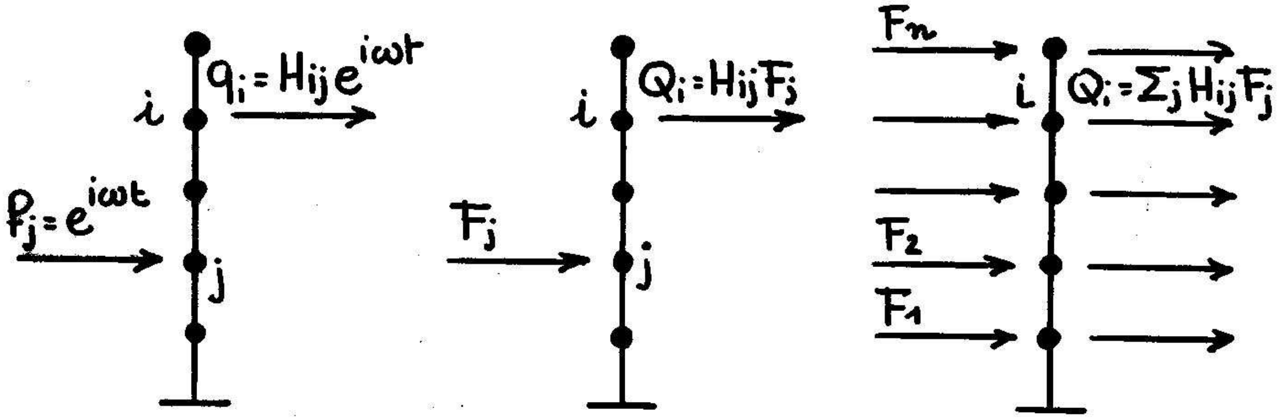
<table>
<tr><td colspan="2">S.D.O.F.

</td></tr>
<tr><td colspan="2">n-D.O.F.

</td></tr>
<tr><td>S.D.O.F.
$q(t)=h(t)*f(t)=\int_0^t h(t-\tau)f(\tau)d\tau$
h = impulse response function</td><td>n-D.O.F.
$\mathbf{q}(t)=\mathbf{h}(t)*\mathbf{f}(t)=\int_0^t \mathbf{h}(t-\tau)\mathbf{f}(\tau)d\tau$
h = impulse response matrix</td></tr>
</table>

FREQUENCY-DOMAIN ANALYSIS

S.D.O.F.



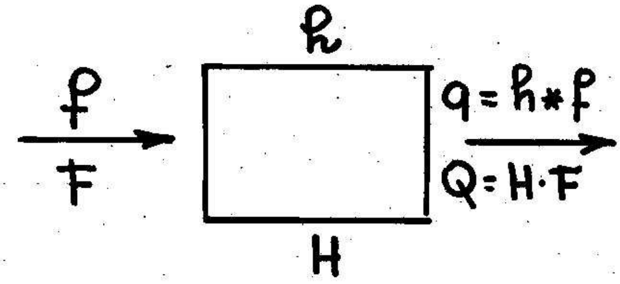
n-D.O.F.



S.D.O.F.

$Q(\omega) = H(\omega) \cdot F(\omega)$

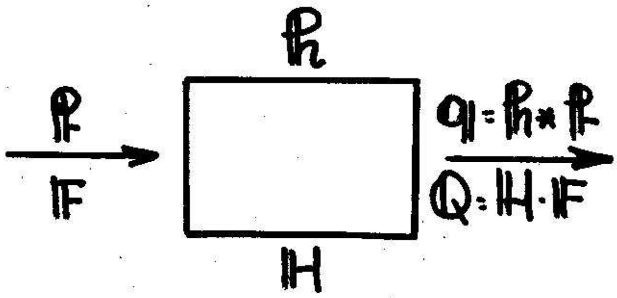
H = complex frequency response function



n-D.O.F.

$\mathbf{Q}(\omega) = \mathbf{H}(\omega) \cdot \mathbf{F}(\omega)$

H = complex frequency response matrix



Damped forced vibrations

Let us consider the equation of motion:

$$\boxed{\mathbf{M}\,\ddot{\mathbf{q}}(t)+\mathbf{C}\,\dot{\mathbf{q}}(t)+\mathbf{K}\,\mathbf{q}(t)=\mathbf{f}(t)} \tag{1}$$

and let us apply the principal transformation law:

$$\mathbf{q}(t)=\boldsymbol{\Psi}\mathbf{p}(t) \tag{2}$$

It follows:

$$\mathbf{M}\boldsymbol{\Psi}\,\ddot{\mathbf{p}}(t)+\mathbf{C}\boldsymbol{\Psi}\,\dot{\mathbf{p}}(t)+\mathbf{K}\boldsymbol{\Psi}\,\mathbf{p}(t)=\mathbf{f}(t)$$

Then, let us pre-multiply by $\boldsymbol{\Psi}^{T}$:

$$\boldsymbol{\Psi}^{T}\mathbf{M}\boldsymbol{\Psi}\,\ddot{\mathbf{p}}(t)+\boldsymbol{\Psi}^{T}\mathbf{C}\boldsymbol{\Psi}\,\dot{\mathbf{p}}(t)+\boldsymbol{\Psi}^{T}\mathbf{K}\boldsymbol{\Psi}\,\mathbf{p}(t)=\boldsymbol{\Psi}^{T}\mathbf{f}(t)\Rightarrow$$
$$\mathbf{L}\,\ddot{\mathbf{p}}(t)+\boldsymbol{\Psi}^{T}\mathbf{C}\boldsymbol{\Psi}\,\dot{\mathbf{p}}(t)+\mathbf{N}\,\mathbf{p}(t)=\boldsymbol{\Psi}^{T}\mathbf{f}(t)$$

Finally, let us pre-multiply by $\mathbf{L}^{-1}$. It follows:

$$\mathbf{L}^{-1}\mathbf{L}\,\ddot{\mathbf{p}}(t)+\mathbf{L}^{-1}\boldsymbol{\Psi}^{T}\mathbf{C}\boldsymbol{\Psi}\,\dot{\mathbf{p}}(t)+\mathbf{L}^{-1}\mathbf{N}\,\mathbf{p}(t)=\mathbf{L}^{-1}\boldsymbol{\Psi}^{T}\mathbf{f}(t)\Rightarrow$$
$$\ddot{\mathbf{p}}(t)+\boldsymbol{\Gamma}\,\dot{\mathbf{p}}(t)+\boldsymbol{\Lambda}\,\mathbf{p}(t)=\mathbf{L}^{-1}\boldsymbol{\Psi}^{T}\mathbf{f}(t) \tag{3}$$

where:

$$\boldsymbol{\Gamma}=\mathbf{L}^{-1}\boldsymbol{\Psi}^{T}\mathbf{C}\boldsymbol{\Psi} \tag{4}$$

Since $\boldsymbol{\Psi}^{T}\mathbf{C}\boldsymbol{\Psi}$ is in general not diagonal, then also $\boldsymbol{\Gamma}$ is in general not diagonal. Thus, Eq. (3) is in general a set of coupled differential equations:

$$\boxed{\begin{array}{l}\ddot{p}_k(t)+\sum_{1}^{n}{}_{\ell}\,\gamma_{k\ell}\;\dot{p}_{\ell}(t)+\omega_k^2\;p_k(t)=\dfrac{1}{m_k}\,\boldsymbol{\psi}_k^{T}\;\mathbf{f}(t)\\ p_k(0)=p_{k0}\;;\dot{p}_k(0)=\dot{p}_{k0}\qquad (k=1,..n)\end{array}} \tag{5}$$

Obviously, Eq. (5) is a set of decoupled equations for **C** = **0**.
In other words, if the structural system is damped, the principal transformation generally does not decouple the equations of motion.

It is possible to show that, considering an initial perturbation, the motion tends to disappear, due to damping, through the contribution of all the vibration modes (not only if $\boldsymbol{\psi}_j$).

Decoupling conditions

Let us assume that **C** is such that **Γ** is diagonal, i.e. $\gamma_{k\ell} = 0$ for $k \neq \ell$. Thus Eq. (3) is decoupled and may be rewritten as:

$$\begin{aligned} &\ddot{p}_k(t) + \gamma_{kk}\, \dot{p}_k(t) + \omega_k^2\, p_k(t) = \frac{1}{m_k}\, \boldsymbol{\psi}_k^T\, \mathbf{f}(t) \\ &p_k(0) = p_{k0}\ ;\ \dot{p}_{k0} = \dot{p}_{k0} \qquad (k = 1, 2, ..n) \end{aligned} \tag{6}$$

Setting:

$$\gamma_{kk} = 2\,\xi_k \omega_k \tag{7}$$

it follows:

$$\boxed{\begin{aligned} &\ddot{p}_k(t) + 2\,\xi_k\, \omega_k\, \dot{p}_k(t) + \omega_k^2\, p_k(t) = \frac{1}{m_k}\, \boldsymbol{\psi}_k^T\, \mathbf{f}(t) \\ &p_k(0) = p_{k0}\ ;\ \dot{p}_{k0}(0) = \dot{p}_{k0} \qquad (k = 1, 2, ..n) \end{aligned}} \tag{8}$$

Thus, if **Γ** is diagonal, the damped forced vibrations of a n-D.O.F. system may be studied (likewise the undamped vibrations) as the forced vibrations on n S.D.O.F., each characterised by a modal damping ratio $\xi_k = \gamma_{kk} / (2\,\omega_k)$.

It is possible to show that, in this case, if the system is initially deformed on the j-th oscillation mode and left free to vibrate, it retains this shape of vibration on passing the time and all its D.O.F.s exhibit a motion with circular frequency $\omega_j = \sqrt{\lambda_j}$ and damping $\xi_j = \gamma_{jj} / (2\omega_j)$.

The systems endowed with such a property are called classically damped. The structures which do not satisfy this property are not classically damped.

Analogously, the damping is said to be classic if **C** is such that **Γ** is diagonal. The damping is not classic if **C** is such that **Γ** is not diagonal.

The necessary and sufficient condition which makes **Γ** not diagonal is (Canghey e O'Kelly, 1965):

$$\mathbf{C} = \mathbf{M} \sum_{1}^{n}{}_k\, a_k \left(\mathbf{M}^{-1}\mathbf{K}\right)^{k-1} \tag{9}$$

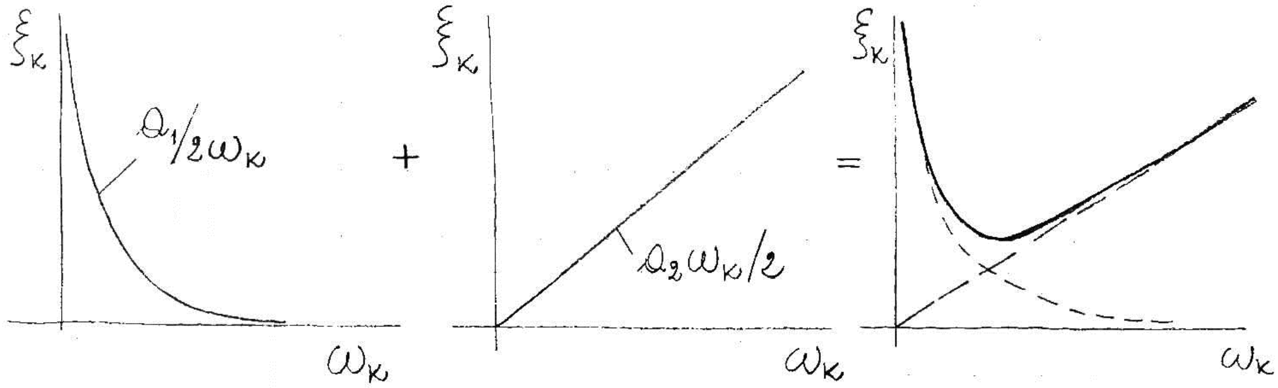
being a_k (k = 1, … n) suitable constants.
In particular, assuming $a_k = 0$ for k > 2, the following sufficient (not necessary) condition results:

$$\mathbf{C} = a_1\mathbf{M} + a_2\mathbf{K} \tag{10}$$

A structural system that satisfies Eq. (10) has a Rayleigh damping or a proportional damping. In such a case:

$$\boldsymbol{\Gamma} = \mathbf{L}^{-1}\,\boldsymbol{\Psi}^{T}\mathbf{C}\boldsymbol{\Psi} = \mathbf{L}^{-1}\,\boldsymbol{\Psi}^{T}\left(a_1\mathbf{M} + a_2\mathbf{K}\right)\boldsymbol{\Psi} =$$
$$= a_1\,\mathbf{L}^{-1}\,\boldsymbol{\Psi}^{T}\mathbf{C}\boldsymbol{\Psi} + a_2\,\mathbf{L}^{-1}\,\boldsymbol{\Psi}^{T}\mathbf{K}\,\boldsymbol{\Psi} =$$
$$= a_1\,\mathbf{L}^{-1}\mathbf{L} + a_2\,\mathbf{L}^{-1}\,\mathbf{N} = a_1\,\mathbf{I} + a_2\,\boldsymbol{\Lambda} \Rightarrow$$

$$\gamma_{kk} = 2\,\xi_k\omega_k = a_1 + a_2\,\omega_k^2 \Rightarrow \xi_k = \frac{a_1}{2\,\omega_k} + \frac{a_2\,\omega_k}{2}$$



In reality, the structures do not possess a classical or proportional damping. Nevertheless, being the definition of **C** very uncertain or difficult to evaluate, it is usual to avoid to evaluation of **C**, writing the equations of motion in their decoupled form, giving to ξ_k values suggested by experience.

Classically damped systems

$$\begin{cases} \mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{C}\dot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{f}(t) \\ \mathbf{q}(0) = \mathbf{q}_0 \; ; \; \dot{\mathbf{q}}(0) = \dot{\mathbf{q}}_0 \end{cases} \tag{1}$$

$$\mathbf{q}(t) = \mathbf{\Psi}\mathbf{p}(t) \Rightarrow \tag{2}$$

$$\begin{cases} \ddot{p}_k(t) + 2\,\xi_k\,\omega_k\,\dot{p}_k(t) + \omega_k^2\,p_k(t) = \dfrac{1}{m_k}\,\boldsymbol{\psi}_k^T\,\mathbf{f}(t) \\ p_k(0) = p_{k0}\;;\;\dot{p}_{k0}(0) = \dot{p}_{k0} \qquad (k = 1, ..n) \end{cases} \tag{3}$$

Time-domain analysis

Assuming for sake of simplicity $p_{k0} = \dot{p}_{k0} = 0 \;\; (k = 1, ..n)$ and using the Duhamel integral (*):

$$p_k(t) = \int_0^t h_{p_k}(t-\tau)\,\boldsymbol{\psi}_k^T\mathbf{f}(\tau)\,d\tau \quad (k = 1,..n) \tag{4}$$

where $\boldsymbol{\psi}_k^T\,\mathbf{f}(t)$ is the k-th modal force and:

$$h_{pk}(t) = e^{-\xi_k\omega_k t}\,\frac{1}{m_k\,\omega_k\sqrt{1-\xi_k^2}}\sin\omega_k\sqrt{1-\xi_k^2}\,t \quad (k = 1, ..n) \tag{5}$$

is the k-th modal response impulse function.

(*) In the case of a S.D.O.F.:

$$\ddot{q}(t) + 2\xi\omega_0\dot{q}_k(t) + \omega_0^2\;q(t) = \frac{1}{m}f(t)$$

$$q(t) = \int_0^t h(t-\tau)\,f(\tau)\,d\tau$$

$$h(t) = e^{-\xi\omega_0 t}\,\frac{1}{m\,\omega_0\sqrt{1-\xi^2}}\sin\omega_0\sqrt{1-\xi_0^2}\,t$$

The modal response impulse matrix is a $n \times n$ diagonal matrix whose k-th term is the k-th modal response impulse function:

$$\mathbf{h_p}(t) = \begin{bmatrix} h_{p1}(t) & 0 & \cdots & 0 \\ 0 & h_{p2}(t) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & h_{pn}(t) \end{bmatrix} \tag{6}$$

Using Eq. (6) the vector of the principal coordinates has the form:

$$\mathbf{p}(t) = \int_0^t \mathbf{h_p}(t-\tau)\boldsymbol{\psi}_k^T \mathbf{f}(\tau)d\tau \tag{7}$$

Using the principal transformation law:

$$\mathbf{q}(t) = \int_0^t \mathbf{h_p}(t-\tau)\boldsymbol{\psi}^T \mathbf{f}(\tau)d\tau \tag{8}$$

Now, let us remember that, whatever the damping may be:

$$\mathbf{q}(t) = \int_0^t \mathbf{h}(t-\tau)\mathbf{f}(\tau)d\tau \tag{9}$$

Comparing Eqs. (8) and (9) it results:

$$\mathbf{h}(t) = \boldsymbol{\psi}\,\mathbf{h_p}(t)\,\boldsymbol{\psi}^T \tag{10}$$

which provides a simple rule to evaluate the response impulse matrix. Obviously, this expression applies if and only if damping is classic.

$$\begin{cases} \mathbf{M}\ddot{\mathbf{q}}(t)+\mathbf{C}\dot{\mathbf{q}}(t)+\mathbf{K}\mathbf{q}(t)=\mathbf{f}(t) \\ \mathbf{q}(0)=\mathbf{q}_0\ ;\ \dot{\mathbf{q}}(0)=\dot{\mathbf{q}}_0 \end{cases} \tag{1}$$

$$\mathbf{q}(t)=\mathbf{\Psi}\mathbf{p}(t)\Rightarrow \tag{2}$$

$$\begin{cases} \ddot{p}_k(t)+2\,\xi_k\,\omega_k\,\dot{p}_k(t)+\omega_k^2\,p_k(t)=\dfrac{1}{m_k}\,\boldsymbol{\psi}_k^T\,\mathbf{f}(t) \\ p_k(0)=p_{k0}\ ;\ \dot{p}_{k0}(0)=\dot{p}_{k0} \qquad (k=1,\ ..n) \end{cases} \tag{3}$$

<u>Frequency-domain analysis</u>

Let us apply the Fourier transform of both members of Eq. (3). It results (*):

$$P_k(t)=H_{p_k}(\omega)\cdot\boldsymbol{\psi}_k^T\mathbf{F}(\omega)\quad (k=1,\ \ ..\ \ n) \tag{11}$$

where $\boldsymbol{\psi}_k^T\,\mathbf{F}(\omega)$ is the Fourier transform of the k-th modal force and:

$$H_{p_k}(\omega)=\frac{1}{m_k\omega_k^2}\frac{1}{1-\dfrac{\omega^2}{\omega_k^2}+2i\xi_k\dfrac{\omega}{\omega_k}}\quad (k=1,\ \ ..\ \ n) \tag{12}$$

is the k-th modal complex frequency response function.

(*) In the case of a S.D.O.F.:

$$\ddot{q}(t)+2\,\xi\,\omega_0\,\dot{q}_k(t)+\omega_0^2\,q(t)=\frac{1}{m}\,f(t)$$

$$Q(\omega)=H(\omega)\,F(\omega)$$

$$H(\omega)=\frac{1}{m\omega_0^2}\frac{1}{1-\dfrac{\omega^2}{\omega_0^2}+2i\xi\dfrac{\omega}{\omega_0}}$$

The modal complex frequency response matrix is a n $\times$ n matrix whose k-th term is the k-th modal complex frequency response function:

$$\mathbf{H_p}(\omega) = \begin{bmatrix} h_{p_1}(\omega) & 0 & \cdots & 0 \\ 0 & H_{p_2}(\omega) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & H_{p_n}(\omega) \end{bmatrix} \tag{13}$$

Using Eq. (13) the Fourier transform of the vector of the principal coordinates has the form:

$$\mathbf{P}(\omega) = \mathbf{H_p}(\omega) \cdot \mathbf{\Psi}^{T} \mathbf{F}(\omega) \tag{14}$$

Using the principal transformation law:

$$\begin{aligned} \mathbf{Q}(\omega) &= \mathbf{\Psi} \, \mathbf{P}(\omega) \Rightarrow \\ \mathbf{Q}(\omega) &= \mathbf{\Psi} \, \mathbf{H_p}(\omega) \, \mathbf{\Psi}^{T} \mathbf{F}(\omega) \end{aligned} \tag{15}$$

Now, let us remember that, whatever the damping may be:

$$\mathbf{Q}(\omega) = \mathbf{H}(\omega) \, \mathbf{F}(\omega) \tag{16}$$

Comparing Eqs. (15) and (16) it follows:

$$\mathbf{H}(\omega) = \mathbf{\Psi} \, \mathbf{H_p}(\omega) \, \mathbf{\Psi}^{T} \tag{17}$$

which provides a simple rule to evaluate the complex frequency response matrix. Obviously this expression applies if and only if damping is classic.

Modal truncation

Let us consider the principal transformation law:

$$\mathbf{q}(t) = \boldsymbol{\psi}\,\mathbf{p}(t) = \sum_{1}^{n} {}_{k}\boldsymbol{\psi}_{k} p_{k}(t) \tag{18}$$

where the k-th principal coordinate $p_k(t)$ is given by the solution of the differential equation:

$$\ddot{p}_k(t) + 2\xi_k \omega_k \dot{p}_k(t) + \omega_k^2 p_k(t) = \frac{1}{m_k} \boldsymbol{\psi}_k^T \mathbf{f}(t) \quad (k = 1,..n) \tag{19}$$

The modal truncation is a technique that replaces the rigorous Eq. (18) by the approximate equation:

$$\mathbf{q}(t) \cong \sum_{1}^{\bar{n}} {}_{k}\boldsymbol{\psi}_{k} p_{k}(t) \tag{20}$$

being $\bar{n} < n$.

Eq. (20) involves two fundamental advantages:

a) it allows to solve a number $\bar{n} < n$ of differential equations (19);
b) it allows to calculate only the first $\bar{n} < n$ eigenvalues and eigencetors (very useful applying iterative algorithms).

Experience shows that in most cases the choice $\bar{n} << n$ provides excellent approximations.